

Particle-Hole Duality in the Lowest Landau Level

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We derive a number of exact relations between response functions of holomorphic, chiral fractional quantum Hall states and their particle-hole (PH) conjugates. These exact relations allow one to calculate the Hall conductivity, Hall viscosity, various Berry phases, and the static structure factor of PH conjugate states from the corresponding properties of the original states. These relations establish a precise duality between chiral quantum Hall states and their PH conjugates. The key ingredient in the proof of the relations is a generalization of Girvin's construction of PH-conjugate states to inhomogeneous magnetic field and curvature. Finally, we make several nontrivial checks of the relations, including for the Jain states and their PH conjugates.

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Introduction.—Particle-hole (PH) transformation for fractional quantum Hall (FQH) states was introduced by Girvin [1]. This transformation relates a FQH state at filling fraction ν to a FQH state at filling fraction $1 - \nu$. In the absence of Landau level mixing the projected lowest Landau level (LLL) Hamiltonian is PH symmetric and, therefore, two states related by a PH transformation have the same energy (up to a shift in the chemical potential). Despite the physical clarity of PH symmetry, the PH-transformed wave functions look quite complicated and are difficult to work with. PH-transformed states contain a different number of particles and have different transport properties and different topological order. In this Letter we will explain that all of the information about a PH-transformed state is encoded in the original state, so that both states are a different representations for essentially the same physics. For this reason we feel it is more appropriate to refer to the PH transformation as a particle-hole duality (PHD).

Recent years have also brought the rise of interest in the role of PHD in the problem of the half-filled Landau level. To resolve the issue of the apparent absence of the PH invariance in the Halperin-Lee-Read [2] theory, Son has proposed a manifestly PH-invariant effective theory of composite fermions with π Berry phase around the composite Fermi surface [3]. This theory can successfully be used to describe Jain states at fillings close to $\nu = 1/2$ and a PH-invariant (or self-dual) version of the Pfaffian state [3,4], which is a viable candidate for the observed $\nu = 5/2$ plateau [5].

PH transformation, as defined by Girvin [1], works in flat space and homogeneous magnetic field. It was recently appreciated that placing a FQH state in an inhomogeneous background magnetic field and curved geometry allows one to extract considerable information about the flat space properties of the state [6–26]. For example, the projected static structure factor (SSF) [27] in leading and subleading order in momentum, and long-wave corrections to Hall

conductivity and Hall viscosity can be calculated from the properties that become apparent in curved space [11,19,28–30]. Integer quantum Hall states in curved geometry are available in (synthetic) photonic systems [31].

In this Letter, we will use the approach of [9,19] to extend Girvin's construction to inhomogeneous magnetic field and curved geometry. Next, we will derive several exact relations between Hall conductivity, Hall viscosity, Berry phases, and the SSF of the holomorphic, chiral FQH states and their PH duals. These relations establish the PHD quantitatively and show that properties of the PH-dual state are completely determined by the original state. The duality is nontrivial since the calculations can be easily done before the PH transformation but are difficult to do after.

Under certain assumptions, the long-wave corrections to Hall conductivity, Hall viscosity, and the SSF are determined by topological quantum numbers [11,19,20,28,29]: filling fraction ν , shift $S = 2\bar{s}$ [6], chiral central charge c_- [32], and the orbital spin variance $\text{var}(s)$ [14,16]. We will explain how the topological quantum numbers transform under the PHD and prove that the aforementioned long-wave corrections are still determined by the (transformed) topological quantum numbers, albeit via different relations. We will check the derived relations against the explicit computation of the corresponding quantities for Jain states done in Son's theory of composite fermions and find complete agreement.

