Light Meets Water in Nonlocal Media: Surface Tension Analogue in Optics

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Shallow water wave phenomena find their analogue in optics through a nonlocal nonlinear Schrödinger (NLS) model in $2+1$ dimensions. We identify an analogue of surface tension in optics, namely, a single parameter depending on the degree of nonlocality, which changes the sign of dispersion, much like surface tension does in the shallow water wave problem. Using multiscale expansions, we reduce the NLS model to a Kadomtsev-Petviashvili (KP) equation, which is of the KPII (KPI) type, for strong (weak) nonlocality. We demonstrate the emergence of robust optical antidark solitons forming Y-, X-, and H-shaped wave patterns, which are approximated by colliding KPII line solitons, similar to those observed in shallow waters.

Many physically different contexts can be brought together through a similar modeling and mathematical description. A common (and rather unlike) example is water waves and nonlinear optics. Two models are inextricably linked with both subjects: the universal Korteweg–de Vries (KdV) and nonlinear Schrödinger (NLS) equations [1]. Furthermore, these models can be reduced from one to the other [2], thus, suggesting that phenomena occurring in water waves may also exist in optics. Here, using such reductions for a nonlocal NLS, we find that surface tension—which causes fluids to minimize the area they occupy—has a direct analogue in optics.

Key to our findings are solitons, i.e., robust localized waves that play a key role in numerous studies in physics [3], applied mathematics [4], and engineering [5]. A unique property of solitons is that they feature a particle like character; i.e., they interact elastically, preserving their shapes and velocities after colliding with each other. Such elastic collisions as well as pertinent emerging wave patterns can sometimes be observed even in everyday life. A predominant example is the one pertaining to flat beaches: in such shallow water wave settings, two-line solitons merging at proper angles give rise to patterns of X-, H-, or Y-shaped waves, as well as other more complicated nonlinear waveforms [6]. All these shallow water wave structures are actually exact analytical multidimensional line soliton solutions of the Kadomtsev-Petviashvili (KP) equation (which generalizes KdV to two dimensions (2D) [1,4]) of the KPII type—a key model in the theory of shallow water waves with weak surface tension [1]. The relative equation with strong surface tension is referred to as KPI.

Here we show that such patterns can also be observed in a quite different physical setting, i.e., the one related to optical beam propagation in media with a spatially nonlocal defocusing nonlinearity. Such media include thermal nonlinear optical media [7,8], partially ionized plasmas [9,10], nematic liquid crystals [11,12], and dipolar bosonic quantum gases [13]. It is shown that approximate solutions of

the nonlocal NLS model satisfy, at proper scales, equations that appear in the context of water waves: a Boussinesq or Benney-Luke [14], as well as a KP equation. For a relatively strong (weak) nonlocality, or background amplitude, the latter is found to be a KPII (KPI), similar to the water wave problem, where a KPII (KPI) results in the case of small (large) surface tension [1]. Our results, thus, suggest an analogue of surface tension in optics. Direct numerical simulations show that approximate antidark line soliton solutions of the nonlocal NLS constructed from the KPII line soliton solutions form patterns observable in shallow water [6]. Pertinent Y-, X-, or H-wave patterns may be realized in an experimental setup similar to the one used for the observation of antidark solitons [15].

The evolution of optical beams in nonlinear defocusing media is governed by the following paraxial wave equation (cf. Ref. [16] for derivation and relevant adimensionalizations):

$$i u_t + \frac{1}{2}\Delta u - n u = 0, \tag{1}$$

where subscripts denote partial derivatives, u is the complex electric field envelope, $\Delta \equiv \partial_x^2 + \partial_y^2$ is the transverse Laplacian, and real function n denotes the nonlinear, generally nonlocal, medium response. For instance, in optics, n is the nonlinear change of the refractive index depending on the intensity $I = |u|^2$ [7,8], in plasmas it is the relative electron temperature perturbation [9,10], in liquid crystals it is the optically induced angle perturbation [11,12], and so on. Here, we consider that n obeys the following diffusion-type equation:

$$d^2 \Delta n - n + |u|^2 = 0, \tag{2}$$

where d is a spatial scale (setting the diffusion length) that measures the degree of nonlocality. Note that for $d = 0$, Eqs. (1) and (2) reduce to the defocusing 2D NLS equation [16]. Importantly, Eqs. (1) and (2), the nonlocal NLS

model of principal interest herein, has been used satisfactorily to model experiments on liquid solutions exhibiting thermal nonlinearities [17,18], while it has also been used in studies of plasmas [9,10] and nematic liquid crystals [11,12].

