Entanglement and Nonlocality in Infinite 1D Systems

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We consider the problem of detecting entanglement and nonlocality in one-dimensional (1D) infinite, translation-invariant (TI) systems when just near-neighbor information is available. This issue is deeper than one might think *a priori*, since, as we show, there exist instances of local separable states (classical boxes) which admit only entangled (nonclassical) TI extensions. We provide a simple characterization of the set of local states of multiseparable TI spin chains and construct a family of linear witnesses which can detect entanglement in infinite TI states from the nearest-neighbor reduced density matrix. Similarly, we prove that the set of classical TI boxes forms a polytope and devise a general procedure to generate all Bell inequalities which characterize it. Using an algorithm based on matrix product states, we show how some of them can be violated by distant parties conducting identical measurements on an infinite TI quantum state. All our results can be easily adapted to detect entanglement and nonlocality in large (finite, not TI) 1D condensed matter systems.

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Imagine a scenario where a number of scientists are sent on a space exploration mission. Confined to separate vessels, they can only probe their immediate surroundings and communicate the outcomes of their experiments. We do not need to specify the exact nature of those experiments, but one could think, for instance, that each scientist is locally interacting with the vacuum state of a global quantum field.

After conducting such experiments in different places, what the scientists find out is that they always obtain the same statistics, no matter where they are, as long as the relative position between their vessels is the same. Unable to explore the whole Universe, they postulate that this property must hold elsewhere, in addition to the regions they already visited. To model this assumption physically, we picture these scientists probing different sites of an infinite translation-invariant (TI) system.

For further elucidation, we consider the simplest such scenario where the scientists live in a world that has one spatial dimension, so experiments are conducted at equidistant points on a straight line. The question we want to address is the following: From the information gathered by a small neighborhood of scientists, what global properties can they infer about the whole—infinite, unexplored—one-dimensional (1D) TI system? In this Letter, we will focus on two: (i) entanglement (namely, whether the local quantum state describing the neighborhood is incompatible with an underlying multiseparable state for the whole system) and (ii) Bell nonlocality (namely, whether it is impossible to simulate the statistics of the whole system with a classical device).

Entanglement and nonlocality are two hallmark features of our world which signify a clear departure from classical physics [1,2]. The problem of certifying whether a given state is entangled and/or nonlocal is important in order to determine the type of correlations that are furnished by the state or to characterize the state as a useful resource for various quantum processing tasks. Most research in the entanglement of TI quantum systems has been focused on the entanglement between two distant sites [3–5] or between a region of the chain and the rest of it [6]. The multiseparability of quantum spin chains has been studied in, e.g., Refs. [7–9], and the permutation-invariant systems studied in Refs. [10,11], when placed on a line, can be seen as translation invariant as well. Unfortunately, the certification of entanglement or nonlocality in the aforementioned works requires the knowledge of correlations between arbitrarily distant sites, impossible to acquire in the gedanken experiment described above. Prior works on the nonclassicality of infinite translation-invariant systems have focused on how to detect Bell nonlocality directly, i.e., by showing that the probed regions cannot be described classically [12,13]. The problem of global nonlocality detection in 1D TI systems via local measurements has been studied for a finite number of parties [14,15].

In this Letter, we study the problem of certifying entanglement and nonlocality in 1D infinite TI systems from the information available to a finite number of parties exploring the chain. As we will show, our results also apply to the verification of entanglement and nonlocality in large (finite but not TI) 1D quantum many-body systems, so they may be particularly relevant for condensed matter experiments. In this regard, since all our entanglement witnesses and Bell inequalities depend on only near-neighbor two-body correlators, the quantum states maximally minimizing them can be prepared by cooling a condensed matter system described by a local TI Hamiltonian [16].

Conceptual setup.—Consider infinitely many sites distributed equidistantly along a line. Any number of consecutive sites of the chain, say, $1, \ldots, n$, is described by a *state* $\omega_{1,\ldots,n}$. Depending on the level of our description, such a state will correspond to the following. (i) A quantum state $\rho_{1,2,\ldots,n}$. (ii) A conditional probability distribution (also called a *box*) $P_{1,2,\ldots,n}(a_1, a_2, \ldots, a_n | x_1, x_2, \ldots, x_n)$ for the values $a_1, a_2, \ldots, a_n$ of the local properties $x_1, x_2, \ldots, x_n$ at sites $1, 2, \ldots, n$, satisfying the nonsignaling condition [17]. To model separability and locality, we will further require a coarser level of description: (iii) A probability distribution $P_{1,2,\ldots,n}(a_1, a_2, \ldots, a_n)$ for the values $a_1, a_2, \ldots, a_n$ of a local system property at sites $1, 2, \ldots, n$, respectively.