FQH states in inhomogeneous background.—We start with a brief review of the construction [9,19,20] of a LLL FQH state in inhomogeneous magnetic field and curvature. Consider a holomorphic FQH state $\Psi_\nu(\{\xi\})$, where $\{\xi\} = \xi_1, \dots, \xi_N$ denotes the collection of particle positions on the plane in complex coordinates $\xi = x + iy$. We will assume that the magnetic field B is inhomogeneous and the background geometry is curved. Then the unnormalized wave function $\Psi_\nu(\{\xi\})$ takes the following form [9,19]:

$$\Psi_\nu(\{\xi\}) = f_\nu(\{\xi\}) e^{\frac{i}{2} \sum_{i=1}^N Q(\xi_i, \bar{\xi}_i)}, \quad (1)$$

where $f_\nu(\{\xi\})$ is holomorphic in ξ_i , and Q is the magnetic potential [33] defined by

$$\Delta_g Q = -2B, \quad (2)$$

where Δ_g is the Laplace operator for the metric g_{ij} . Throughout the Letter we will fix the coordinates so that $g_{ij} = \sqrt{g} \delta_{ij}$. In these coordinates (also known as the ‘‘conformal gauge’’) the Laplacian is given by $\Delta_g = (4/\sqrt{g}) \partial_z \partial_{\bar{z}}$. When the magnetic field is homogeneous, but the space is curved, the magnetic potential is given by

$$Q = -\frac{\mathcal{K}}{2\ell^2}, \quad (3)$$

where \mathcal{K} is the Kähler potential satisfying $\partial_z \partial_{\bar{z}} \mathcal{K} = \sqrt{g}$, and $\ell = B^{-1/2}$ is the magnetic length. It is of crucial importance that $f_\nu(\{\xi\})$ does not depend on Q or the metric \sqrt{g} . This will not be the case for PH-dual states. Such states comprise a proper subset of chiral FQH states, i.e., FQH states with fully chiral effective edge theories [34].

The central object of interest is the logarithm of the normalization factor

$$\mathcal{Z}_\nu[W] = \int [d\xi] |f_\nu(\{\xi\})|^2 e^{\sum_i W(\xi_i, \bar{\xi}_i)}, \quad (4)$$

where $[d\xi] = d^2\xi_1 \dots d^2\xi_N$ with $d^2\xi = dx dy$, and $W = Q + \log \sqrt{g}$. We assume that for constant magnetic field and flat space, when $W = -|z|^2/2\ell^2$, the state is normalized and $\mathcal{Z}_\nu = 1$. It is not hard to see that $\log \mathcal{Z}_\nu[W]$ is the generating functional of the density correlation functions [19]

$$\langle \rho_\nu(\zeta) \rangle \equiv \langle \Psi_\nu | \rho_\nu(\zeta) | \Psi_\nu \rangle = \frac{1}{\sqrt{g}} \frac{\delta \log \mathcal{Z}_\nu[W]}{\delta W(\zeta)}, \quad (5)$$

where $\rho_\nu(\zeta) = (1/\sqrt{g}) \sum_{i=1}^{N_\nu} \delta(\zeta - \xi_i)$ is the density operator, and N_ν is the number of particles in the state Ψ_ν . In writing $\langle \rho_\nu \rangle$ we will always implicitly assume that the expectation value is taken in the state with the filling factor ν .

The second variation produces the connected two-point function [19]

$$\langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle_c = \frac{1}{\sqrt{g(\zeta')}} \frac{\delta}{\delta W(\zeta')} \langle \rho_\nu(\zeta) \rangle, \quad (6)$$

where $\langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle_c = \langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle - \langle \rho_\nu(\zeta) \rangle \langle \rho_\nu(\zeta') \rangle$. The static structure factor is defined as the Fourier transform (q is the dimensionless momentum)

$$s_\nu(q) = \frac{1}{\bar{\rho}_\nu} \langle \rho_\nu(q) \rho_\nu(-q) \rangle_c, \quad (7)$$

where $\bar{\rho}_\nu = \nu/(2\pi\ell^2)$ is the mean electron density in the homogeneous limit.