The steady-state solution of Eqs. (1) and (2) is composed by the continuous wave (cw), $u = u_0 \exp(-i|u_0|^2 t)$ (u_0 being an arbitrary complex constant) and the constant function $n = |u_0|^2$. Considering small perturbations of this solution behaving like $\exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$, with $\boldsymbol{r} = (x, y)$, we find that the perturbations' wave vector $\boldsymbol{k} = (k_x, k_y)$ and frequency ω obey the dispersion relation:

$$\omega^2 = |\boldsymbol{k}|^2 C^2 (1 + d^2|\boldsymbol{k}|^2)^{-1} + (1/4)|\boldsymbol{k}|^4, \quad (3)$$

where $C^2 = |u_0|^2$ is the wave velocity. Here, it is important to observe the following. First, since $\omega \in \mathbb{R} \forall \boldsymbol{k}$, the steady-state solution is modulationally stable. Second, in the long-wavelength limit ($|\boldsymbol{k}|^2 \ll 1$), Eq. (3) becomes $\omega^2 \approx |\boldsymbol{k}|^2 C^2 + (1/4)\alpha|\boldsymbol{k}|^4$, where $\alpha = 1 - 4d^2|u_0|^2$. This approximate dispersion relation features a striking similarity to the corresponding (approximate) one for shallow water waves, namely [1], $\omega^2 \approx |\boldsymbol{k}|^2 c_0^2 + (1/3)(3\hat{T} - 1)c_0^2 h^2 |\boldsymbol{k}|^4$, where $c_0^2 = gh$ is the velocity, g is the acceleration of gravity, h the depth of water at rest, and $\hat{T} = T/(\rho g h^2)$, with ρ being the density and T the surface tension. Comparing these dispersion relations, the following correspondence is identified: $3\hat{T} \to 4d^2|u_0|^2$, implying that there exists a surface tension analogue in our problem, $\propto d^2|u_0|^2$. This effective surface tension is negative, as is also implied by the fact that the term $\propto d$ in the Hamiltonian $\mathcal{H} = (1/2) \int_{\mathbb{R}^2} (|\boldsymbol{\nabla} u|^2 - d^2(\boldsymbol{\nabla} n)^2 - n^2 + 2n|u|^2)d\boldsymbol{r}$ of Eqs. (1) and (2) decreases the potential energy of the system, opposite to the water wave case where the surface tension increases the respective potential energy [19].

These arguments can be further solidified by analyzing the fully nonlinear problem: similar to water waves [1], we will derive KPI and KPII equations, depending on the strength of the effective surface tension, i.e., the parameter α, which sets the dispersion coefficient in KP. This can already be identified from the linear theory as follows. Using $|\boldsymbol{k}|^2 = k_x^2 + k_y^2$, the long-wavelength limit of Eq. (3) reads $\omega = \pm C k_x [1 + (k_y/k_x)^2]^{1/2}[1 + (\alpha/4C^2)k_x^2 + \mathcal{O}(k_y^2)]$, with $\pm$ corresponding to right- and left-going waves. Assuming $|k_y/k_x| \ll 1$ and $k_y^2 \sim \mathcal{O}(k_x^4)$, we find $(1/C)\omega k_x = \pm[k_x^2 + (\alpha/4C^2)k_x^4 + (1/2)k_y^2]$. Then, using $\omega \to i\partial_t, k_{x,y} \to -i\partial_{x,y}$, the linear PDE associated to this dispersion relation is $\partial_x[\pm q_t + Cq_x - (\alpha/8C)q_{xxx}] + (C/2)q_{yy} = 0$. This is a linear KP equation, with a dispersion coefficient depending on the effective surface tension through α, similar to shallow water waves, where the respective dispersion coefficient depends on $\hat{T}$ [1].