The quantum state ρ of an infinite chain is *multiseparable* if it can be decomposed in the form $\rho_{1,\ldots,n} = \int d\vec{\varrho} P(\varrho_1, \ldots, \varrho_n) \varrho_1 \otimes \cdots \otimes \varrho_n$, for all n, where $P(\varrho_1, \ldots, \varrho_n)$ is a probability density and $\varrho_1, \ldots, \varrho_n$ are single-site density matrices. We will refer to $P(\varrho_1, \ldots, \varrho_n)$ as a *separable decomposition* for the state $\rho_{1,\ldots,n}$. Alternatively, the set of multiseparable states is the set of all states which can be generated via quantum one-site operations and classical communication, i.e., without the need of making the subsystems interact.

Analogously, the box P of an infinite chain is *local* or *classical*—that is, it does not violate any Bell inequality—if, for any n, the box $P_{1,2,\ldots,n}(a_1, a_2, \ldots, a_n | x_1, x_2, \ldots, x_n)$ admits a local hidden variable model [2], namely, if there exists a probability distribution $\mu(\lambda)$ over a hidden variable λ such that $P_{1,\ldots,n}(a_1, a_2, \ldots, a_n | x_1, x_2, \ldots, x_n) = \sum_\lambda \mu(\lambda) Q_1(a_1|x_1, \lambda) Q_2(a_2|x_2, \lambda) \ldots Q_n(a_n|x_n, \lambda)$, where $Q_k(a_k|x_k, \lambda)$ is a probability distribution for outcome a_k at site k. Intuitively, the set of local boxes is the set of all black boxes which can be simulated via classical devices.

In this Letter, we will be interested mostly in TI states. An *infinite TI state* Ω for the whole chain is defined as an infinite *sequence* of states $(\Omega_{1,2,\ldots,s})_s$ satisfying $\Omega_{k,\ldots,k+m} = \Omega_{k+1,\ldots,k+m+1}$ for all m, k.

Our goal is, given access to $\Omega_{1,2,\ldots,r}$, to determine global properties of the infinite TI state Ω, such as its entanglement (when Ω is a quantum state) or its nonlocality (when Ω corresponds to a box). However, our results can also be applied to finite and non-TI systems by means of the following—widely known—*symmetrization procedure*, that allows us to construct an infinite TI state Ω, given an n-site state $\omega_{1,2,\ldots,n}$.

Consider a state Γ of an infinite chain composed by infinitely many copies of ω, i.e., $\Gamma \equiv \omega_{1,\ldots,n} \otimes \omega_{1,\ldots,n} \otimes \cdots$. The symbol $\otimes$ denotes the composition law of the state under consideration (tensor product for Hilbert spaces, multiplication for probability distributions, etc.). The state Γ is clearly invariant under translations of n sites. We can construct an infinite TI state Ω by summing Γ with states obtained by translating Γ by $k \in \{1, 2, \ldots, n\}$ sites, each time with probability $(1/n)$; see Fig. 1. State Ω is

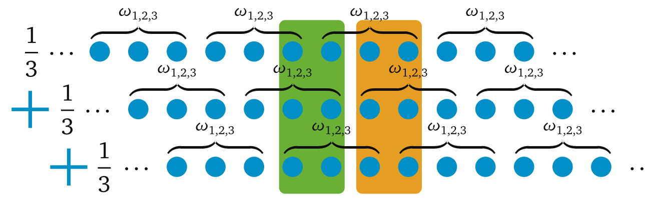

FIG. 1. The symmetrized state for $n = 3$. The green and yellow rectangles highlight the partial terms that contribute to two neighboring two-site reduced states, respectively. These are seen to be equal (since they are the sum of the same three partial terms).

called a *symmetrization* of $\omega_{1,2,\ldots,n}$, and the marginal $\Omega_{1,2,\ldots,r}$ of $r \leq n$ sites is given by

$$\Omega_{1,\ldots,r} = \frac{1}{n}\left(\sum_{k=1}^{n-r+1} \omega_{k,\ldots,r+k-1} + \sum_{k=1}^{r-1} \omega_{n-r+k+1,\ldots,n} \otimes \omega_{1,\ldots,k}\right). \quad (1)$$

