Universality of the Turbulent Velocity Profile

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For nearly a century, the universal logarithmic law of the mean velocity profile has been a mainstay of turbulent fluid mechanics and its teaching. Yet many experiments and numerical simulations are not fit exceedingly well by it, and the question whether the logarithmic law is indeed universal keeps turning up in discussion and in writing. Large experiments have been set up in various parts of the world to confirm or deny the logarithmic law and accurately estimate von Kármán's constant, the coefficient that governs it. Here, we show that the discrepancy among flows in different (circular or plane) geometries can be ascribed to the effect of the pressure gradient. When this effect is accounted for in the form of a higher-order perturbation, universal agreement emerges beyond doubt and a satisfactorily simple formulation is established.

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Turbulent flow in a parallel channel or duct is the prototype problem of wall-bounded turbulence. Be it between parallel plane walls or in a circular pipe, this is the contained turbulent flow with highest geometrical symmetry, and also the one whose physical properties are best understood. In the ideal case of infinite extension in the other directions, the mean velocity is directed parallel to the walls and is a function of the wall-normal coordinate only; namely, it is a function of a single variable, the velocity profile $u(z)$.

Although the equations governing the time evolution of the flow are known, the Navier-Stokes equations, the only workable way to extract its time average is to actually run a time-resolved simulation for a long time and take its average (so called direct numerical simulation, or DNS), equivalent to running an experiment for a long enough time and taking its average. In the case of a parallel flow, there are so few parameters that a universal shape of the velocity profile can be identified. The classical route to do so is based on dimensional analysis coupled with a few critical assumptions.

Nearly a century ago, Prandtl [1] recognized by his mixing-length argument that the velocity profile in a duct or pipe would have to be approximately logarithmic. The theory was then given its present form based on dimensional analysis by Millikan [2]. The physical parameters affecting the phenomenon, in the case of an incompressible newtonian fluid like water or air, are the density ρ and kinematic viscosity ν of the fluid, a typical dimension h of the container (which we shall assume to be the distance from the wall to the symmetry axis, *i.e.*, the radius of a pipe or the semi-distance of two parallel plane walls), and the externally imposed pressure gradient p_x. The latter is tied to the mean shear stress τ_w exerted on the container walls by a simple force balance: if A is the area of the duct's cross section and LP its lateral area, product of length L and perimeter P, static equilibrium requires that $-Ap_xL = LP\tau_w$, or

$$p_x = -4\tau_w/D_H, \tag{1}$$

where the quantity $D_H = 4A/P$ goes by the name of hydraulic diameter (because in a circular pipe it coincides with its diameter, and in other cases, it has the role of diameter of an equivalent pipe with the same volume to area ratio). Therefore, for a given geometry, only one of τ_w or p_x must be specified; nevertheless, these two quantities are related differently in different geometries, and in what follows, the distinction will be crucial.

Using the dimensional parameters ρ, ν, h, τ_w, and the wall-normal running coordinate z, at most two independent dimensionless groupings can be assembled. Traditionally, a characteristic velocity is defined first, as $u_\tau = \sqrt{\tau_w/\rho}$, and then a "viscous length" $l = \nu/u_\tau$. The ratio h/l coincides with the Reynolds number $\mathrm{Re}_\tau = hu_\tau/\nu$, and it must be large or the flow would not be turbulent. The classical asymptotic theory of the turbulent velocity profile after Millikan is constructed in the limit of $\mathrm{Re}_\tau \to \infty$. The basic ansatz is that near the wall, in the region $z \ll h$ known as the "wall layer", the velocity profile turns out independent of height h and is thus a dimensionless function of a single variable,

$$u^+ = f(z^+), \tag{2}$$

where $u^+ = u/u_\tau$ and $z^+ = z/l = zu_\tau/\nu$. (Quantities denoted by a $^+$ are also commonly described as being measured in "wall units".)

Conversely, for $z \gg l$, the velocity profile becomes independent of length l, or equivalently of the fluid's viscosity, but with a catch: because of Galilean invariance, having lost the reference of a wall whose possible motion we assume the flow has become independent of, only velocity differences can be significant. Therefore, the difference between the centerline velocity $U = u(h)$ and the generic velocity $u(z)$ is defined to be a dimensionless function of the single variable $Z = z/h$ as

$$U^+ - u^+ = F(Z). \tag{3}$$

$U - u$ is known as the velocity *defect*, and the region $z \gg l$ as the "defect layer".