To derive the full nonlinear version of the KP equation, we resort to multiple scales. We, thus, consider small-amplitude slowly varying modulations of the steady state

and seek solutions of Eqs. (1) and (2) in the form of the asymptotic expansions:

$$u = u_0\sqrt{\rho} \exp(-i|u_0|^2 t + i\epsilon^{1/2}\Phi), \quad (4)$$

$$\rho = 1 + \sum_{j=1}^{\infty} \epsilon^j \rho_j, \qquad n = |u_0|^2 + \sum_{j=1}^{\infty} \epsilon^j n_j, \quad (5)$$

where $0 < \epsilon \ll 1$ is a formal small parameter, while phase Φ and amplitudes ρ_j and n_j are unknown real functions of the slow variables $X = \epsilon^{1/2}x$, $Y = \epsilon^{1/2}y$, and $T = \epsilon^{1/2}t$. Substituting the expansions (4) and (5) into Eqs. (1) and (2), we obtain the following results. First, the real part of Eqs. (1) and (2) yields the leading-order equations at $\mathcal{O}(\epsilon^{3/2})$ and $\mathcal{O}(\epsilon)$,

$$\rho_{1T} + \tilde{\Delta}\Phi = 0, \qquad n_1 = |u_0|^2\rho_1, \quad (6)$$

and the first-order equations at $\mathcal{O}(\epsilon^{5/2})$ and $\mathcal{O}(\epsilon^2)$,

$$\rho_{2T} + \tilde{\boldsymbol{\nabla}} \cdot (\rho_1 \tilde{\boldsymbol{\nabla}}\Phi) = 0,$$
$$d^2\tilde{\Delta}n_1 - n_2 + |u_0|^2\rho_2 = 0, \quad (7)$$

connecting the amplitudes $\rho_{1,2}$ and $n_{1,2}$ with the phase Φ; here, $\tilde{\Delta} \equiv \partial_X^2 + \partial_Y^2$ and $\tilde{\boldsymbol{\nabla}} \equiv (\partial_X, \partial_Y)$. Second, the imaginary part of Eq. (1) combined with Eqs. (6) and (7) yields

$$\Phi_{TT} - C^2\tilde{\Delta}\Phi + \epsilon\left[\frac{1}{4}\alpha\tilde{\Delta}^2\Phi + \frac{1}{2}\partial_T(\tilde{\boldsymbol{\nabla}}\Phi)^2\right.$$
$$\left. + \tilde{\boldsymbol{\nabla}} \cdot (\Phi_T\tilde{\boldsymbol{\nabla}}\Phi)\right] = \mathcal{O}(\epsilon^2). \quad (8)$$

Equation (8) incorporates fourth-order dispersion and quadratic nonlinear terms resembling the Boussinesq and Benney-Luke [14] equations, which describe bidirectional shallow water waves [1]. Similar to the water wave problem, we now use a multiscale expansion method to derive the KP equation under the additional assumptions of quasi-two-dimensionality and unidirectional propagation. In particular, we introduce the asymptotic expansion $\Phi = \Phi_0 + \epsilon\Phi_1 + \cdots$, where functions Φ_ℓ ($\ell = 0, 1, \ldots$) depend on the variables $\xi = X - CT$, $\eta = X + CT$, $\mathcal{Y} = \epsilon^{1/2}Y$, and $\mathcal{T} = \epsilon T$. Substituting this expansion into Eq. (8), at the leading order in ϵ, we obtain the wave equation $\Phi_{0\xi\eta} = 0$, implying that Φ_0 can be expressed as a superposition of a right-going wave $\Phi_0^{(R)}$ depending on ξ and a left-going one $\Phi_0^{(L)}$ depending on η, namely,

$$\Phi_0 = \Phi_0^{(R)}(\xi, \mathcal{Y}, \mathcal{T}) + \Phi_0^{(L)}(\eta, \mathcal{Y}, \mathcal{T}). \quad (9)$$

