P H Y S I C A L   R E V I E W   L E T T E R S

Statistics of Fractionalized Excitations through Threshold Spectroscopy

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We show that neutral anyonic excitations have a signature in spectroscopic measurements of materials: The low-energy onset of spectral functions near the threshold follows universal power laws with an exponent that depends only on the statistics of the anyons. This provides a route, using experimental techniques such as neutron scattering and tunneling spectroscopy, for detecting anyonic statistics in topologically ordered states such as gapped quantum spin liquids and hypothesized fractional Chern insulators. Our calculations also explain some recent theoretical results in spin systems.

Quantum mechanics allows for the possibility of phases not characterized by symmetry breaking, including the complex of the fractional quantum Hall effects [1,2]. These are characterized from a theoretical point of view by subtle characteristics of their quantum entanglement known as topological order [3–5]. This phenomenon may also appear in quantum spin liquids, of which we have several candidates [6–8]. From an experimental point of view, the most interesting phenomenon in such phases is not the lack of long-range order—a negative characteristic—but rather the existence of fractionalized quasiparticle excitations that have anyonic statistics [9–11]. In this Letter, we suggest a way to measure these quantum statistics through the threshold behavior of spectroscopic cross sections for creating these excitations. We will focus on gapped phases with Abelian excitations.

It is interesting to ask how we can measure the phases that anyons accumulate when they wind around one another; is it necessary to guide microscopic particles along specific paths to make sure they circle around one another a certain number of times? Proposals for methods can be tested soon, since anyons have been proven to exist in theoretical states that model the fractional quantum Hall effect and possibly spin liquids; see [12] for theoretical calculations in model wave functions, and see [13] for calculations in realistic states utilizing degenerate ground states on a torus. Experimentally, there have been attempts using interferometers with two arms enclosing trapped anyons [14], and other proposals such as measuring non-Abelian statistics through the entropy associated with the quasiparticles [15]. However, neutron scattering, which measures the dynamical correlation function, is a much more established technique for magnetic materials, like the candidates for spin liquids. Neutron scattering can create multiple excitations, which would then move around

one another (in a partly random way), and the statistics could be hidden in the time dependence of this process.

Neutron scattering has already been used to demonstrate signatures of fractionalization in candidate spin liquids [16–18]. Here the characteristic feature is the absence of sharp dispersive lines (e.g., magnons). They are replaced by a broad continuum, interpreted as arising from the creation of multiple fractionalized quasiparticles, leading to a spread in energies because of the different ways of sharing momentum among them. Thus, it seems that, at low energies, several fractional particles are automatically created. How can one tell whether these particles have fractional "anyon" statistics and accumulate phase factors as they braid around one another, without controlling them?

In classic work, Wigner demonstrated that the onset behavior of cross sections near production thresholds is often dominated by long-range interactions [19]. Anyons effectively have a long-range interaction due to the statistical phases—so it seems possible that the statistical phase can be revealed in such measurements when the anyons do not have more dominant long-range interactions like Coulomb interactions. We derive the behavior of the cross section near the threshold for two anyons analytically, showing that there is a distinct power law that reveals the nature of the anyons. The structure factor is given by

$$S(\vec{q}, \omega) \propto (\omega - \Delta)^{\alpha}, \tag{1}$$

where α is the statistical parameter for the anyons (ranging from 0 for bosons to 1 for fermions) and Δ is the minimum energy needed to create two anyons at a given wave vector. When $\alpha = 0$, short-range interactions modify the power law. We also perform numerics to verify the robustness of the behavior to
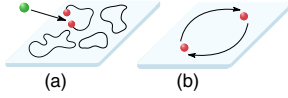
FIG. 1. (a) In phases with topological order, scattering processes can create fractionalized anyonic excitations above some threshold energy. (b) The statistics of the excitations mediate long-range effects between them.

interactions. Extending our results, we explore the corresponding behavior for three particles. This, too, displays an interesting dependence on the statistics, reflecting the braiding of three particles. We then discuss which types of excitations one could identify using this method experimentally in spin liquids. We will also use our method to rederive recent results for the dynamic structure factor in topologically ordered phases.