It follows from (2) that in flat space, derivatives with respect to W and B can be traded with each other. Going to momentum space we recover [19]

$$s_\nu(q) = \frac{q^2}{2} \frac{2\pi \delta \langle \rho_\nu(q) \rangle}{\nu \delta B(-q)} = \frac{q^2 \sigma_\nu^H(q)}{2 \sigma_\nu^H(0)}, \quad (8)$$

where we used the Středa formula $\delta \langle \rho_\nu \rangle / \delta B = \sigma_\nu^H$ [35], and the dc Hall conductance $\sigma_\nu^H(0) = \nu/2\pi$.

We will also need to know how the electron density depends on the spatial curvature. This dependence is captured by the function $\eta_\nu(q) = (2\pi/\nu) \delta \langle \rho_\nu \rangle / \delta R$ [19]. In general, $\eta_\nu(q)$ has the following momentum expansion:

$$\eta_\nu(q) = \frac{\mathcal{S}}{4} - \frac{b}{4\nu} q^2 + O(q^4), \quad (9)$$

where the constant b is an *a priori* nonuniversal parameter. However in the LLL it is determined by the topological quantum numbers [11]

$$b = \nu \bar{s} (1 - \bar{s}) + \frac{\tilde{c}}{12}, \quad (10)$$

where $\tilde{c} = c_- - 12\nu \text{var}(s)$. It is also known to control the Berry curvature on the moduli space of higher genus surfaces [18]. Note that the kinematic Hall viscosity [36] follows from the zero momentum limit of the curvature response $\eta_\nu^H/\bar{\rho}_\nu = \eta_\nu(0)$ [20]. At the same time using the expression for the scalar curvature $R = -\Delta_g \log \sqrt{g}$, and the general relation, valid for any metric-independent operator \mathcal{O} [19]

$$-\frac{\ell^2}{2} \Delta_g \frac{\delta \langle \mathcal{O} \rangle}{\delta \sqrt{g}} = \left(1 - \frac{\ell^2}{2} \Delta_g\right) \langle \mathcal{O} \rho_\nu \rangle_c \quad (11)$$

it is possible to show that [19]

$$s_\nu(q) = \frac{q^2/2}{1 + q^2/2} (q^2 \eta_\nu(q) + 1). \quad (12)$$

These relations imply

$$s_\nu(q) = \frac{1}{2} q^2 + \frac{\mathcal{S}-2}{8} q^4 - \left(\frac{b}{8\nu} + \frac{\mathcal{S}-2}{16} \right) q^6 + \dots \quad (13)$$

This relation, together with Eq. (10), was derived for the Laughlin states in [19], and conjectured to hold for states of the form (1) with $\text{var}(s) = 0$ in [20]. We conjecture that it holds generally for chiral FQH states.

Finally, combining (8) and (12) we establish an exact relation between Hall conductivity and $\eta_\nu(q)$

$$\frac{\sigma_\nu^H(q)}{\sigma_\nu^H(0)} = \frac{1}{1 + q^2/2} (q^2 \eta_\nu(q) + 1). \quad (14)$$

Relations (8), (12), and (14) hold to all orders in q , under the assumption of the absence of Landau level mixing and long-range interactions. Together with (10) these relations

imply that the first 3 terms in the momentum expansion of $\sigma_\nu^H(q)$ and $s_\nu(q)$ are completely determined by the topological quantum numbers.

Particle-hole transformation in inhomogeneous background.—Following Girvin [1], we define $\Psi_{1-\nu}(\{z\})$ as a state of holes at filling ν , which, when viewed as a state of electrons, has filling $1-\nu$. Let z_1, \dots, z_M be the coordinates of electrons and ξ_1, \dots, ξ_N be the coordinates of holes. Then the PH dual state is defined as

$$\Psi_{1-\nu}(\{z\}) = \sqrt{\frac{(N+M)!}{N!M!}} \int [d\xi] \Psi_1(\{z\}, \{\xi\}) \Psi_\nu^*(\{\xi\}), \quad (15)$$

where $\Psi_\nu(\{\xi\})$ is given by (1) and Ψ_1 is the $\nu = 1$ state. The overall factor is required to ensure that the PH-dual state is normalized to 1 in constant magnetic field and flat space. A defining property of the PH transformation is that it is an involution

