Exact Solution for the Interacting Kitaev Chain at the Symmetric Point

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The Kitaev chain model with a nearest neighbor interaction U is solved exactly at the symmetry point $\Delta = t$ and chemical potential $\mu = 0$ in an open boundary condition. By applying two Jordan-Wigner transformations and a spin rotation, such a symmetric interacting model is mapped onto a noninteracting fermion model, which can be diagonalized exactly. The solutions include a topologically nontrivial phase at $|U| < t$ and a topologically trivial phase at $|U| > t$. The two phases are related by dualities. Quantum phase transitions in the model are studied with the help of the exact solution.

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As a prototype of one-dimensional (1D) systems possessing Majorana zero modes (MZMs) [1] at two edges, the Kitaev chain model [2] has recently attracted a lot of attention [3–5]. This noninteracting spinless fermion model was initially solved in a ring with a periodic boundary condition. Because of the increased interest in the effect of the interaction in MZMs [6–15], the model has been generalized to include a nearest neighbor interaction. As examined by Fidkowski and Kitaev [16], such an interacting term may give rise to a transition from topological to trivial phases in one dimension, and the noninteracting classification of fermionic systems [17–19] may "collapse." The interacting Kitaev chain has been studied by many authors [20–23], including by using numerical methods [24–31]. On the other hand, the model does not have an analytic exact solution in the general case. The exact ground states are available in a special set of tuned parameters [32]. However, the parameter space in their solvable model does not include any phase transition point. In this Letter, we shall present an exact solution to the interacting Kitaev chain model at a symmetric point (see Fig. 1). We show that the symmetric model is integrable and present the solutions of the ground state and all excited states and demonstrate phase transitions from topologically nontrivial to trivial phases.

Model.—We consider spinless fermions in a chain of length L with open boundary condition. The Hamiltonian of such an interacting Kitaev chain reads

$$H = \sum_{j=1}^{L-1}[-t(c_j^\dagger c_{j+1} + \mathrm{H.c.}) + U(2n_j - 1)(2n_{j+1} - 1) - \Delta(c_j^\dagger c_{j+1}^\dagger + \mathrm{H.c.})] - \mu\sum_{j=1}^{L}\left(n_j - \frac{1}{2}\right), \quad (1)$$

where $c_j(c_j^\dagger)$ is the fermion annihilation (creation) operator on site j, $n_j = c_j^\dagger c_j$ is the fermion occupation number operator, t is the hopping integral, Δ is the p-wave superconducting pairing potential, μ is the chemical potential controlling the electron density, and U is the nearest neighbor interaction. Without loss of generality, both t and Δ are chosen to be real and positive. The parameter transformation of $\mu \to -\mu$ can be realized by the particle-hole conjugation $c_j \to (-1)^j c_j^\dagger$. Therefore, $\mu = 0$ corresponds to the particle-hole symmetry, which can be characterized by the particle-hole conjugation operator Z_2^p defined as follows:

$$Z_2^p = \prod_j [c_j + (-1)^j c_j^\dagger], \quad (2)$$

Z_2^p is conserved if and only if $\mu = 0$. It is easy to verify that $(Z_2^p)^2 = (-1)^L$ and $(Z_2^p)^\dagger Z_2^p = 1$. Hereafter, we shall assume that L is an even number, so that $(Z_2^p)^2 = 1$ and $Z_2^p = \pm 1$. Another good quantum number is the fermion number parity Z_2^f defined as

$$Z_2^f = e^{i\pi\sum_j n_j} = (-1)^{\hat{N}}, \quad (3)$$

where $\hat{N} = \sum_j n_j$ is the number of fermions in the system. It is obvious that $(Z_2^f)^2 = 1$ and $[H, Z_2^f] = 0$. Both Z_2^p and Z_2^f will be used to characterize the ground states of the model Eq. (1) in different phases.