Setup and eigenvalue problem.—We consider a 2D system in which gapped fractionalized excitations can be created by the action of a local operator (Fig. 1). For example, we can consider inelastic neutron scattering on a gapped spin liquid, resulting in the creation of n fractionalized excitations, which in general obey anyonic exchange and mutual statistics. The double differential scattering cross section $d^2\sigma/d\Omega d\omega$ thus obtained is proportional to the dynamic structure factor $S^{\alpha\beta}(\vec{q}, \omega)$, defined as the Fourier transform of the correlation function $\langle S^\alpha(r, t)S^\beta(0, 0)\rangle$, where $\alpha, \beta = \{x, y, z\}$ [20]. Since the excitations are gapped, we can focus on the effective n-particle system to calculate $S(\vec{q}, \omega)$ at energies below those involving additional excitations.

An anyon can be viewed [10] as a composite particle carrying "electric" charge and attached to an infinitesimal "magnetic" flux tube (where the charge and flux pertain to an emergent gauge field, not ordinary electromagnetism). The braiding phases arise as effective Aharonov-Bohm phases as these particles are exchanged or taken around one another. We will work in the magnetic or boson gauge, where the Hamiltonian acts on bosonic wave functions and the statistics is encoded in the Hamiltonian as an interaction through minimal coupling of the gauge field $\vec{a}$.

For the case of two identical fractionalized excitations [21], $\vec{a} = (\hbar c\alpha/q)\nabla\theta$, where θ is the angle between the particles, q is the charge, and α is the statistics parameter which varies from 0 for bosons to 1 for fermions. Thus, for two excitations with a quadratic dispersion, the effective Hamiltonian is

$$H = \frac{P_{\vec{R}}^2}{4m} + \frac{p_r^2}{m} + \frac{(p_\phi - \hbar\alpha)^2}{mr^2} + V(r, \phi) \quad (2)$$

in the center of mass frame, where $\vec{R}$ is the center of mass coordinate, m, r, ϕ are the mass and relative coordinates of the particles, and $V(r, \phi)$ is the effective interaction between the two particles.

One can show (see below and Supplemental Material [22]) that the behavior of the cross section close to the threshold is not altered by the potential V except for bosons. Therefore, we temporarily set $V(r, \phi) = 0$. Then we can solve the eigenvalue problem for H_r using the

separation of variables. The complete normalized solution to the eigenvalue problem for $H = H_R + H_r$ is

$$\Psi(\vec{R}, r, \phi) \sim \sqrt{\frac{k}{L^3}} J_{|l-\alpha|}(kr)\exp(il\phi)\exp(i\vec{K}\cdot\vec{R}), \quad (3)$$

where $l = 2n$, $n \in \mathbb{Z}$; relative momentum $k = \sqrt{(E_r m/\hbar^2)}$, center of mass momentum $|\vec{K}| = 2\sqrt{(E_R m/\hbar^2)}$, total energy $E = E_r + E_R$, and $J_{|l-\alpha|}$ is a Bessel function of the first kind. The box normalization of $\sqrt{k/L^3}$ is strictly valid only when $k > 0$ and in the large L limit. We see that different sorts of anyons have different probabilities to be close to one other. In particular, bosons are the only particles which can be at the same point ($r = 0$), since $J_{|l-\alpha|}(0) > 0$ only if $l = \alpha = 0$; other anyons satisfy a hard-core condition for all l. Spectral functions for creating localized excitations can be expected to show signs of these differences, which reflect repulsive angular momentum barriers.

Spectral function.—We consider the Lehmann representation [23] of a zero temperature spectral function $S(\vec{Q}, \omega)$ associated to a spectroscopic measurement

$$S(\vec{Q}, \omega) = \sum_{|\psi_f\rangle} |\langle\psi_f|\hat{O}_{\vec{Q}}|\psi_i\rangle|^2\delta(\omega + \omega_i - \omega_f). \quad (4)$$

Here, $|\psi_f\rangle$ are two anyon energy eigenstates and $|\psi_i\rangle$ is the topologically ordered ground state. We will look at the behavior close to a momentum $\vec{q}$ with quadratic dispersion. We assume the operator $\hat{O}(\vec{R})$ creates a superposition of all the states of two anyons whose center of mass is located at $\vec{R}$ and that are separated from each other by a distance a, i.e., $\hat{O}(\vec{R})|\psi_i\rangle = \int d\phi|\vec{R}, a, \phi\rangle$. This is motivated from the creation of excitations in lattice models with topological order such as the toric code, where a local operator such as σ_z creates excitations on neighboring sites. Breaking rotational invariance does not affect the final answer at low energies where s-wave eigenstates ($l = 0$) dominate. In general, $\hat{O}(\vec{R})$ creates some local perturbation to the many-body ground state (like a certain spin texture), and the matrix element reflects the overlap between this state and two-anyon many-body eigenstates. However, it can be shown that the resulting energy dependence near the threshold is insensitive to details (see Supplemental Material [22]). We set $\hbar = 1$ henceforth.