Many experimental setups in condensed matter physics do not allow the experimenter to probe each individual site of an n-site spin chain. Instead, one can estimate, via neutron diffraction, the average two-site correlators or *static structure factors* [18]:

$$\tilde{\omega}_{[r]}^{(n)} \equiv \frac{1}{n-r+1} \sum_{k=1}^{n-r+1} \omega_{k,k+r-1}. \quad (2)$$

Given a large chain with structure factors $\{\omega_{[r]}^{(n)}\}_r$, the symmetrization procedure (1) hence implies that there exists a TI state Ω with the property

$$\Omega_{1,r} = \omega_{[r]}^{(n)} + O\left(\frac{r}{n}\right). \quad (3)$$

Moreover, if $\omega_{1,2,\ldots,n}$ is a separable quantum state or a local or quantum box, then so is Ω. It follows that, if the structure factors of the system violate an entanglement witness or Bell inequality for TI systems by an amount greater than $O(r/n)$, then the n-site chain must be, respectively, entangled or nonlocal. This means that, as long as we restrict ourselves to devising two-body entanglement witnesses and Bell inequalities, the conclusions which we will extract regarding TI systems also apply to large 1D condensed matter systems. Moreover, the witnesses constructed this way will be experimentally friendly, since by construction they are maximally violated by the ground states of local TI Hamiltonians.

With the required notation and tools in place, we now turn to addressing our two main goals, namely, the detection of entanglement and nonlocality in large 1D chains by using only local information.

Entanglement detection in large 1D chains.— Given a partial quantum state $\rho_{1,\ldots,r}$, obtained by ignoring all but r

consecutive sites of an infinite TI quantum state, how can we ascertain whether the total state $\rho_{-\infty,\ldots,\infty}$ is entangled? Of course, if $\rho_{1,\ldots,r}$ itself is entangled, e.g., if it is not positive under partial transposition (PPT) [19], then nothing needs to be done. On the other hand, it is possible, as we illustrate below, that the total quantum state is *entangled* even when $\rho_{1,\ldots,r}$ is multiseparable. Given just access to $\rho_{1,\ldots,r}$, the only question we can hope to answer is whether there exists a total multiseparable TI state from which the given partial state can be obtained by ignoring sites, i.e., whether $\rho_{1,\ldots,r}$ admits a *TI and separable (TIS) extension*.

Before addressing this problem, let us consider a related one: Given an r-site probability distribution $P_{1,\ldots,r}(x_1,\ldots,x_r)$, decide whether it can be realized as the marginal of an infinite TI distribution Q. The solution of this problem has been known for some time [20–23] and remains part of the folklore of TI systems: $P_{1,\ldots,r}(x_1,\ldots,x_r)$ admits a TI extension if and only if

$$P_{1,\ldots,r-1}(x_1,\ldots,x_{r-1}) = P_{2,\ldots,r}(x_1,\ldots,x_{r-1}). \qquad (4)$$

To see why this is true, consider the conditional probability distribution $P(x_n|x_1,\ldots,x_{r-1}) \equiv [P_{1,\ldots,r}(x_1,\ldots,x_r)/P_{1,\ldots,r-1}(x_1,\ldots,x_{r-1})]$ (if the denominator is 0, then any distribution is allowed). We can recursively extend the probability distribution $P_{1,\ldots,r}(x_1,\ldots,x_r)$ to a sequence $(Q_{1,\ldots,s})_s$ of probability distributions for increasingly larger chains $s \geq r$ by means of the recurrence relation:

$$Q(x_1,\ldots,x_{s+1}) = Q(x_1,\ldots,x_s)P(x_{s+1}|x_{s-r+2},\ldots,x_s). \quad (5)$$

It is readily checked that $Q_{1,2,\ldots,r} = P_{1,2,\ldots,r}$, namely, $P_{1,2,\ldots,r}$ is a marginal of $Q_{1,2,\ldots,s+1}$ for $s \geq r$. From Eq. (5), it also follows that $Q_{1,2,\ldots,s+1}$ is a TI sequence provided that Eq. (4) holds. A characterization of the extreme points of the set $\mathcal{T}_r$ of r-site TI marginals, i.e., those TI marginals which cannot be expressed as convex combinations of other marginals, can be found in Ref. [24].