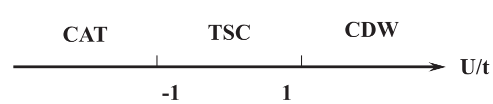

FIG. 1. Phase diagram of the symmetric interacting Kitaev chain with $\Delta = t$ and $\mu = 0$. The Shrödinger-cat-like state (a superposition of two trivial superconducting states with different occupation numbers), a topological superconducting state, and a charge density wave state (defined in text). Two critical points occur at $U = \pm t$.

At $U = 0$, the model is reduced to the usual noninteracting Kitaev chain model [2], which can be diagonalized exactly. For interacting cases, $U \neq 0$, an exact solution is not available in the literature so far, except that the ground states have been constructed by Katsura *et al.* [32] when chemical potential μ is tuned to a particular function of the other parameters (t, Δ, U). In this Letter, we shall study the interacting Kitaev model at the symmetric point of $\Delta = t$ and $\mu = 0$, and solve the model exactly by giving all the eigenstates. Note that a similar symmetric model has been constructed in the context of the Majorana linear chain without an analytic solution [33].

Majorana fermion representation.—We shall study the Hamiltonian in Eq. (1) in the Majorana fermion representation. Following Katsura *et al.* [32], we split one complex fermion operator into two Majorana fermion operators $c_j = \frac{1}{2}(\lambda_j^1 + i\lambda_j^2)$ and $c_j^\dagger = \frac{1}{2}(\lambda_j^1 - i\lambda_j^2)$. The Majorana fermion operators are real $(\lambda_j^a)^\dagger = \lambda_j^a$ and satisfy the anticommutation relations $\{\lambda_j^a, \lambda_l^b\} = 2\delta_{ab}\delta_{jl}$, where a, $b = 1$, 2. Thus, the Hamiltonian in Eq. (1) becomes

$$H = \sum_{j=1}^{L-1}\left[-\frac{i}{2}(t+\Delta)\lambda_{j+1}^1\lambda_j^2 - \frac{i}{2}(t-\Delta)\lambda_j^1\lambda_{j+1}^2 - U\lambda_j^1\lambda_j^2\lambda_{j+1}^1\lambda_{j+1}^2\right] - \frac{i}{2}\mu\sum_{j=1}^{L}\lambda_j^1\lambda_j^2. \quad (4)$$

At $U \neq 0$, the above Hamiltonian contains both quadratic and quartic terms, and cannot be diagonalized straightforwardly.

Mapping to a noninteracting chain.—The Hamiltonian in Eq. (4) can be mapped to a noninteracting model consisting of quadratic terms only, at $\Delta = t$ and $\mu = 0$. The mapping is composed of two Jordan-Wigner transformations [34,35] and a spin rotation. First, we construct spin operators by the first Jordan-Wigner transformation,

$$S_j^x = \frac{1}{2}\lambda_j^1 e^{i\pi\sum_{l<j} n_l}, \quad (5a)$$

$$S_j^y = -\frac{1}{2}\lambda_j^2 e^{i\pi\sum_{l<j} n_l}, \quad (5b)$$

$$S_j^z = \frac{i}{2}\lambda_j^1\lambda_j^2. \quad (5c)$$

Thus, the Hamiltonian in Eq. (4) can be written in terms of spin operators S_j^x and S_j^z,

$$H = \sum_{j=1}^{L-1} -4tS_j^xS_{j+1}^x + 4US_j^zS_{j+1}^z, \quad (6)$$

which is a typical XZ spin chain.

Second, we rotate all the spins by $\pi/2$ around the x axis using the rotation operator $R = e^{-i(\pi/2)\sum_j S_j^x}$. Then two new spin operators can be defined as $\tilde{S}_j^x \coloneqq RS_j^xR^{-1} = S_j^x$ and $\tilde{S}_j^y \coloneqq RS_j^yR^{-1} = S_j^z$. The XZ chain becomes an XY chain,

$$H = \sum_{j=1}^{L-1} -4t\tilde{S}_j^x\tilde{S}_{j+1}^x + 4U\tilde{S}_j^y\tilde{S}_{j+1}^y. \quad (7)$$

Such an XY spin chain has been exactly solved by Lieb, Schultz, and Mattis with the help of the Jordan-Wigner transformation [36].