In addition, at order $\mathcal{O}(\epsilon)$, we obtain the equation

$$4C^2\Phi_{1\xi\eta} = -C(\Phi_{0\xi\xi}^{(R)}\Phi_{0\eta}^{(L)} - \Phi_{0\xi}^{(R)}\Phi_{0\eta\eta}^{(L)})$$

$$+ \left[\partial_\xi \left(-2C\Phi_{0T}^{(R)} + \frac{\alpha}{4}\Phi_{0\xi\xi\xi}^{(R)} - \frac{3C}{2}\Phi_{0\xi}^{(R)2} \right) - C^2\Phi_{0yy}^{(R)} \right]$$

$$+ \left[\partial_\eta \left(2C\Phi_{0T}^{(L)} + \frac{\alpha}{4}\Phi_{0\tilde{\eta}\tilde{\eta}\tilde{\eta}}^{(L)} + \frac{3C}{2}\Phi_{0\tilde{\eta}}^{(L)2} \right) - C^2\Phi_{0yy}^{(L)} \right].$$

When integrating this equation, secular terms arise from the square brackets, which are functions of ξ or η alone, not both. Removal of these secular terms leads to two uncoupled nonlinear evolution equations for $\Phi_0^{(R)}$ and $\Phi_0^{(L)}$. Then, using $\Phi_T = -n_1$ obtained from the leading-order part of Eq. (8) together with Eq. (7), it is found that the amplitude ρ_1 can also be decomposed to a left- and a right-going wave, i.e., $\rho_1 = \rho_1^{(R)} + \rho_1^{(L)}$, which satisfy the following KP equations:

$$\partial_{\mathcal{X}} \left(\pm\rho_{1T}^{(R,L)} - \frac{\alpha}{8C}\rho_{1\mathcal{X}\mathcal{X}\mathcal{X}}^{(R,L)} + \frac{3C}{4}\rho_1^{(R,L)2} \right) + \frac{C}{2}\rho_{1yy}^{(R,L)} = 0,$$

where $\mathcal{X} = \xi$ ($\mathcal{X} = \eta$) for the right- (left-) going wave. Next, for right-going waves, we use the transformations $\mathcal{T} \to -(\alpha/8C)\mathcal{T}, \mathcal{Y} \to \sqrt{3|\alpha|/4C^2}\mathcal{Y}$, and $\rho_1^{(R)} = -(\alpha/2C^2)U$ and express KP in its standard dimensionless form [1,4]:

$$\partial_{\mathcal{X}}(U_\mathcal{T} + 6UU_\mathcal{X} + U_{\mathcal{X}\mathcal{X}\mathcal{X}}) + 3\sigma^2 U_{yy} = 0, \quad (10)$$

where $\sigma^2 = -\text{sgn}\alpha = \text{sgn}(4d^2|u_0|^2 - 1)$. Importantly, Eq. (10) includes both versions of the KP equation, KPI and KPII [4]. Indeed, for $\sigma^2 = 1 \Rightarrow \alpha < 0$, i.e., for $4d^2|u_0|^2 > 1$, Eq. (10) is a KPII equation; on the other hand, for $\sigma^2 = -1 \Rightarrow \alpha > 0$, i.e., $4d^2|u_0|^2 < 1$, Eq. (10) is a KPI equation. Thus, for a fixed cw intensity $|u_0|^2$, a strong (weak) nonlocality d^2, as defined by the above inequalities, corresponds to KPII (KPI); the same holds for a fixed degree of nonlocality d^2 and a larger (smaller) cw intensity $|u_0|^2$. Thus, both our linear and nonlinear analyses establish a "homeomorphism" between optics and shallow water waves: in this latter context, weak surface tension (typical for water waves) corresponds to $\sigma^2 = 1$ in Eq. (10) (i.e., to KPII), while strong surface tension is pertinent to $\sigma^2 = -1$ (i.e., to KPI) [1,4].

