Nontrivial Critical Fixed Point for Replica-Symmetry-Breaking Transitions

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The transformation of the free-energy landscape from smooth to hierarchical is one of the richest features of mean-field disordered systems. A well-studied example is the de Almeida–Thouless transition for spin glasses in a magnetic field, and a similar phenomenon—the Gardner transition—has recently been predicted for structural glasses. The existence of these replica-symmetry-breaking phase transitions has, however, long been questioned below their upper critical dimension, $d_u = 6$. Here, we obtain evidence for the existence of these transitions in $d < d_u$ using a two-loop calculation. Because the critical fixed point is found in the strong-coupling regime, we corroborate the result by resumming the perturbative series with inputs from a three-loop calculation and an analysis of its large-order behavior. Our study offers a resolution of the long-lasting controversy surrounding phase transitions in finite-dimensional disordered systems.

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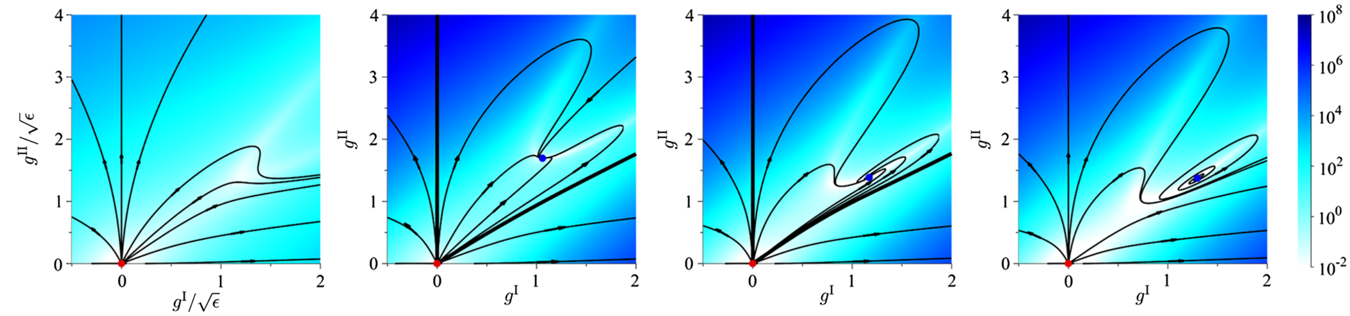
Introduction.—Spontaneous symmetry breaking can dramatically change material properties. Breaking translational symmetry turns liquids into crystalline solids, breaking gauged phase symmetry gives rise to superconductivity, and breaking non-Abelian gauged symmetry endows elementary particles with mass. In a host of disordered models, a symmetry of the most peculiar type can break. Upon cooling, the mean-field free-energy of these systems develops a finite complexity; the number of metastable states grows exponentially with system size. The similitude between copies (replicas) of the system then depends on whether or not they belong to a same cluster of metastable states. In particular, right at the transition point, each replica of the system is on the brink of falling into one cluster or another, resulting in critical fluctuations of the similarity between uncoupled copies. Remarkably, such replica symmetry breaking (RSB) accounts for the emergence of glassiness in mean-field models ranging from liquids to optimization problems and neural networks [1]. Mean-field criticality, however, bears the seed of its own destruction. Below the upper critical dimension, d_u, violent critical fluctuations challenge the very validity of the approximation within which they were conceived. The existence of a continuous transition into an RSB phase in dimensions $d < d_u$ is thus not a foregone conclusion, and its fate in disordered systems remains hotly debated [2].

An illustrious example of this dispute centers around mean-field models of spins with quenched impurities in an external magnetic field, known to exhibit a de Almeida–Thouless (dAT) transition [3]. This transition accompanies the emergence of continuous RSB with a hierarchically rough landscape, which eventually becomes fractal in the low-temperature limit [4]. Its upper critical dimension, $d_u = 6$ [5], however, is well above 3 and the existence of the transition in real physical systems has long been questioned [6–13]. Recent advances in the mean-field description of structural glasses have unveiled a new facet of this problem. Solid glasses are also predicted to undergo a critical RSB transition—known as a Gardner transition—upon cooling, compressing, or shearing [4,14,15]. A growing body of evidence further relates the Gardner transition to the anomalous behavior of amorphous solids compared to their crystalline counterpart, and to a nontrivial critical scaling upon approaching the jamming limit [4,16–20]. The question of whether dAT and Gardner transitions survive finite-dimensional fluctuations has thus gained renewed impetus.