Finally, following Lieb, Schultz, and Mattis, we use the second Jordan-Wigner transformation,

$$\tilde{S}_j^x = \frac{1}{2}\tilde{\lambda}_j^1 e^{i\pi\sum_{l<j}\tilde{n}_l}, \quad (8a)$$

$$\tilde{S}_j^y = -\frac{1}{2}\tilde{\lambda}_j^2 e^{i\pi\sum_{l<j}\tilde{n}_l}, \quad (8b)$$

$$\tilde{S}_j^z = \frac{i}{2}\tilde{\lambda}_j^1\tilde{\lambda}_j^2, \quad (8c)$$

to transform the XY chain model in Eq. (7) to a quadratic fermion Hamiltonian, which is given by

$$H = \frac{i}{2}\sum_{j,l=1}^{L}\tilde{\lambda}_j^1 B_{jl}\tilde{\lambda}_l^2, \quad (9)$$

where $B_{jl} = 2U\delta_{j,j+1} - 2t\delta_{j,j-1}$ is a $L \times L$ real matrix. $\tilde{\lambda}_j^{1,2}$ can be written in terms of the original Majorana fermion operators $\lambda_j^{1,2}$, explictly,

$$\tilde{\lambda}_j^1 = \begin{cases} \left(\prod_{l=\mathrm{odd}}^{j-2} i\lambda_l^2\lambda_{l+1}^1\right)\lambda_j^1, & j = \mathrm{odd}, \\ \left(\prod_{l=\mathrm{odd}}^{j-3} i\lambda_l^1\lambda_{l+1}^2\right)i\lambda_{j-1}^1\lambda_j^1, & j = \mathrm{even}, \end{cases} \quad (10a)$$

and

$$\tilde{\lambda}_j^2 = \begin{cases} \left(\prod_{l=\mathrm{odd}}^{j-2} i\lambda_l^1\lambda_{l+1}^2\right)i\lambda_j^1\lambda_j^2, & j = \mathrm{odd}, \\ \left(\prod_{l=\mathrm{odd}}^{j-3} i\lambda_l^2\lambda_{l+1}^1\right)\lambda_{j-1}^2 i\lambda_j^1\lambda_j^2, & j = \mathrm{even}, \end{cases} \quad (10b)$$

With the help of Eqs. (10), one is able to show that $\tilde{\lambda}_j^{1,2}$ are Majorana fermion operators by examining the relations $(\tilde{\lambda}_j^a)^\dagger = \tilde{\lambda}_j^a$ and $\{\tilde{\lambda}_j^a, \tilde{\lambda}_l^b\} = 2\delta_{ab}\delta_{jl}$, so that the two sets of operators $\{\lambda_j^a\}$ and $\{\tilde{\lambda}_j^a\}$ must be related by a unitary transformation [37].

Thus, when $\Delta = t$ and $\mu = 0$, the interacting fermion model given in Eq. (4) with arbitary U can be mapped to the noninteracting fermion model in Eq. (9) through the unitary transformation given in Eq. (10). This is the central result of this Letter. Note that Gangadharaiah *et al.* [6] also pointed out that the interacting fermion model can be reduced to a free gapless fermion gas at the critical point $U = \Delta = t$ and $\mu = 0$.

Exact diagonalization.—The quadratic form of the Hamiltonian can be exactly diagonalized by singular value decomposition (SVD) as follows. The nonsymmetric matrix B given in Eq. (9) can be written in the SVD form $B = U\Lambda V^T$ [32], where Λ is a non-negative diagonal matrix whose diagonal elements Λ_k give rise to the singular values of B. U and V are real orthogonal matrices and transform the Majorana fermion operators as $\tilde{\lambda}_k^1 = \sum_{j=1}^L U_{jk}\tilde{\lambda}_j^1$ and $\tilde{\lambda}_k^2 = \sum_{j=1}^L V_{jk}\tilde{\lambda}_j^2$. The self-conjugate and anticommutation relations remain the same, $(\tilde{\lambda}_k^a)^\dagger = \tilde{\lambda}_k^a$ and $\{\tilde{\lambda}_k^a, \tilde{\lambda}_q^b\} = 2\delta_{ab}\delta_{kq}$.