Since by assumption $h \gg l$, there is a range of z where the two conditions $z \gg l$ and $z \ll h$ can be simultaneously verified. We are then in the "overlap layer", where the velocity profile is independent of both h and l. Having exhausted the available dimensional quantities, only a dimensionless *constant* can be constructed in this layer. This is von Kármán's constant

$$\kappa = \frac{u_\tau}{z u_z} \tag{4}$$

(involving the velocity derivative u_z in place of the velocity because, just as in the defect layer, velocity can only be determined up to an additive constant). It follows by integration of (4) that in the overlap layer the velocity profile is logarithmic

$$u^+ = \kappa^{-1}\log(z^+) + B = \kappa^{-1}\log(Z) + C, \tag{5}$$

where B and C are suitable integration constants allowing a matching with the wall and defect layers, and tied to one another as $C = B + \kappa^{-1}\log(h/l) = B + \kappa^{-1}\log(\mathrm{Re}_\tau)$. Notice that out of this reasoning, while C and $F(Z)$ depend on geometry, the κ and B constants and the so called "law of the wall" $f(z^+)$ must be universal.

The above theory has been the pillar of the description of wall-bounded turbulence for nearly a century, but there are problems. Whereas since the first experimental measurements of Nikuradse [3] it was clear that von Kármán's constant κ is of the order of 0.4, attempts to determine it more precisely from dedicated experiments [4,5] and, once they became available, from DNS proved elusive. A number of authors developed alternative explanations, most prominent being the incomplete-similarity theory of Barenblatt [6], leading to a power law with a Reynolds-dependent exponent, and recently it was suggested [7] that von Kármán's constant κ is not universal but indeed it takes on three different values for a channel with parallel plane walls, for a circular pipe and for the zero-pressure-gradient boundary layer (approximated as a parallel flow). The difficulty is exemplified in Fig. 1, which reports, at a particular Reynolds number $\mathrm{Re}_\tau = 1000$, the numerically obtained velocity profiles for circular-pipe, plane pressure-driven, and plane Couette flow and their logarithmic derivatives, as reproduced from DNS data available in the literature. Clearly a single set of logarithmic-law constants cannot fit all three. In addition, the velocity law does not look especially close to being logarithmic, as is even more evident from its derivative which would have to be constant if it was.

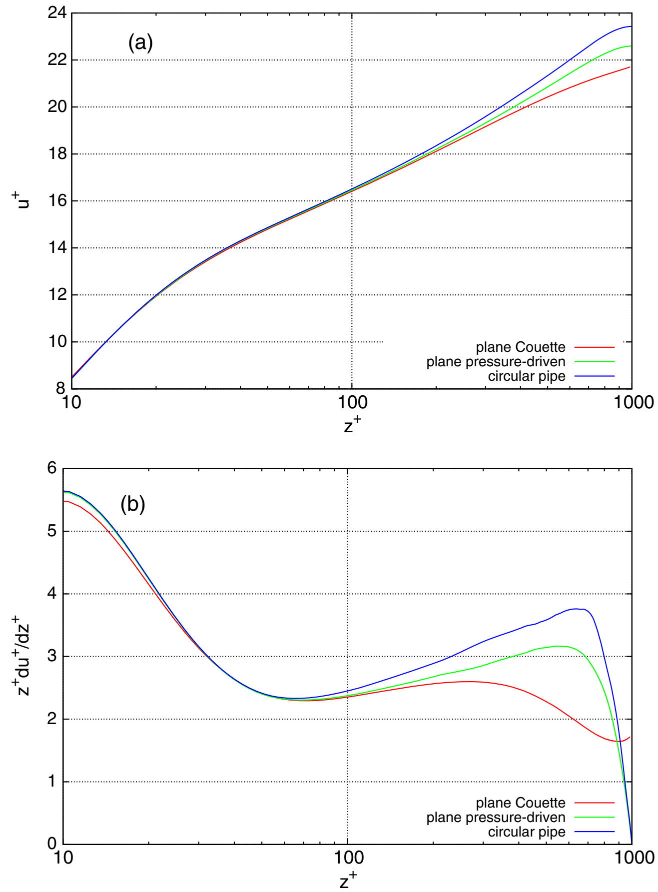

FIG. 1. Velocity profile in wall units versus wall-normal coordinate on a logarithmic scale (a) and its logarithmic derivative (b), for three different geometries at $\mathrm{Re}_\tau = 1000$. DNS data for the circular pipe flow are taken from El Khoury *et al.* [8]. DNS data for the pressure-driven plane parallel flow are taken from Lee and Moser [9]. DNS data for plane Couette flow are taken from Pirozzoli *et al.* [10].