Using the solution of the classical TI marginal problem, we will next derive a characterization of the set of states admitting a TIS extension. Assume that $\rho_{1,\ldots,r}$ does indeed admit a TIS extension ρ, and let $P(\varrho_1,\ldots,\varrho_n)$ define a separable decomposition for the state $\rho_{1,\ldots,n}$. Applying the symmetrization procedure to $P(\varrho_1,\ldots,\varrho_n)$, we obtain a TI distribution that we can regard as the separable decomposition of a chain state $\bar{\rho}$, whose reduced state $\bar{\rho}_{1,\ldots,r}$ is $O(r/n)$-close to $\rho_{1,\ldots,r}$. Since n was arbitrary, we conclude that, if $\rho_{1,\ldots,r}$ admits a TIS extension, then we can take its separable decomposition to be TI. Invoking Eq. (4), we thus have that an r-site quantum state $\rho_{1,\ldots,r}$ admits a TIS extension iff it satisfies

$$\rho_{1,\ldots,r} = \int d\vec{\varrho} P(\varrho_1,\ldots,\varrho_r)\varrho_1 \otimes \cdots \otimes \varrho_r, \qquad (6)$$

with $P_{1,\ldots,r-1}(\varrho_1,\ldots,\varrho_{r-1}) = P_{2,\ldots,r}(\varrho_1,\ldots,\varrho_{r-1})$.

Unfortunately, this characterization of TI separability is not very practical to detect entanglement. Indeed, given the state $\rho_{1,\ldots,r}$, how does one argue that it does *not* admit a decomposition of the form in Eq. (6)? This motivates us to look for simpler criteria to decide the existence of TIS extensions.

As a first attempt, we can apply the intuition from the characterization of TI probability distributions. Notice that if the state $\rho_{1,\ldots,r}$ has a TIS extension, then it must be separable and satisfy

$$\rho_{1,\ldots,r-1} = \rho_{2,\ldots,r}. \qquad (7)$$

Are these conditions also sufficient to guarantee the existence of a TIS extension?

Let $\{\sigma_i\}_{i=x,z,y}$ denote the Pauli matrices. Using the Jordan-Wigner transformation [25], it can be shown that $\mathrm{tr}(\rho_{1,2}\sigma_y \otimes \sigma_x) \leq (2/\pi)$ for TI states ρ (see [24] for the proof). Now, the separable state $\varrho = \frac{1}{2}(|+i\rangle\langle+i| \otimes |+\rangle\langle+| + |-i\rangle\langle-i| \otimes |-\rangle\langle-|)$, with $\sigma_x|\pm\rangle = \pm|\pm\rangle$, $\sigma_y|\pm i\rangle = \pm|\pm i\rangle$, satisfies $\varrho_1 = \varrho_2 = \frac{1}{2}\mathbb{I}$, but $\mathrm{tr}(\varrho\sigma_y \otimes \sigma_x) = 1$. Thus, separability plus condition Eq. (7) do not even guarantee the existence of a TI extension, separable or not.

This last observation, however, suggests a stronger criterion for the existence of a TIS extension, namely, to demand the state $\rho_{1,\ldots,r}$ to be both separable and the reduced state of an infinite TI state. Unfortunately, this criterion, although necessary, is still not sufficient to guarantee a TIS extension. To construct a counterexample, we will first give two states, one TI and the other TIS, which can be seen as optimally witnessing translation invariance and translation invariance plus multiseparability.

First, in Supplemental Material [24], we identify a TI state ρ^1 that saturates the inequality $\mathrm{tr}(\rho_{1,2}\sigma_y \otimes \sigma_x) \leq (2/\pi)$, with $\rho^1_{1,2} = \frac{1}{4}\mathbb{I}_4 + (1/2\pi)(\sigma_y \otimes \sigma_x + \sigma_x \otimes \sigma_y) + (1/\pi^2)\sigma_z^{\otimes 2}$.