Based on the above analysis, we now utilize the exact solutions of the KP, Eq. (10), and construct approximate solutions of the original system of Eqs. (1) and (2); such solutions read

$$u \approx u_0 \left(1 - \epsilon\frac{\alpha}{2|u_0|^2}U \right)^{1/2} \exp(-i|u_0|^2 t)$$

$$\times \exp \left(\frac{i}{2}\alpha\epsilon^{-1/2} \int_0^{\mathcal{T}} U d\mathcal{T}' \right), \quad n \approx |u_0|^2 - \frac{1}{2}\alpha U. \quad (11)$$

Clearly, for $\alpha < 0$ ($\alpha > 0$), i.e., for solutions satisfying KPII (KPI), u in Eq. (11) has the form of a hump (dip) on top

(off) of the cw background and is, thus, an antidark (dark) soliton.

Notice that in the local limit of $d = 0$ (i.e., $\alpha = 1$), we solely obtain the KPI model in which line solitons are unstable: as was shown in plasma physics and hydrodynamics [20], line solitons develop undulations and eventually decay into lumps [21]. In the same venue, but now in optics, the asymptotic reduction of the defocusing 2D NLS to KPI [22,23] and the instability of the line solitons of the latter was used to better understand the transverse instability of rectilinear dark solitons: indeed, these structures also develop undulations and eventually decay into vortex pairs [23,24].

Here, we focus on the stable soliton solutions of the KP equations, namely, the (antidark) line solitons of the KPII equation and the (dark) lump of KPI. The one-line soliton solution traveling at an angle to the $\mathcal{Y}$ axis is

$$U(\mathcal{X}, \mathcal{Y}, \mathcal{T}) = 2\kappa^2 \text{sech}^2(\mathcal{Z}), \quad (12)$$

where $\mathcal{Z} \equiv \kappa[\mathcal{X} + \lambda\mathcal{Y} - (4\kappa^2 + 3\lambda^2)\mathcal{T} + \delta]$, with κ, λ, and δ being free parameters. On the other hand, the two-line soliton can be expressed in the following form:

$$U = 2\partial_{\mathcal{X}}^2 \ln(1 + e^{\mathcal{Z}_1} + e^{\mathcal{Z}_2} + e^{\mathcal{Z}_1 + \mathcal{Z}_2 + A_{12}}), \quad (13a)$$

$$\exp(A_{12}) = \frac{4(\kappa_1 - \kappa_2)^2 - (\lambda_1 - \lambda_2)^2}{4(\kappa_1 + \kappa_2)^2 - (\lambda_1 - \lambda_2)^2}, \quad (13b)$$

where $\mathcal{Z}_i \equiv \kappa_i[\mathcal{X} + \lambda_i\mathcal{Y} - (4\kappa_i^2 + 3\lambda_i^2)\mathcal{T} + \delta_i]$.

As was shown and observed in the context of shallow water waves [6], when two-line solitons of the KPII intersect, a plethora of patterns can emerge. We focus here on the ones most frequently observed in shallow waters. To do this, we fix $\epsilon = 0.2$, $d^2 = 1/3$, and $u_0 = 1$ and choose two-line solitons with specific parameters so that the angle of interaction will lead to different patterns. We evolve these initial waves up to $t = 600$ so that they have enough time to interact. In Fig. 1, we show the resonant interaction of two-line solitons resulting in a Y-type wave. The
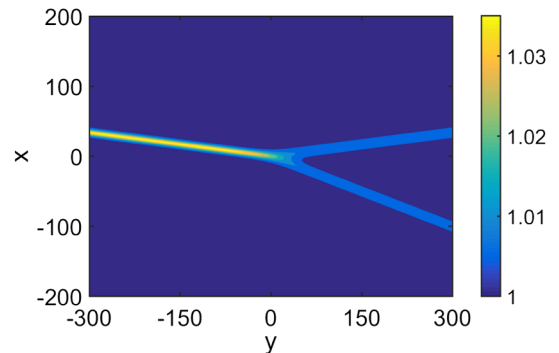

FIG. 1. A Y-type interaction for $2\kappa_1 = \kappa_2 = 1$ and $\lambda_1 = 3\lambda_2 = \frac{1}{4}$.

parameters leading to these interactions are summarized in the figure captions.