The diagonalized Hamiltonian reads

$$H = \frac{i}{2}\sum_k \tilde{\lambda}_k^1 \Lambda_k \tilde{\lambda}_k^2 = \sum_k \Lambda_k\left(\tilde{c}_k^\dagger \tilde{c}_k - \frac{1}{2}\right), \quad (11)$$

where $\tilde{c}_k = \frac{1}{2}(\tilde{\lambda}_k^1 + i\tilde{\lambda}_k^2)$ and $\tilde{c}_k^\dagger = \frac{1}{2}(\tilde{\lambda}_k^1 - i\tilde{\lambda}_k^2)$ are complex fermion operators. Thus, the energy spectrum is given by Λ_k, and reads

$$\Lambda_k = \sqrt{(t-U)^2\cos^2 k + (t+U)^2 \sin^2 k}, \quad (12)$$

where each k value gives rise to a single particle eigenstate in the rotated $(\tilde{\lambda})$ representation, which will be called the "k mode." The values of k and corresponding eigenstates can be determined similar to the noninteracting case [31] (See the Supplemental Material [38] for details). The spectrum Λ_k is gapful except at two quantum critical points $U = \pm t$.

There always exist $(L-1)$ real solutions and a single complex solution to k. When $|U| > t$, the complex solution $k = k_0^I$ will give rise to corresponding singular value $\Lambda_{k_0^I}$, which is separated from the bulk energy continuum and has the asymptotic form at large L as follows:

$$\Lambda_{k_0^I} \simeq \left(1 - \left|\frac{t}{U}\right|\right)\left|\frac{t}{U}\right|^{L/2}. \quad (13)$$

When $|U| < t$, the complex solution $k = k_0^{II}$ will give rise to the corresponding singular value $\Lambda_{k_0^{II}}$,

$$\Lambda_{k_0^{II}} \simeq \left(1 - \left|\frac{U}{t}\right|\right)\left|\frac{U}{t}\right|^{L/2}. \quad (14)$$

Ground states.—In a finite system, the ground state $|0\rangle$ is nondegenerate and the energy spectrum is gapped except $U = \pm t$. However, in the thermodynamic limit $L \to \infty$, $\Lambda_{k_0} \to 0$ exponentially, where $k_0 = k_0^I$ (k_0^{II}) for $|U| > t$ $(|U| < t)$. Thus, the first excited state $|1\rangle = c_{k_0}^\dagger |0\rangle$ is degenerate with the ground state $|0\rangle$ in the thermodynamic limit.

With the help of the spin XZ model given in Eq. (6), where long-range spin correlation exists along the S^x or S^z direction [39], one is able to compute the long-range density correlation $\rho_{ij} = \langle (2n_i - 1)(2n_j - 1)\rangle$ in the bulk as follows:

$$\lim_{|i-j|\to\infty} \rho_{ij} = \begin{cases} \sqrt{1-(t/U)^2}, & U < -t, \\ 0, & |U| < t, \\ (-1)^{i-j}\sqrt{1-(t/U)^2}, & U > t. \end{cases} \quad (15)$$

On the other hand, we have $\langle 0|n_j|0\rangle = \langle 1|n_j|1\rangle = \frac{1}{2}$. Therefore, the number fluctuation ΔN can be estimated for eigenstates $|0\rangle$ and $|1\rangle$: when $U < -t$, $(\Delta N)^2/N^2 \to \sqrt{1-(t/U)^2}$; otherwise, $(\Delta N)^2/N^2 \propto (1/N)$. There are three different parameter regions for the symmetric interacting Kitaev chain model $U < -t$, $-t < U < t$ and $U > t$. (i) When $U < -t$, the ground state is a Shrödinger-cat-like (CAT) state, which is a superposition of two trivial superconductor states with different occupation numbers, $\langle \hat{N}\rangle / L \sim (1 \pm \sqrt{1-(t/U)^2})/2$ [40]. (ii) When $-t < U < t$, the ground state is a topological superconductor (TSC) state. (iii) When $U > t$, the ground state is a charge density wave (CDW) state.