The impact of fluctuations on RSB transitions was first examined using the perturbative renormalization group (RG) approach that proved so successful for Ising and other universality classes. A loop expansion of the field theory appropriate for the dAT and Gardner universality class, however, finds that the critical fixed point is absent to lowest, one-loop order for $d < d_u$ [21–23]. This has led many to conclude that such transitions then either become discontinuous or simply vanish. Yet the lack of dimensional robustness is challenged by numerical evidence supporting the existence of a critical dAT transition in $d = 4$ [24,25] and of a critical Gardner transition in $d = 3$ [26,27]. An alternate interpretation is that the critical fixed point resides in the strong-coupling regime of the field theory for all d, thus preventing a one-loop calculation from identifying it. A precedent is the Caswell-Banks-Zaks (CBZ) fixed point of the non-Abelian gauge theory for elementary particles in $3 + 1$ space-time dimensions [29]. This fixed point is missed at one-loop order but captured at two-loop order. Although the CBZ fixed point generically lies in the strong-coupling regime, which falls beyond the designed range of



(a) One-loop RG, $d < 6$ (b) Two-loop RG, $d = 3$ (c) Two-loop RG, $d = 5$ (d) Two-loop RG, $d = 5.5$

FIG. 1. RG flows in the space of couplings for (a) the one-loop calculation in $d < 6$ and the two-loop calculation in (b) $d = 3$, (c) $d = 5$, and (d) $d = 5.5$. Arrows denote flow toward longer length scales; background shading denotes the intensity of the flow quantified by $(\beta^{\mathrm{I}})^2 + (\beta^{\mathrm{II}})^2$, and normalized by ϵ^{-3} in (a). The Gaussian fixed point (red dot) is unstable for $d < 6$. In (b) and (c) a nontrivial fixed point (blue dot) is stable and lies at strong couplings. Its basin of attraction is delineated by two thick lines: one precisely along $g^{\mathrm{I}} = 0$ and the other approximately along $g^{\mathrm{I}} \approx g^{\mathrm{II}}$. Outside this basin, the flow runs toward infinity, which is often characteristic of discontinuous transitions. Note that for $d = 5$, the flow spirals into the nontrivial fixed point, while for $d = 5.5$ both fixed points are unstable.

a perturbative calculation, its existence has been corroborated by adiabatically connecting it to a perturbative fixed point [30], supported by lattice simulations even in the strong-coupling regime [31], and established beyond reasonable doubt in supersymmetric theories [32,33]. Two-loop calculations may thus find fixed points that are missed by one-loop analysis, but additional lines of evidence are then needed to confirm the result.

In this Letter, we present field-theoretic calculations that capture the physics of both dAT transitions in spin glasses and Gardner transitions in structural glasses. Like for the CBZ fixed point, our two-loop calculation identifies a critical fixed point for $d < d_u$ that is missed by the one-loop RG flow. Resummation of the perturbative series at three-loop order supplemented by an analysis of its large-order behavior further supports the robustness of this critical fixed point for the dAT–Gardner universality class.

Field-theory setup.—The finite-dimensional generalization of the mean-field Edwards-Anderson order parameter for glasses is the replicated overlap field, $q_{ab}(\mathbf{x})$. This field characterizes the similarity at positions $\mathbf{x}$ between pairs of distinct replicated configurations through an n-by-n symmetric matrix with a null diagonal; the zero replica limit, $n \to 0$, is taken at the end of the calculations in order to properly average over disorder [34]. In general, field fluctuations can be subdivided into longitudinal, anomalous, and replicon modes [35,36]. At dAT and Gardner transitions only replicon modes become critical (massless); the other two remain short-ranged (massive) and can thus be neglected at long distances. We henceforth only focus on the replicon field, $\phi_{ab}(\mathbf{x})$, defined by the condition $\sum_{b=1}^{n} \phi_{ab} = 0$ for all $a = 1, \ldots, n$, thus leaving $n(n-3)/2$ degrees of freedom.