X-type waves can also emerge, as shown in Fig. 2. These structures are essentially discriminated by their "stems": a short, intermediate, and a long stem are, respectively, depicted in the top, middle, and bottom panels of Fig. 2; notice that the long stem's height is higher than that of the incoming line solitons. In addition, we can produce long stem interactions where the stem height is lower than the tallest incoming line soliton; cf. Fig. 3. We refer to these patterns as *H* type.

While we chose the above patterns as they appear more frequently in water, other more exotic, weblike structures are also supported by the KPII equation [25–27], and may, in principle, also be produced in optics. Furthermore, these solutions, while approximate, also hold well beyond the small-amplitude limit: similar results (not shown here) were obtained even for $\epsilon = 0.7$, with the only additional effect being emission of noticeable radiation. This is due to the robustness of KPII solitons, which is also verified by the observation of these patterns in shallow water, even after the waves break [6].

From the viewpoint of experiments, observations of antidark solitons were reported in Refs. [15,28]. The *Y*, *X*, and *H* waves may be observed experimentally using a setup similar to that of Ref. [15]. In particular, one may employ at first a cw laser beam, which is split into two parts via a beam splitter. One branch goes through a cavity system to form a pulse (as happens in typical pulsed lasers); this pulse branch undergoes phase engineering, i.e., passes through a phase mask so that the characteristic phase jump of the antidark soliton is inscribed. Then, the cw and the phase-engineered pulse are incoherently coupled inside the nonlocal medium, e.g., a nematic liquid crystal described by Eqs. (1) and (2). This process forms one antidark soliton, as in Ref. [15]. To observe the *Y*, *X*, or *H* patterns predicted above, two such antidark solitons have to be combined inside the crystal. The angle between the two incident beams, which should be appropriately chosen so that a specific pattern be formed, can be controlled by a rotating mirror in one of the branches.

Finally, let us consider the KPI case ($\alpha > 0$). KPI also exhibits line soliton solutions, as above, which are, however, unstable; it is, thus, most known for its solution that decays algebraically in both spatial coordinates, i.e., the lump

$$U(\mathcal{X}, \mathcal{Y}, \mathcal{T})$$
$$= 4\frac{-(\mathcal{X}+a\mathcal{Y}+3(a^2-b^2)\mathcal{T})^2+b^2(\mathcal{Y}+6a\mathcal{T})^2+1/b^2}{[(\mathcal{X}+a\mathcal{Y}+3(a^2-b^2)\mathcal{T})^2+b^2(\mathcal{Y}+6a\mathcal{T})^2+1/b^2]^2},$$

where a, b are free real parameters. Lumps have not yet been observed in water. In Fig. 4, we show a direct
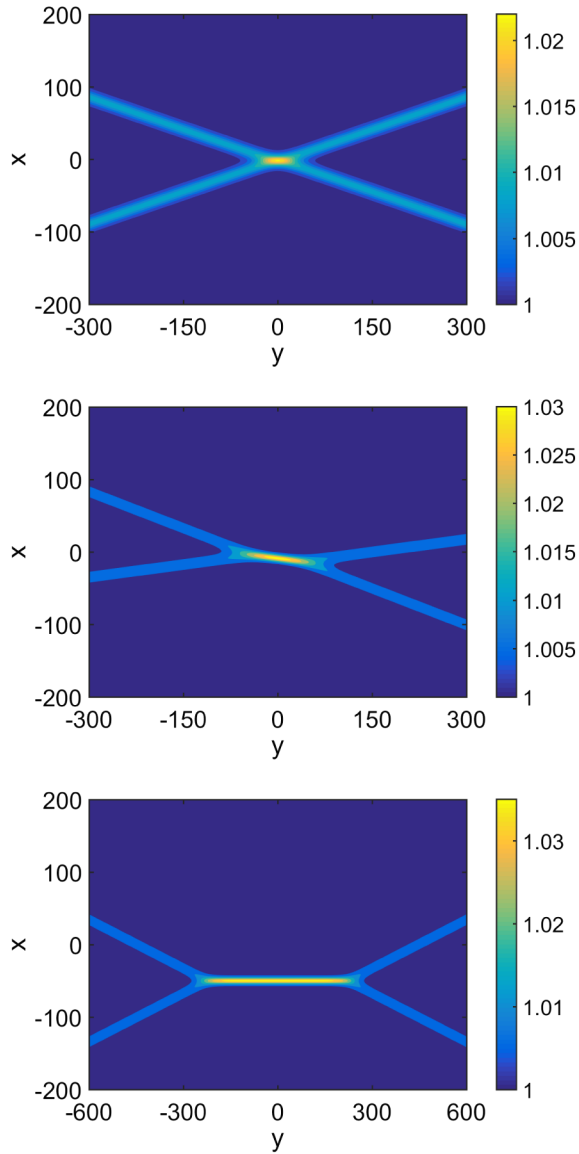