For a finite system, the first excited state $|1\rangle$ can be distinguished from the ground state $|0\rangle$ by the fermion number parity Z_2^f and the particle-hole conjugation Z_2^p, when $|1\rangle$ is not degenerate with $|0\rangle$. Since $[Z_2^f, H] = [Z_2^p, H] = 0$, any nondegenerate eigenstate $|\Psi_n\rangle$ of H is also an eigenstate of Z_2^f and Z_2^p, namely, $Z_2^f|\Psi_n\rangle = \pm|\Psi_n\rangle$ and $Z_2^p|\Psi_n\rangle = \pm|\Psi_n\rangle$. With the help of the exact solution, one is able to show that (i) when $|U| < t$, $\langle 1|Z_2^f|1\rangle = -\langle 0|Z_2^f|0\rangle$ and $\langle 1|Z_2^p|1\rangle = \langle 0|Z_2^p|0\rangle$, (ii) when $|U| > t$, $\langle 1|Z_2^f|1\rangle = \langle 0|Z_2^f|0\rangle$ and $\langle 1|Z_2^p|1\rangle = -\langle 0|Z_2^p|0\rangle$. Now let $L \to \infty$ to approach the thermodynamic limit, these Z_2^f and Z_2^p values can be used to characterize different phases with two degenerate states $|0\rangle$ and $|1\rangle$. For the TSC phase, Z_2^f has opposite values (± 1) while Z_2^p has the same value in the two degenerate ground states. For the CDW and CAT phases, Z_2^p has opposite values while Z_2^f has the same value. Then one can draw down the conclusion that the zero mode obtained in the TSC phase is a fermionic mode since it is made of odd number of fermions, while the zero mode obtained in the CDW or CAT phase is bosonic.

Duality symmetries.—There exist interesting dual relations between $|U| > t$ and $|U| < t$ phases when $\Delta = t$ and $\mu = 0$, which impose quantum critical points at $U = \pm t$. Such dualities can be seen clearly by rewriting Eq. (9) as follows:

$$H = \frac{i}{2}\left(\sum_{j=1}^{L-1} 2t\tilde{\lambda}_j^2\tilde{\lambda}_{j+1}^1 + 2U\tilde{\lambda}_j^1\tilde{\lambda}_{j+1}^2\right). \quad (16)$$

For $U > 0$, by interchanging the Majorana fermion operators in Eq. (16),

$$\tilde{\lambda}_j^1 \leftrightarrow \tilde{\lambda}_j^2, \quad (17a)$$

Hamiltonian H has the same form with parameters changes as

$$t \leftrightarrow U. \tag{17b}$$

Thus, the duality between TSC and CDW phases has been established and $U = t$ must be the phase transition point separating these two phases.

For $U < 0$, by interchanging the Majorana fermion operators as follows,

$$\tilde{\lambda}_j^1 \leftrightarrow (-1)^j \tilde{\lambda}_j^2, \tag{18a}$$

H will keep the same form with parameter changes as

$$t \leftrightarrow -U. \tag{18b}$$

Equations (18) set up the duality between TSC and CAT phases, and $U = -t$ must be the critical point separating these two phases.