$$\Psi_{1-(1-\nu)}(\{\xi\}) = \pm \Psi_\nu(\{\xi\}). \quad (16)$$

Property (16) is ensured by the following identity. First, we define an n -particle reduced density matrix [37]

$$\begin{aligned} \mathcal{P}_\eta^{(n)}(\xi_1, \dots, \xi_n; \xi'_1, \dots, \xi'_n) \\ = \frac{N_\eta!}{n!(N_\eta - n)!} \frac{1}{\mathcal{Z}_\eta[W]} \int [d\hat{\xi}] \Psi_\eta(\hat{\xi}_{n+1}, \dots, \hat{\xi}_{N_\eta}, \xi_1, \dots, \xi_n) \\ \times \Psi_\eta^*(\hat{\xi}_{n+1}, \dots, \hat{\xi}_{N_\eta}, \xi'_1, \dots, \xi'_n). \end{aligned} \quad (17)$$

For $\eta = 1$, this density matrix is a projector to the LLL satisfying

$$\Psi_\nu(\{\xi\}) = \int [d\xi'] \mathcal{P}_1^{(N_\nu)}(\{\xi\}, \{\xi'\}) \Psi_\nu(\{\xi'\}). \quad (18)$$

Equation (16) follows by applying PH conjugation to $\Psi_{1-\nu}$ and expressing the resulting convolution integral for $\Psi_{1-(1-\nu)}$ in terms of (18). We make extensive use of the following formula relating the two-particle reduced density matrices between PH-dual states [38]

$$\begin{aligned} \mathcal{P}_{1-\nu}^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2) \\ = \mathcal{P}_1^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2) + \mathcal{P}_\nu^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2) \\ + \frac{1}{2} \mathcal{P}_1^{(1)}(\xi_1; \xi_2) \mathcal{P}_\nu^{(1)}(\xi_2; \xi_1) + \frac{1}{2} \mathcal{P}_1^{(1)}(\xi_2; \xi_1) \mathcal{P}_\nu^{(1)}(\xi_1; \xi_2) \\ - \frac{1}{2} \langle \rho_1(\xi_1) \rangle \langle \rho_\nu(\xi_2) \rangle - \frac{1}{2} \langle \rho_1(\xi_2) \rangle \langle \rho_\nu(\xi_1) \rangle. \end{aligned} \quad (19)$$

Integrating over position ξ_2 reduces this to a simple formula relating the electron density (in inhomogeneous background)

$$\langle \rho_\nu \rangle + \langle \rho_{1-\nu} \rangle = \langle \rho_1 \rangle. \quad (20)$$

Equations (19) and (20) reveal the PHD, and are the central results of the present Letter. Next we will discuss the physical consequences of the duality.

Particle-hole duality.—The Hall conductivity and curvature response in the PH-dual state can be found using (20). Taking a derivative with respect to the magnetic field $B(q)$, and applying the Středa formula we obtain an exact relation between the Hall conductivities

$$\sigma_\nu^H(q) + \sigma_{1-\nu}^H(q) = \sigma_1^H(q). \quad (21)$$

Similarly we find

$$\nu \eta_\nu(q) + (1-\nu) \eta_{1-\nu}(q) = \eta_1(q). \quad (22)$$

These are exact relations connecting the linear response functions of PH conjugate pair states, and hold for *all* LLL wave functions connected by (15). They are new results of the present Letter.

Next, we turn to the normalization factor. It follows directly from the definition of the reduced density matrix and the reproducing formula (18), as well as the definition of the generating functional (4) that

$$\frac{\mathcal{Z}_{1-\nu}}{\mathcal{Z}_\nu} = \mathcal{Z}_1 \Rightarrow \log \mathcal{Z}_{1-\nu} - \log \mathcal{Z}_\nu = \log \mathcal{Z}_1, \quad (23)$$

where we have dropped the argument of \mathcal{Z}_ν for brevity. Equation (23) is an *exact* relation between the generating functionals for a pair of PH-dual states. Equation (23) clearly illustrates the duality, and is a new result of the present Letter.