Second, in Ref. [24] it is shown that all states $\rho_{1,2} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ with a TIS extension satisfy

$$\mathrm{tr}\left(\rho_{1,2}\sum_{i,j=1}^{3} T_{ij}\sigma_i \otimes \sigma_j\right) \leq \frac{1}{2}\max_{\theta\in[0,2\pi]} \|e^{i\theta}T + e^{-i\theta}T^\dagger\|, \quad (8)$$

where $T_{ij} \in \mathbb{R}$. Taking $T_{i,j} = \delta_{i,2}\delta_{j,1}$, where δ is the Kronecker delta, implies that all states which have TIS extensions satisfy $\mathrm{tr}(\rho_{1,2}\sigma_y \otimes \sigma_x) \leq \frac{1}{2}$. This bound is tight, since it can be saturated by the TIS state $\rho^0 \equiv \frac{1}{3}\sum_{s=1}^{3} \varrho^{s+1} \otimes \varrho^s$, where $\varrho^1 = \varrho^4$, and ϱ^1, ϱ^2, and ϱ^3 are described, respectively, by the Bloch vectors $(1/\sqrt{2})(1,1,0)$, $(1/\sqrt{2})(-1,1,0)$, and $(1/\sqrt{2})(1,-1,0)$ [26].

Now, consider the family of TI states $\rho^\lambda \equiv \lambda\rho^1 + (1-\lambda)\rho^0$. Clearly, for $\lambda \in (0,1]$, all those states violate the entanglement witness $\langle\sigma_y \otimes \sigma_x\rangle \leq \frac{1}{2}$. Also, it can be verified that $\rho^{\lambda}_{1,2}$ is PPT for $\lambda \leq (2\pi^2/12 + 12\pi - \pi^2) \approx 0.4956$. It follows that, for $\lambda \in (0, 0.4956]$, the states $\rho^{\lambda}_{1,2}$ are separable [27] and TI, but all their TI extensions are entangled. Similar effects have been reported in Ref. [28], where the authors construct bipartite separable states (local boxes) which admit only entangled (nonlocal) tripartite extensions.

Thus, even though $\rho^{\lambda}_{1,2}$ is not entangled, its two-body correlators tell us that there exists a finite system size n such that $\rho^{\lambda}_{1,\ldots,n}$ is. This raises another interesting question, namely, how large n must be. Consider a witness of the form $\mathrm{tr}(W\rho_{1,2}) \leq S$ and suppose that $\rho_{1,2}$ violates it by an amount $\Delta > 0$, i.e., $\mathrm{tr}(\rho_{1,2}W) = S + \Delta$. If there exists a TI extension ρ of $\rho_{1,2}$ such that $\rho_{1,\ldots,n}$ is separable, then applying the symmetrization procedure in Eq. (1) to $\rho_{1,\ldots,n}$ would produce a separable TI state $\tilde{\rho}$ with $\tilde{\rho}_{1,2} = (n-1/n)\rho_{1,2} + (1/n)\rho_1 \otimes \rho_1$. Since $\tilde{\rho}$ is separable and TI, it must satisfy $\mathrm{tr}(\tilde{\rho}_{1,2}W) \leq S$, from which it follows that $n \leq [S - \mathrm{tr}(W\rho_1^{\otimes 2})/\Delta] + 1$.

Therefore, contrary to the ordinary entanglement detection setup, the degree of violation of a linear entanglement witness has a clear operational meaning in the TI scenario thanks to an intrinsic notion of size: Its inverse is proportional to the number n of consecutive sites which n parties must share in order to hold an entangled resource. This quantitative relation between nonseparability and size can be seen to hold for arbitrary entanglement witnesses, not just bipartite ones. It also extends to the realm of Bell nonlocality that we will study next.

Detecting nonlocality in large 1D chains.— Before tackling the characterization of nonlocality in TI systems, we will argue that certain infinite TI quantum systems are indeed nonclassical. Take any bipartite quantum state $\rho \in \mathcal{B}(\mathcal{H}^{\otimes 2})$ which allows two parties to violate a Bell inequality B, and consider an infinite chain where each site k holds two systems with Hilbert spaces $\mathcal{H}_1^{(k)}$, $\mathcal{H}_2^{(k)}$, with $\dim(\mathcal{H}_1^{(k)}) = \dim(\mathcal{H}_2^{(k)}) = \dim(\mathcal{H})$. If we distribute a copy of $\rho \in \mathcal{B}(\mathcal{H}_1^{(k)} \otimes \mathcal{H}_2^{(k+1)})$ to all neighboring pairs (k, $k+1$), we end up with a TI chain configuration with the property that any pair of nearest neighbors can violate B.