It is interesting that the fermion number parity Z_2^f and the particle-hole conjugation Z_2^p will interchange to each other under the duality transformations. In order to see this, we rewrite Z_2^f and Z_2^p in terms of $\tilde{\lambda}_j^a$,

$$Z_2^f = \prod_{j=\text{odd}}^{L-1} (i\tilde{\lambda}_j^1 \tilde{\lambda}_{j+1}^2), \tag{19a}$$

and

$$Z_2^p = \prod_{j=\text{odd}}^{L-1} (i\tilde{\lambda}_j^2 \tilde{\lambda}_{j+1}^1). \tag{19b}$$

Therefore, we have the dual relation,

$$Z_2^f \leftrightarrow Z_2^p. \tag{20}$$

It is noted that the fermion number parity in the rotated representation $\tilde{Z}_2^f$ is self-dual, say,

$$\tilde{Z}_2^f \leftrightarrow \tilde{Z}_2^f. \tag{21}$$

Phase transitions.—The exact solution to the Hamiltonian H also allows us to explore the quantum phase transition between neighboring phases. The bulk spin correlation functions for the XY spin chain have been evaluated by McCoy [39] as well as Capel and Perk [41]. In a previous paper, we proposed to use an edge correlation function to characterize the phase transition between the TSC and SC phases [31], which is defined as follows:

$$G_{1L} = \langle i\lambda_1^1 \lambda_L^2 \rangle. \tag{22}$$

In the rotated representation, it reads

$$G_{1L} = \langle i\tilde{\lambda}_1^1 \tilde{\lambda}_L^2 \tilde{Z}_2^f Z_2^f \rangle. \tag{23}$$

The detailed calculations for G_{1L} can be found in the Supplemental Material [38]. For a generic ground state in the thermodynamic limit, the edge correlation function behaves as follows:

$$\lim_{L\to\infty} G_{1L} \propto \begin{cases} 1 - \left(\frac{|U|}{t}\right)^2, & |U| < t, \\ 0, & |U| \geq t. \end{cases} \tag{24}$$

The edge correlation function G_{1L} is finite only in the TSC state in the thermodynamic limit. Around the quantum phase transition points $U = \pm t$,

$$G_{1L} \propto (t - |U|)^z, \tag{25}$$

with critical exponent $z = 1$, which is the same as that for the TSC to SC transition.

In summary, we have studied in this Letter the interacting Kitaev chains with an open boundary condition at the symmetric case $\Delta = t$ and $\mu = 0$. Exact solutions of all the eigenvalues and corresponding eigenstates are obtained. We find three different ground states: a Shrödinger-cat-like state at $U < -t$, a topological superconducting state at $-t < U < t$, and a charge density wave state at $U > t$. Duality symmetries between the CAT and TSC and between TSC and CDW are found. The quantum phase transitions in the system are described by the edge correlation function, and the critical exponent is found to be $z = 1$. In addition to the ground state properties discussed in this Letter, dynamics, thermodynamics, and spectral function can also be studied through the exact solution. The interaction effect in the TSC phase can be observed by measurement of tunneling conductance $dI/dV \propto \rho(1,\omega)$ at the edge. The edge density of states $\rho(1,\omega)$ is of the form $\rho(1,\omega) = A\delta(\omega) + B(\omega)\theta(\omega - \Delta_g)$, where Δ_g is the bulk energy gap, $A \propto 1 - (U/t)^2$ and $B(\omega) \propto \omega(\omega^2 - \Delta_g^2)^\alpha$ with $\alpha \to \frac{1}{2}$ when $U \to 0$ and $\alpha \to -\frac{1}{2}$ when $|U| \to t$. The transition from positive to negative value of α exhibits a qualitative difference in spectra and can be observed in tunneling experiments. Such a drastic transition within the TSC phase (see Supplemental Material [38] for details) suggests that the dynamic properties associated with excited states may undergo a transition although the ground states are protected by the energy gap.

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[38] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.118.267701, in the first part, we prove that the new operators in the rotated representation $\tilde{\lambda}$ are real and satisfy the anticommutation relations. These properties of the operators $\tilde{\lambda}$ show that they are Majorana fermion operators. In the second part, we give the expression of the operators Z_2^f and Z_2^p in terms of $\tilde{\lambda}$ in the rotated representation. In the third part, we calculate the edge correlation, which characterize the topological superconductor. In the fourth part, we calculate the spectrum function in the topological superconductor state and discuss its experimental consequence.
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