$$S(\vec{q}, \omega) = c\sum_{|\psi_f\rangle}\left|\int d\vec{R}d\phi e^{i\vec{q}\cdot\vec{R}}\langle\psi_f|\vec{R}, a, \phi\rangle\right|^2$$

$$\times \delta\left(\omega + \frac{k^2}{m} - \frac{|\vec{K}|^2}{4m} - \Delta\right) \quad (5)$$

$$= cmJ_\alpha^2(a\sqrt{\Omega})\Theta(\Omega) \quad (6)$$

$$\approx cm(a^2\Omega)^\alpha\Theta(\Omega), \quad (7)$$

where $\Omega = m(\omega - \Delta) - |\vec{q}|^2/4$, Δ is the energy gap to the threshold of two-particle excitations, and in the last step we have made a low-energy approximation. An infrared cutoff is set to avoid spurious divergences, and c is an energy-independent constant.

Above the gap, the cross section for bosons has a sharp onset, whereas fermions show a linear increase with energy and semionic excitations ($\alpha = 0.5$) show a characteristic square-root dependence. It is important to note that the difference arises due to the effect of statistics on the matrix elements and is not an effect from the density of states. This can be seen easily for bosons and fermions, where the two-particle density of states is the same in the thermodynamic limit, since Pauli exclusion for fermions excludes only a one-dimensional curve in the three-dimensional constant energy manifold available to two bosons.

Although we have considered only the case of identical excitations, the results generalize to the case of two distinct particles interacting through mutual statistics. There are a few minor differences, such as the presence of two distinct masses and the ability of (formerly) "bosonic" angular momentum l to assume odd values.

Universality and effect of interactions between anyons.— These results apply more generally than appears from the calculation. The Schrödinger equation applies to isolated gapped excitations as long as they have a quadratic dispersion, which we get near the threshold. In general, the anyons will interact at a short distance, i.e., $V(r, \phi) \neq 0$. However, in the absence of resonances, weak short-range interactions will generically not affect the power law of the dynamic structure factor at low energies. This can be seen from the fact that the anyon eigenstates are rigid when the anyons are close together and depend on the value of the kinetic energy only at large distances, where interactions are not important. This means that $S(\vec{q}, \omega)$ changes by only an overall energy-independent factor near the threshold (see Supplemental Material [22]).

However, bosons behave differently in the presence of interactions, since the noninteracting point is fine-tuned. Put differently, since bosons lack a statistical repulsion that prevents them from getting close to each other, they are susceptible to short-range repulsive interactions. To quantify this, we obtain $S(\vec{q}, \omega)$ for a system with two excitations which are hard-core bosons, i.e., bosons which interact with a hard-core potential $V(r)$ which is infinite for $r \leq b$ and zero everywhere else. The general form of the eigenstates is

$$\Psi(R, r, \phi) = \exp(i\vec{K} \cdot \vec{R}) \exp(il\phi)[A_l J_l(kr) + B_l N_l(kr)], \tag{8}$$

where we may focus on $l = 0$ at low energies.

The effect of the potential can be incorporated into a phase shift defined by $\tan \delta_0 = -B_0/A_0 = J_0(kb)/N_0(kb)$. Normalization of the eigenstates yields $A_0 = \cos \delta_0 \sqrt{k/2L}$, where L is the radius of the system. We can now obtain $S(\vec{q}, \omega)$ using Eq. (8):

$$S(\vec{q}, \omega) = \frac{cm}{1 + \tan^2 \delta_0} |J_0(a\sqrt{\Omega}) - \tan \delta_0 N_0(a\sqrt{\Omega})|^2 \Theta(\Omega) \tag{9}$$

$$\approx \frac{cm \log^2(\frac{a}{b})}{[\log(\frac{\Omega b^2}{4}) + 2\gamma]^2 + \pi^2} \Theta(\Omega), \tag{10}$$

where Ω and a are the same as in the free anyon case, γ is the Euler-Mascheroni constant, and we have made a low-energy approximation in the last step. We see that the hard-core interaction drastically changes the low-energy behavior for bosons, and one can also show that any finite repulsive interaction produces a similar effect. A corresponding analysis for semions and fermions shows that interactions do not affect their low-energy behavior, as expected.