In order to investigate the putative critical point, we seek infrared-stable fixed points of the RG flow within the critical surface on which the replicon field remains massless. Within this surface, the field theory is governed by the bare action, $S = \int d\mathbf{x}\mathcal{L}$, with [37]

$$\mathcal{L} = \frac{1}{2}\sum_{a,b=1}^{n} (\nabla\phi_{ab})^2 - \frac{1}{3!}\left(g^{\mathrm{I}}_{\mathrm{bare}} \sum_{a,b=1}^{n} \phi_{ab}^3 + g^{\mathrm{II}}_{\mathrm{bare}} \sum_{a,b,c=1}^{n} \phi_{ab}\phi_{bc}\phi_{ca} \right), \tag{1}$$

which is the most generic cubic action for replicon modes [22]. The effective description of the system then depends on the energy scale, μ, probed. This dependence is encoded in the RG flow of dimensionless couplings, $g^{\mathcal{X}}(\mu)$ with $\mathcal{X} \in \{\mathrm{I}, \mathrm{II}\}$, that are related to bare couplings, $g^{\mathcal{X}}_{\mathrm{bare}}$, in Eq. (1) [34]. The flow is governed by $\beta^{\mathcal{X}} \equiv \mu \partial g^{\mathcal{X}}/\partial\mu$, and stops at fixed points, where $\beta^{\mathrm{I}}(g^{\mathrm{I}}_\star, g^{\mathrm{II}}_\star) = \beta^{\mathrm{II}}(g^{\mathrm{I}}_\star, g^{\mathrm{II}}_\star) = 0$. Note that for all d a Gaussian fixed point with $g^{\mathrm{I}}_\star = g^{\mathrm{II}}_\star = 0$ exists, but it is stable only for $d > d_u$ [21].

Two-loop RG.—Inspired by the CBZ fixed point, we compute the β functions to two-loop order for the replica field theory in Eq. (1), using the dimensional regularization scheme [29,34,38–43]. As expected [21–23], no stable fixed point can be found at one-loop order for $d < 6$ [Fig. 1(a)]. For $d < d_0 \approx 5.41$, however, the two-loop RG flow locates a stable fixed point with a finite basin of attraction [Figs. 1(b) and 1(c)]. A system lying within this basin eventually approaches the fixed point upon rescaling and is thus critical. By contrast, a system that remains outside the basin cannot continuously transition into an RSB phase, and may instead exhibit a discontinuous transition. Remarkably, the boundary of the basin is closely approximated by the tree-level condition for a critical transition into a RSB phase, i.e., $1 < g^{\mathrm{II}}/g^{\mathrm{I}} < \infty$ [44].

The eigenvalues, λ_1 and λ_2, of

$$\left.\begin{bmatrix} \frac{\partial\beta^{\mathrm{I}}}{\partial g^{\mathrm{I}}} & \frac{\partial\beta^{\mathrm{I}}}{\partial g^{\mathrm{II}}} \\ \frac{\partial\beta^{\mathrm{II}}}{\partial g^{\mathrm{I}}} & \frac{\partial\beta^{\mathrm{II}}}{\partial g^{\mathrm{II}}} \end{bmatrix}\right|_{(g^{\mathrm{I}}, g^{\mathrm{II}}) = (g^{\mathrm{I}}_\star, g^{\mathrm{II}}_\star)} \tag{2}$$