However, it should not be forgotten that the theory of the logarithmic behavior of the overlap layer is an asymptotic approximation for $\mathrm{Re}_\tau \to \infty$. In this sense, every similarity theory is incomplete, but like all asymptotic approximations, it can be improved with the addition of higher-order terms.

A number of authors have proposed higher-order asymptotic expansions of the turbulent profile, starting with Tennekes [11] and Yajnik [12], a recent review being offered by Panton [13,14].

The most likely expansion parameter was immediately identified as $\mathrm{Re}_\tau^{-1} = l/h$, but in some cases, it was $u_\tau/U \approx (\log \mathrm{Re}_\tau)^{-1}$ instead, or a combination of both. Many authors (*e.g.*, [15]) went to a great length to identify a suitable set of differential equations from which the expansion could be derived, often introducing closure assumptions (of the eddy-viscosity or mixing-length kind) for this purpose [16]. Afzal [17] observed that Millikan's overlap argument can be extended to obtain higher-order results without introducing any arbitrary assumption of this kind.

It can be remarked that these authors mostly focused their attention on the Reynolds-number dependence of the velocity profile, rather than on the different behavior of circular-pipe and plane parallel flow, and did not have access to Couette flow data which only became available recently. As a consequence, all asymptotic expansions of parallel flow treated p_x and τ_w as interchangeable parameters. (Not so the asymptotic theories of the turbulent boundary layer, many of which were developed after Mellor [18]; Gorin and Sikovsky [19], for instance, explicitly emphasized that in the boundary layer p_x and Re_τ^{-1} are independent small parameters. That this is so becomes evident in the near-separation regime, where τ_w becomes negligible and p_x stands out as the only surviving parameter.)

When different geometries are compared to each other, the roles of p_x and τ_w can be separated even in purely parallel flow. Here, continuing in the vein of Millikan's dimensional analysis, we can base a higher-order asymptotic approximation on the identification of the most influential dimensional parameters and then of their dimensionless groupings. We have seen that the influence of the physical parameters ν (viscosity), h (geometry), and p_x [pressure gradient, tied to τ_w and to geometry by the relationship (1)] tends to be lost in the overlap layer, but, whereas viscosity is "small" in the sense that $u_\tau z/\nu \gg 1$, and geometry is "far" in the sense that $z \ll h$, the pressure gradient is uniform and acts all along the velocity profile. Therefore, we modify Millikan's ansatz of independence that led to (4) into the new statement:

In the overlap layer, the dimensionless quantity zu_z/u_τ is almost independent of h and ν, but is acted upon by p_x as by a small perturbation.

Of course that the pressure gradient can affect the turbulent velocity profile is not in itself a new idea, as noted before with reference to the turbulent boundary layer, and indeed the experimental discrepancies between pipe and plane-channel flow have been qualitatively ascribed to its effect before, but the effect of pressure gradient has rather been interpreted as producing a variation of the value of von Kármán's constant [7]. From the general standpoint of dimensional analysis, the inclusion of an additional dimensional parameter (p_x) in Millikan's ansatz could mean that zu_z/u_τ is no longer a constant κ^{-1} but a new arbitrary function of a single variable. However, by specifying that p_x is a small perturbation, we impose this to be a linear function of p_x. Then there is only one dimensionally correct linear extension of (4), and it is

$$\frac{u_z}{u_\tau} = \frac{1}{\kappa z} - A_1 \frac{p_x}{\tau_w}, \tag{6}$$

whence, by integration, the velocity profile (5) becomes

$$\begin{aligned} u^+ &= \kappa^{-1}\log(z^+) + A_1 g \mathrm{Re}_\tau^{-1} z^+ + B \\ &= \kappa^{-1}\log(Z) + A_1 g Z + C. \end{aligned} \tag{7}$$

Here, A_1 is a new universal constant, and the geometry parameter $g = -hp_x/\tau_w = 4h/D_H$ makes its appearance from the nondimensionalization. It takes on the values of $g = 2$ for circular pipe flow, $g = 1$ for pressure-driven flow between still plane walls and $g = 0$ for turbulent Couette flow between countermoving plane walls. (Fractional values of g can also be achieved by imposing mixed Couette-Poiseuille boundary conditions, and naturally occur in a boundary layer.) In words, the assumption of a linear dependence on p_x makes the velocity correction linear in z as a consequence (as also remarked in [19] for the scope of an expansion of the turbulent boundary layer). As may be seen from (7), the new term proportional to p_x acts as a small perturbation $O(\mathrm{Re}_\tau^{-1})$ in the wall layer, consistent with the idea that (2) and (5) were just the leading term of an asymptotic expansion in reciprocal powers of Re_τ, and becomes $O(1)$ in the defect layer where it blends smoothly