Long-range interactions can, however, affect the low-energy behavior depending on how fast they decay. For power-law interactions of the form $1/r^\beta$, the threshold behavior is unchanged when $\beta > 2$, since the interaction is still dominated by the repulsive angular momentum [Eq. (2)] at large distances [19]. However, Coulomb interactions (if present) will dominate and wash out the effects due to the statistics.

Numerics.—We check some of the results numerically by considering an effective model of two anyons hopping on a square lattice which can be constructed from a many-body system by projecting into the two-anyon subspace. We assume a local operator creates anyons on neighboring sites of the lattice and perform an exact diagonalization to obtain the corresponding spectral function. Figure 2(a) shows $S(0, \omega)$ for fermions on a 600×600 square lattice interacting with nearest-neighbor repulsion U and next-nearest-neighbor repulsion V. Interactions drastically affect the high-energy behavior but leave the low-energy linear onset unchanged as expected for fermionic excitations. The linear onset can be seen clearly in the log-log plot in Fig. 2(b), where the exponent for all three cases is approximately 0.95 with the difference between them being less than 0.01. The exponents slowly converge towards the expected value of 1 as the system size is increased.

We also obtain the spectral function for general anyons which are modeled as charges living on the sites of the lattice which move together with fluxes living on adjacent plaquettes [24]. Since we use periodic boundary conditions, there are large gauge transformations of the vector potential which result in doubling the Hilbert space for semions. Figure 2(c) shows $S(0, \omega)$ for particles of various statistics. Although we are restricted to small system sizes and additionally the presence of a broadening which occurs as a result of smoothing discrete data points (over an $\epsilon = 0.1$ range of energies in the 20×20 system as opposed to $\epsilon = 0.005$ for the previous 600×600 system), the behavior at low energies is qualitatively similar to the analytic predictions for two anyons in the continuum [Eqs. (7) and (10)]. Bosons shows a decreasing behavior which fits in with the fact that their low-energy dependence is dominated by $J_0(x)$. Extracting the power law for semions, we get an exponent of around 0.45, which fits in with the expected square-root dependence. As expected, the density of states [Fig. 2(d)] is the same for the different statistics.

Three particles.—We now consider a system with three identical fractionalized excitations ($n = 3$) where we see nontrivial effects due to the braiding of three particles around each other. Unlike bosons and fermions, a system of three anyons is no longer exactly solvable even without
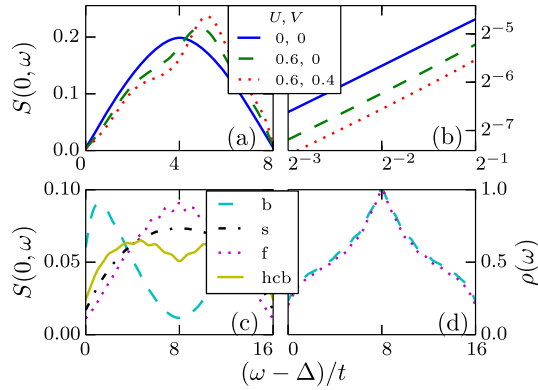
FIG. 2. (a) Two-particle spectral function from exact diagonalization on a 600×600 square lattice for fermions with nearest-neighbor repulsion U and next-nearest-neighbor repulsion V. The high-energy behavior is drastically affected by interactions, but the low-energy linear onset is unchanged as seen in the log-log plot on the top right, where the exponents for all three cases is 0.95. (c) $S(0, \omega)$ for noninteracting bosons (b), semions (s), fermions (f), and hard-core bosons (hcb) on a 20×20 lattice. (d) Density of states for bosons and fermions.

interactions [$V(r, \phi) = 0$], but the low-energy dependence of $S(\vec{q}, \omega)$ can still be obtained for certain anyons. After separating out the center of mass motion, the resulting system [25] can be described using hyperspherical coordinates consisting of a radial coordinate ρ and three angular coordinates θ, ϕ, ψ. The eigenfunctions of the angular equation can be complicated, since they are not determined by symmetry as in the two-particle case. The angular form of the wave function can be related (see Supplemental Material [22]) to the ground state of the same anyons in a harmonic oscillator potential which has been studied [25]. In some cases this ground state can be found analytically, while in other cases the eigenvalue is irrational.