In this construction, the nonclassicality of the whole chain is established by proving that the probed sites 1,2 do not admit a local hidden variable model. Is this necessarily the case, or are there situations where the probed sites are classical but nonetheless incompatible with an infinite classical TI box? We will need a complete characterization of nonlocality in 1D TI systems in order to answer this question.

Assume that the data available are of the form $P_{1,\ldots,r}(a_1,\ldots,a_r|x_1,\ldots,x_r)$, where $a_k \in \{1,\ldots,d\}$ and $x_k \in \{0,\ldots,m-1\}$, with the promise that it arises from an infinite TI box. The task is to decide whether there exists a TI box P, compatible with the experimental data and such that $P_{1,\ldots,n}(a_1,\ldots,a_n|x_1,\ldots,x_n)$ admits a local hidden variable model for all n. By Fine's theorem [29], the existence of a local hidden variable model for P is equivalent to the existence of a global probability distribution $Q(\vec{a}_1,\ldots,\vec{a}_\infty)$, with $\vec{a}_k \in \{1,\ldots,d\}^m$ such that

$$Q(a_1^{x_1} = b_1,\ldots,a_n^{x_n} = b_n) = P_{1,\ldots,n}(b_1,\ldots,b_n|x_1,\ldots,x_n), \tag{9}$$

for all n. As in the characterization of TIS, we apply the symmetrization procedure over $Q_{1,\ldots,n}$ in the limit $n \to \infty$ and find that we can assume the global distribution Q to be TI.

As a vector of probabilities, the distribution $P_{1,\ldots,r}(a_1,\ldots,a_r|x_1,\ldots,x_r)$ is a linear function L of $Q(\vec{a}_1,\ldots,\vec{a}_r)$, whose only constraint is that it is the marginal of a TI distribution. Since this is equivalent to satisfying Eq. (4), it follows that we can characterize $P_{1,\ldots,r}(a_1,\ldots,a_r|x_1,\ldots,x_r)$ via linear programing [30].

The set of all marginal distributions $P_{1,\ldots,r}(a_1,\ldots,a_r|x_1,\ldots,x_r)$ arising from a 1D classical TI chain thus forms a *convex polytope*, i.e., the convex hull of a finite number of vertices. This in itself is a very surprising result: Because of the presence of infinitely many parties, there is no *a priori* reason to expect this set to be a polytope. Actually, in the 2D case, the boundary of the corresponding set has both flat and smoothly curved parts and does not admit an exact computational characterization [31].

Each polytope has a dual description in terms of a finite set of linear inequalities or *facets*. The transformation between these two descriptions can be done algorithmically albeit generally with high complexity. Using the software PANDA [32], we enumerated the facets of the classical polytope describing two-input–two-output nearest-neighbor and next-to-nearest-neighbor distributions [that is, $P_{1,2}(a_1,a_2|x_1,x_2)$, $P_{1,3}(a_1,a_3|x_1,x_3)$, with $x_k \in \{0,1\}$, $a_k \in \{1,2\}$]. The polytope has 32 372 facets, which reduce to 2102 inequivalent inequalities after taking one-site relabelings and the reflection of the chain into consideration. Two examples are given by

$$I_T \equiv -2E_0 - 4E_1 - 2E_{00}^{1,2} + 2E_{01}^{1,2} + 2E_{10}^{1,2} + 2E_{11}^{1,2} + E_{00}^{1,3} + E_{11}^{1,3} \geq -4, \tag{10}$$

$$I_G \equiv -4E_0 - 6E_1 - 3E_{00}^{1,2} + 2E_{01}^{1,2} + 3E_{10}^{1,2} + 2E_{11}^{1,2} + 2E_{00}^{1,3} + E_{10}^{1,3} + E_{11}^{1,3} \geq -6, \tag{11}$$

where $E_x \equiv \langle A_x^1 \rangle = \sum_{a=0,1} P_1(a|x)(-1)^a$ and $E_{xy}^{i,j} \equiv \langle A_x^i A_y^j \rangle = \sum_{a,b=0,1} P_{i,j}(a,b|x,y)(-1)^a(-1)^b$. Here A_x^i denotes the observable corresponding to measuring property x at site i and assigning it the numerical value $(-1)^a$.