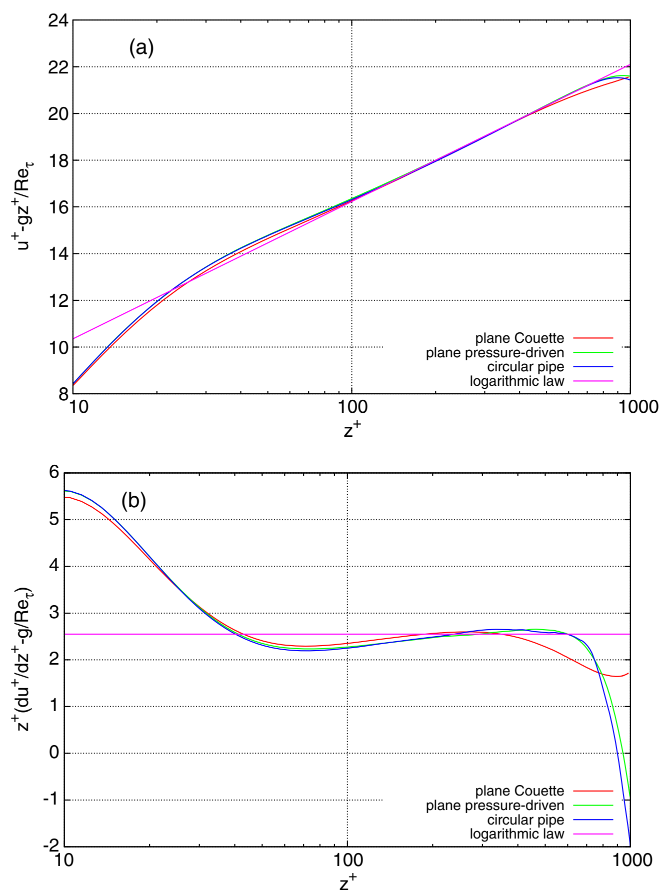

FIG. 2. Velocity profile (a) and its logarithmic derivative (b) after subtraction of the pressure-gradient term $A_1 g z^+/\mathrm{Re}_\tau$, compared with the logarithmic law with coefficients (8).

with the function $F(Z)$ of (3), which is allowed to vary in its own right from one geometry to another. This is analogous to the method of matched asymptotic expansions developed by van Dyke [20] for the laminar boundary layer: the first-order $[O(\mathrm{Re}_\tau^{-1})]$ asymptotic correction to the inner solution becomes the first-order $[O(Z)]$ term of the Taylor series for the outer solution.

Out of a fitting of available numerical and experimental data at several values of the Reynolds number, the value of A_1 comes fairly close to unity; thus, it is tempting to set it exactly equal to 1. Eventually, a good overall fit of (7) to the available data is provided by

$$\kappa = 0.392; \qquad A_1 = 1; \qquad B = 4.48. \tag{8}$$

In order to illustrate the effect of this modification, Fig. 2 contains the same velocity profiles as Fig. 1, each diminished of the first-order correction $A_1 g \mathrm{Re}_\tau^{-1} z^+$; these profiles must coincide much more closely across the three geometries if (7) is correct. As can be seen they do, not only in slope but also in vertical position, and the common behavior of the three curves is at the same time closer to logarithmic than before. [The initial, nonlogarithmic part of the curve up to $z^+ \approx 200$ is the trace of the universal law of the wall (2) with its own higher-order correction, but this finer correction sits on the edge of what can be estimated within the accuracy of present simulations and experiments [21].]

In conclusion, the logarithmic law of the turbulent velocity profile is indeed universal across different geometries of wall-bounded flow, provided the perturbative effect of pressure gradient is accounted for. The present Eq. (7) does not contradict the classical Eq. (5) but rather extends it with the inclusion of a higher-order term, and in fact tends to (5) for $\mathrm{Re}_\tau \to \infty$. Nevertheless, this higher-order term is essential when using the logarithmic law for practical engineering purposes or when estimating von Kármán's constant κ from numerical or experimental data taken at practical values of the Reynolds number. Its omission in the classical theory justifies the doubts that have arisen in the literature, whereas including it definitely shows that the logarithmic law is valid and the value of κ is universal.

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