Assuming now that Ψ_ν has the form (1), variation of $\log \mathcal{Z}_{1-\nu}$ over $W(\zeta, \bar{\zeta})$ is given by

$$\frac{\delta}{\delta W(\zeta)} \log \mathcal{Z}_{1-\nu} = \langle \rho_1 \rangle + \langle \rho_\nu \rangle = \langle \rho_{1-\nu} \rangle + 2 \langle \rho_\nu \rangle. \quad (24)$$

We emphasize that since (24) does not have the same form as (5), the wave function $\Psi_{1-\nu}$ does not have the form (1). More precisely, we see that $f_{1-\nu}$ has to depend on W . In other words, the dual states couple differently to the inhomogeneous magnetic field. It appears that the condition for f_ν to be independent of W has to do with the chirality of a state [39]. For instance, all conformal block trial states share this property. This complication indicates that identity (23) is not sufficient to extract all of the observables in a PH-dual state in terms of the observables in the original state, because the relationship between the observables and variations of $\log \mathcal{Z}_{1-\nu}$ is more complicated for $\Psi_{1-\nu}$ states.

Now we will derive an analog of (8) for the dual states. By definition, the two-point density correlation function is related to the two-particle density matrix via

$$\langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle = \langle \rho_\nu(\zeta) \rangle \delta(\zeta - \zeta') + 2 \mathcal{P}_\nu^{(2)}(\zeta, \zeta'; \zeta, \zeta').$$

In the translation-invariant limit, the one-particle density matrix for the ν state is known to be [37]

$$\mathcal{P}_\nu^{(1)}(\zeta; \zeta') = \bar{\rho}_\nu e^{\zeta \bar{\zeta}' / 2l^2 - |\zeta'|^2 / 2e^2}. \quad (25)$$

In this limit, Eq. (19) then becomes

$$\begin{aligned} & \langle \rho_{1-\nu}(\zeta) \rho_{1-\nu}(\zeta') \rangle_c \\ &= \langle \rho_\nu(\zeta) \rho_\nu(\zeta') \rangle_c + \frac{\bar{\rho}_1 - 2\bar{\rho}_\nu}{\bar{\rho}_1} \langle \rho_1(\zeta) \rho_1(\zeta') \rangle_c. \end{aligned} \quad (26)$$

Taking the Fourier transform, we find a beautiful exact relation between the projected static structure factors $\bar{s}_\nu = s_\nu - s_1$ for a pair of PH-dual states [40]

$$\bar{\rho}_\nu \bar{s}_\nu(q) = \bar{\rho}_{1-\nu} \bar{s}_{1-\nu}(q). \quad (27)$$

This relation is another novel result of the Letter. Equation (27) also follows from the relation between the pair-correlation functions [42] for the dual states [43]. We stress here that (27) requires only holomorphicity of the wave function.

Now we are in position to relate the Hall conductivity to the SSF of the PH-dual state

$$\sigma_{1-\nu}^H(q) = \sigma_{1-\nu}^H(0) \frac{2}{q^2} (s_1(q) - \bar{s}_{1-\nu}(q)). \quad (28)$$

The simplest way to obtain (28) is to use (21) and (27), and assume (8).

Next we will derive an analog of (12) for PH-dual states. Using (21), (22), and (14) we find

$$\bar{s}_{1-\nu}(q) = s_1(q) - \frac{q^2/2}{1 + q^2/2} (q^2 \eta_{1-\nu}(q) + 1). \quad (29)$$

Excluding $s_{1-\nu}(q)$ from (28) and (29) we come to a surprising conclusion—the relation between $\sigma_{1-\nu}^H(q)$ and $\eta_{1-\nu}(q)$ is precisely the same as before the PH transformation (14), up to replacing ν by $1 - \nu$. Equation (14) is thus invariant under PH transformation.