In order to estimate the quantum value of an inequality given by $I \equiv \sum_{x,y=0,1} \frac{1}{2} C_x E_x + C_{xy}^{AB} E_{xy}^{1,2} + C_{xy}^{AC} E_{xy}^{1,3}$, we associate a quantum system of dimension $d = 4$ to each of the sites. $d = 4$ is chosen because we could not violate any inequality by using quantum systems with lower dimensions on each site. We then identify the observables A_0, A_1 at each site with the operators $A_0 \equiv M(0,0)$, $A_1 \equiv M(\theta,\phi)$, where

$$M(\theta,\phi) \equiv \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ \sin(\theta) & -\cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\phi) & \sin(\phi) \\ 0 & 0 & \sin(\phi) & -\cos(\phi) \end{pmatrix}. \quad (12)$$

This way, fixing θ, ϕ, we can map the original Bell inequality to the 3-local Hamiltonian

$$H \equiv \sum_{i=1}^{\infty} \sum_{x,y=0,1} \frac{1}{2} C_x A_x^i + C_{xy}^{AB} A_x^i \otimes A_y^{i+1} + C_{xy}^{AC} A_x^i \otimes A_y^{i+2}. \quad (13)$$

The minimum quantum value of the Bell inequality (under the corresponding measurement settings) corresponds to the ground state energy per site of this Hamiltonian. The computation of the latter was carried out over infinite matrix product states (MPS) using a combination of the time evolving block decimation (TEBD) method [33] and the tool OpenSource MPS [34], which implements a variant of the density matrix renormalization group (DMRG) method [35]. Using these tools, we find the violations $I_T = -4.1847$ with $\theta = 0.077$, $\phi = 1.874$ and $I_G = -6.1798$ with $\theta = 6.236$, $\phi = 4.175$. More inequalities, including violations using DMRG and lower bounds on the nonsignaling and quantum values, can be found in Ref. [24].

The violations obtained above may not be optimal, and a seesawlike method, similar to Ref. [36], can be used to enhance them. In such a method, the optimization is divided into two rounds: In one round the measurements are held fixed and the optimization is over the state, and in the other round the state is held fixed and the measurements are optimized. By repeating these two rounds, a seesaw method usually converges to better violations than naive methods such as the one used above. While optimization over states by fixing the measurements can be done using TEBD or DMRG, the optimization over measurements with a fixed state involves further complications, because the measurement operators make the objective function bilinear. Fortunately, this bilinearity can be removed if, in addition to the four-dimensional quantum state, each party is given access to a classical TI register, as described by the protocol given in Ref. [24]. Using this protocol, the violation of I_G can be increased to -6.1907. At first glance, it may seem surprising that, by giving them access to shared randomness, the parties are able to increase their violation. Note, though, that the extreme points of TI distributions are not necessarily deterministic.

We verified that the TI local value of I_T cannot be beaten by local tripartite boxes $P_{1,2,3}(a_1, a_2, a_3|x_1, x_2, x_3)$ with $P_{1,2}(a, a'|x, x') = P_{2,3}(a, a'|x, x')$. This means that a TI box P can violate Eq. (10) only when the tripartite box $P_{1,2,3}$ describing the state of three consecutive sites does not admit a classical model. In other words, I_T is just detecting standard tripartite nonlocality. I_G, however, is different. While the tripartite box $Q_{1,2,3}$ generated by nearest and next-to-nearest neighbors of the state achieving the violation is not local, TI local noise can be added to Q to turn it into a new TI box $\tilde{Q}$, with $\tilde{Q}_{1,2,3}$ tripartite local, while keeping a violation of $I_G(\tilde{Q}) \approx -6.1525$. Similarly to the entanglement case, even though the behavior of the tripartite box $\tilde{Q}_{1,2,3}$ can be reproduced with classical devices, for some n no local hidden variable model can possibly describe an n-site box with marginals $\tilde{Q}_{1,2,3}$.

Conclusions.— In this Letter, we showed how to derive global properties of infinite 1D TI systems when only local information is available. We provided a characterization of the reduced density matrices of TI multiseparable states and used it to derive entanglement witnesses for infinite TI qubit chains. Along the way, we constructed examples of TI states with a separable nearest-neighbors density matrix which nonetheless admit only entangled TI extensions. Regarding nonlocality, we fully characterized the set of r-partite boxes obtained by probing the sites of a classical infinite TI chain. Similarly to the entanglement case, we identified a classical tripartite box which admits only nonclassical TI extensions.

For future research, it would be interesting to develop effective methods to bound the nonlocality of TI quantum and nonsignaling systems. Also, it would be desirable to extend some of our results to higher spatial dimensions.

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