FIG. 2. *X*-type interactions. Top: Short stem for $\kappa_1 = \kappa_2 = \frac{1}{2}$ and $\lambda_1 = -\lambda_2 = \frac{2}{3}$. Middle: Intermediate stem for $\kappa_1 = \kappa_2 = \frac{1}{2}$, $\lambda_1 = -\frac{1}{4} - 10^{-2}$, and $\lambda_2 = \frac{3}{4}$. Bottom: Long stem for $\kappa_1 = \kappa_2 = \frac{1}{2}$ and $\lambda_1 = -\lambda_1 + 10^{-10} = \frac{1}{2}$.
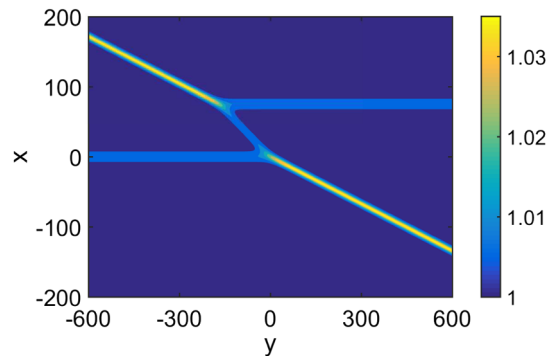

FIG. 3. An *H*-type interaction with $2\kappa_1 = \kappa_2 = 1$, $\lambda_1 = \frac{1}{2} - 10^{-7}$, and $\lambda_2 = 0$.
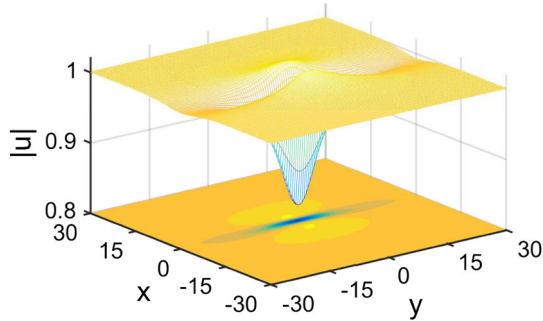
FIG. 4.   A typical lump solution with $a = 0$, $b = 1$.

simulation for the dark lump, and we refer the reader to the recent work [29] for details on multilump solutions and their interactions.

Concluding, we have established a "homeomorphism" between nonlocal nonlinear media and shallow water waves. In particular, we have shown that there exists a surface tension analogue in optics, depending on the nonlocality strength. This has been identified from the linear theory and has been rigorously analyzed in the fully nonlinear regime by the asymptotic reduction of a nonlocal NLS system to the KP equations. We have demonstrated that by depending on the effective surface tension, i.e., the degree of nonlocality, novel structures can appear in optical nonlocal media. Thus, fascinating phenomena that appear in water waves can also be observed in optics; an example studied here is the emergence of X-, H-, or Y-shaped waves resulting from the resonant interactions of stable line antidark solitons that have beeb found to exist in strongly nonlocal media. For weakly nonlocal ones, dark lumps have also been predicted to occur. The structures predicted in this work may, in principle, be experimentally observed either by choosing a material of specific nonlocality or in a specific material altering the magnitude of the cw background, in a setup similar to the one used for the observation of antidark solitons [15].

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