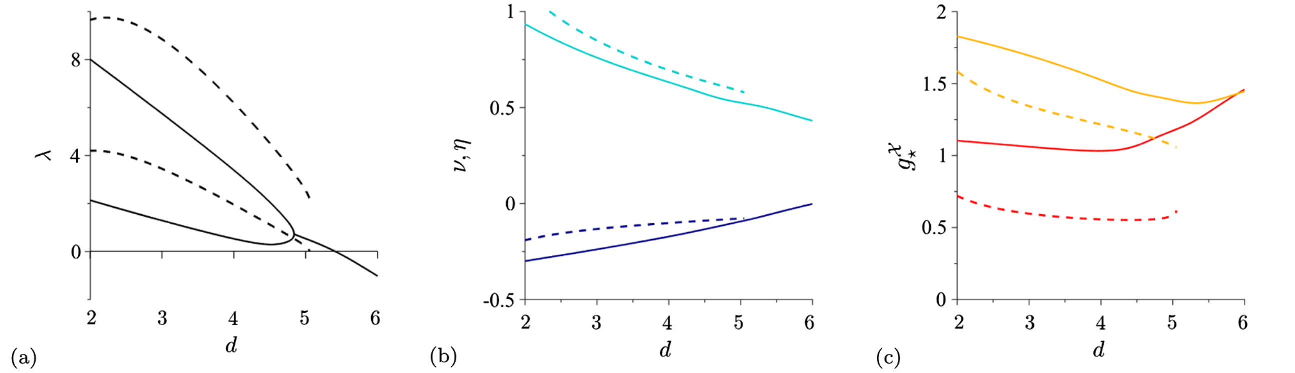


FIG. 2. Critical parameters at the nontrivial fixed point derived within two-loop (solid lines) and three-loop with Borel resummation (dashed lines) RG schemes as functions of the spatial dimension d. (a) Real parts of the stability exponents around the fixed point within the critical surface, λ_1 and λ_2. (b) Critical exponents, ν (cyan) and η (navy blue). (c) Fixed-point values of running couplings, g^{I} (red) and g^{II} (orange). At two-loop order, the nature of the fixed point changes at $d_s \approx 4.84$ and $d_0 \approx 5.41$. The two stability exponents merge at $d = d_s$, at which point they acquire imaginary parts; hence, the flow spirals into the (stable) fixed point [Fig. 1(c)], while for $d > d_0$ the real part of these eigenvalues becomes negative and the flow spirals out of the (unstable) fixed point [Fig. 1(d)]. Upon inclusion of three-loop contributions, Borel resummation indicates that the fixed point is robustly stable for $d \lesssim 5.05$ but does not exhibit any spiraling flow.

give the stability exponents that control subleading corrections from irrelevant deformations near the critical point. Figure 2(a) indicates that these exponents acquire an imaginary component for $d > d_s \approx 4.84$; hence, the RG flow then spirals toward the fixed point [Fig. 1(c)]. As has been observed in other disordered systems [45–47], such complex exponents can emerge from the nonunitarity of the replica field theory, and give rise to an oscillatory decay of the appropriate correlation functions in the critical region. Conformality gets lost with the change in the spiral direction at $d = d_0$ and no stable fixed point can be found for $d \in (d_0, d_u)$ [Fig. 1(d)]. In the absence of additional nontrivial fixed points with which to collide [48], this scenario provides a natural mechanism for exchanging dominance between the Gaussian and the genuinely nonperturbative fixed points as one goes from $d > d_u$ down to physical dimensions.

We also compute the critical exponents, ν and η, that govern the divergence of the correlation length and the decay of two-point correlation functions at the critical point, respectively [Fig. 2(b)]. The former is obtained from the relevant deformation by the quadratic coupling that drives the system away from the critical surface. Estimates of ν and η agree qualitatively with the trend observed in $d = 4$ simulations [24]; η is negative and ν is larger than its mean-field value, $\nu_{\mathrm{MF}} = \frac{1}{2}$.

Resummation.—Because the critical couplings are of order unity for all $d < d_u$ [Fig. 2(c)], resummation is needed to assess the existence of the fixed point. (Without a careful resummation, even the $d \leq 3$ Wilson-Fisher fixed point for the Ising universality class disappears [49].) A field-theoretic perturbative series is indeed generically not convergent but rather asymptotic. More precisely, a formal series in terms of the coupling constant,

$$f(g^2) = \sum_k f_k g^{2k}, \tag{3}$$

typically has coefficients that exhibit a factorial growth, i.e., $f_k \sim k!(-1/A)^k$, with a large-order constant A given in terms of the saddle-point action [50,51]. Although a truncation to the first couple of terms may yield a good approximation in the weak-coupling regime, the series itself is not mathematically well defined.

Borel resummation is the most common scheme used to give epistemological traction to a fixed point. The approach starts from the observation that a Borel transform, $\tilde{f}_B(g^2) \equiv \sum_k (f_k/k!) g^{2k}$, has a finite radius of convergence, $|A|$. Using the identity $k! = \int_0^\infty dt e^{-t} t^k$ the original series [Eq. (3)] can formally be expressed as $f(g^2) = \int_0^\infty dt e^{-t} \tilde{f}_B(t g^2)$. The analytic continuation of the Borel transform onto the whole positive axis then unambiguously defines the function f. There is typically no problem to this analytic continuation when $A > 0$; hence, the series is then deemed Borel summable.