In particular, for the interesting case of three fractionalized particles with $\alpha = \frac{1}{3}$, the eigenstates are $\psi(\rho, \theta, \phi, \psi) = [J_2(\sqrt{2E}\rho)/\rho]g(\theta, \phi, \psi)$, where E is the energy of the relative motion and g is a function which is independent of the energy. This leads to a threshold behavior of $S(\vec{q}, \omega)$ which increases as $(\omega - \Delta)^2$.

Applications.—Our results [Eqs. (7) and (10)] could be used to study the statistics of excitations in a spin liquid or a fractional Chern insulator by measuring the scattering cross section as a function of energy close to the threshold. At low energies, the excitations would be described by Schrödinger's equation (as assumed above), because the dispersion can generically be expanded to quadratic order around the minimum. In practice, one needs temperatures much lower than the gap and a mean free path much longer than the wavelength. Then, one can measure the structure factor slightly above the threshold, so that disorder which dominates the behavior near the threshold can be neglected, but not so far above the threshold that the universal behavior is lost.

Consider an experiment on a chiral spin liquid [26]. They have one type of topological excitation, which are

semions. A single semion is not accessible from the ground state by local processes, such as inelastic scattering by neutrons due to the conservation law that the total topological charge must be trivial. However, a neutron could excite two semions, because semions are their own antiparticles, which then leads to the dramatic consequence of a square-root onset with energy at the threshold.

In the case of materials in the toric code universality class (i.e., Z_2 spin liquids), one could also measure anyonic statistics. The possible topological charges of excitations are m and e (which are bosons but have a mutual phase of -1) and a composite em, which is a fermion.

The most interesting thresholds (consistent with the topological conservation law requiring e's and m's to be created in pairs) are the production of an $(me) - (me)$ pair, which would show a linear onset because me is a fermion, and a triple, $e - m - (me)$. (The pairs $m - m$ and $e - e$ are both mutual bosons, so their structure factor would have the same form as that possessed by local excitations such as two triplon excitations in a valence bond state.) There are two difficulties here: First, there may not be a stable particle of type me (in a spin liquid, the stable spinon might either have type m or me); second, the excitations also carry spin; therefore, the pair of me particles can be in either a triplet channel or a singlet channel (which would have the same behavior as a pair of bosons). This can be avoided with a weak magnetic field to favor parallel spins.

It is also appropriate to consider the quantum Hall effect, for which the propagator of a pair of anyons is calculated in Ref. [27]. Universality does not emerge in a straightforward way for this case, because the quasiparticles in such states are generally electrically charged. The Coulomb force then formally dominates any effective statistical force at large distances, and it will be quantitatively significant even if it is weakly screened. In addition, the motion of charged quasiparticles is inhibited by the background magnetic field. The presence of the magnetic field causes the appearance of discrete responses in spectroscopy instead of a threshold, and additionally the quasiparticles are hindered from moving far apart enough to see the universal effects of the statistics. These problems are avoided in fractional Chern insulators [28], where we can expect to see the universal threshold behavior.

There are also measurable signatures in tunneling spectroscopy which can involve neutral particles or cases where the Coulomb interaction between the excitations is screened. We expect a different exponent here, since the particles do not have a well-defined momentum any more. For the simplest case where the tunneling tip resembles a Fermi liquid, we find that the tunneling current goes as $(\omega - \Delta)^{1+\alpha}$.

Several recent works [17,29–31] contain calculations of dynamic structure factors in topologically ordered phases of spin systems. We now consider two prominent examples. Qi, Xu, and Sachdev [29] describe a bosonic $\mathbb{Z}_2$ spin liquid on the triangular lattice where the low-energy behavior near the bottom of the band is obtained through a large-N analysis of a sigma model. There is a constant onset above

the gap for noninteracting bosonic spinons which changes to an inverse-log behavior on adding interactions, exactly as expected. For the case of the $Q_1 = Q_2$ spin liquid on the kagome lattice [32], the spinons have a quadratic dispersion around the high symmetry M point, and the structure factor [17] there shows a sharp onset above the gap, as expected for bosonic excitations.

Discussion.—We have shown that the production rates for fractionalized excitations (for example, in neutron scattering or tunneling experiments) contain a signature of anyons that allows one to measure the statistical parameter of the anyons experimentally: The cross section follows a power law determined by this parameter. This may also be accessible in systems of ultracold atoms in optical lattices following recent ideas of how to access their spectral functions [33].

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