In order to adapt the above scheme to a replica field theory with two cubic couplings, we define $(g^{\mathrm{I}}, g^{\mathrm{II}}) \equiv g(\cos\theta, \sin\theta)$ and regroup the double series, with the power of g^2 counting loop order:

$$\begin{aligned} f(g^{\mathrm{I}}, g^{\mathrm{II}}) &= \sum_{k_1, k_2 = 0; k_1 + k_2 = \mathrm{even}}^{\infty} f_{k_1, k_2} (g^{\mathrm{I}})^{k_1} (g^{\mathrm{II}})^{k_2} \\ &= \sum_{k=0}^{\infty} g^{2k} \left[\sum_{k_1 = 0}^{2k} f_{k_1, 2k - k_1} (\cos\theta)^{k_1} (\sin\theta)^{2k - k_1} \right] \\ &\equiv \sum_{k=0}^{\infty} f_k(\theta) g^{2k}. \end{aligned} \tag{4}$$

The Borel-summability of the series is then governed by the angle-dependent large-order behavior $f_k(\theta) \sim k![-1/A(\theta)]^k$. Consequently, as has been observed for the Abelian gauge theory with background fields [52], Borel-summability depends on the ratio of two couplings, as encoded in the saddle-point solution to the classical equations of motion for replicons [53].

Among nontrivial saddles, we assume [55,56] that the saddle of the form

$$\phi^{\star}_{ab}(\mathbf{x};\theta) = \frac{1}{g} F(\mathbf{x}) v^{(\theta)}_{ab} \tag{5}$$

dictates the value $A(\theta)$. Here, F is a spherically symmetric function that solves

$$\nabla^2 F = F - F^2, \tag{6}$$

obtained numerically through the pseudospectral method [57,58], and $v^{(\theta)}_{ab}$ is the replicon component of the Parisi RSB ansatz [1,59–61]. Computing the action of the resulting saddle [34] indicates that a solution exists if and only if $1 < \tan\theta < \infty$, with

$$A(\theta) = \frac{c_d}{\cos^2\theta(\tan\theta - 1)}, \tag{7}$$

where c_d is a d-dependent positive constant. The series is thus Borel summable within the wedge $1 < g^{\mathrm{II}}/g^{\mathrm{I}} < \infty$, consistent with the mean-field consideration [44] and the two-loop basin of attraction obtained in Fig. 1. This result thus validates our perturbative treatment of the strong-coupling regime within the basin of attraction.

Given the large-order behavior at hand, we further compute the critical properties of the fixed point by resumming the three-loop series, analytically continuing the Borel transform through the conformal mapping [34,62]. Comparing the two-loop and the resummation results upon inclusion of three-loop contributions [34] (Fig. 2) confirms that the fixed point is robustly conserved for $d \lesssim 5.05$. The critical exponents from the two schemes further qualitatively agree with one another.

Conclusion.—The nontrivial critical fixed point identified here governs both dAT and Gardner transitions in $d < d_u$. An RSB transition for the underlying universality class is thus possible over a broader d range than previously thought [67–69]. The RG flow diagrams (Fig. 1) and the large-order behavior, however, make it clear that not all microscopic models belong to the basin of the attraction of the critical fixed point. This realization offers a possible explanation for the absence of dAT criticality in the Edwards-Anderson model in $d = 3$. The model may simply remain outside the basin of attraction, and thus be governed either by a discontinuous transition into the RSB phase or by the two-state droplet picture [6–10]. Enlarging the range of disordered spin systems used for studying RSB criticality would clarify this last point.

Our results further highlight various future research directions. First, they guide efforts in systematizing nonperturbative RG methods [70] and controlling conformal bootstrap techniques for nonunitary theories [71–74]. Both approaches should find the nontrivial critical fixed point when applied to the replica field theory. Second, conflicting results have been obtained for the lower critical dimension, d_l, from a heuristic interface argument [75] and from a correlation-function argument [76]. The dimensional dependence of the infrared divergence associated with soft modes thus deserves further scrutiny. Third, extending the current approach will enable the study of the RG trajectory between the critical point identified here and the multicritical fixed point found perturbatively for the spin-glass transition in the absence of an external magnetic field [5], where longitudinal and anomalous modes become massless concurrently with the replicons.

Data relevant to this work can be accessed by following the link in Ref. [77].

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