Berry curvature.—Next we turn to the dependence of the PH-dual states on parameters such as adiabatically varying fluxes of magnetic field or the modular parameter of a torus τ . Denote any of these parameters in complex coordinates as x and \bar{x} . Berry curvature can be computed under the assumption that the state Ψ_ν is holomorphic in the coordinates on the parameter space, except for the real-analytic normalization factor. The *normalized* states have the form [16,18,44]

$$\psi_\nu(\{\xi, \bar{\xi}\}; x, \bar{x}) = \frac{1}{\sqrt{\mathcal{Z}_\nu[x, \bar{x}]}} \Psi_\nu(\{\xi, \bar{\xi}\}; x). \quad (30)$$

Then the holomorphic component of the Berry connection is determined entirely by the normalization factor

$$A_x \equiv i \langle \psi_\nu | \partial_x | \psi_\nu \rangle = \frac{i}{2} \partial_x \log \mathcal{Z}_\nu, \quad (31)$$

which follows by using the identity $\partial_x \langle \psi_\nu | \psi_\nu \rangle = 0$ to trade derivatives of Ψ_ν for derivatives of \mathcal{Z}_ν . Thus, for such

holomorphic states the Berry curvature is a Kähler form with the Kähler potential $\mathcal{U}_\nu = \log \mathcal{Z}_\nu$, and is given by

$$\Omega_\nu = \frac{i}{2} (\partial_x \partial_{\bar{x}} \mathcal{U}_\nu) dx \wedge d\bar{x}. \quad (32)$$

This structure is nearly preserved for the PH-dual state. A straightforward calculation shows that the Kähler potential is $\mathcal{U}_{1-\nu} = \log(\mathcal{Z}_1/\mathcal{Z}_\nu)$, which is *not* the logarithm of the normalization as before. Thus, in contrast to the formula (23), the Berry curvature obeys

$$\Omega_\nu + \Omega_{1-\nu} = \Omega_1. \quad (33)$$

PHD and Chern-Simons terms.—The first few terms in the long wave expansion of $\sigma_\nu^H(q)$, $s_\nu(q)$, and $\eta_\nu(q)$ are determined by the topological quantum numbers, which appear as the coefficients of the Chern-Simons terms in the effective action [11,19,28].

The Chern-Simons part of the effective action is given by [14,16]

$$\mathcal{W}_{\text{CS}}^\nu = \frac{\nu}{4\pi} \int (A + \bar{s}\omega) d(A + \bar{s}\omega) - \frac{\tilde{c}}{48\pi} \int \omega d\omega, \quad (34)$$

where $\tilde{c} = c_- - 12\nu \text{var}(s)$ and other coefficients are the topological quantum numbers discussed in the Introduction. We have also introduced ω_μ —a spatial part of the spin connection satisfying $\partial_1 \omega_2 - \partial_2 \omega_1 = \frac{1}{2} \sqrt{g} R$. This effective action encodes the linear response functions. Notably, the Hall conductance, shift, and the Hall viscosity, averaged over the sample, are given by

$$\sigma_\nu^H = \frac{\nu}{2\pi}, \quad S = \frac{\nu^{-1}N - N\phi}{\chi/2}, \quad \eta_\nu^H = \frac{\bar{s}}{2} \bar{\rho}_\nu + \frac{\tilde{c}}{24A} \chi, \quad (35)$$

where A and χ are, respectively, the area and the Euler characteristic of the sample, N is the number of electrons, and N_ϕ is the total magnetic flux in units of the flux quantum. When a FQH state is constructed as a single conformal block in a conformal field theory [16,45] $\tilde{c} = c_-$. However, in general (and notably for Jain states) $\text{var}(s)$ does not vanish [14].

The action of PH transformation on the Chern-Simons part of the effective action is

$$\mathcal{W}_{\text{CS}}^\nu + \mathcal{W}_{\text{CS}}^{1-\nu} = \mathcal{W}_{\text{CS}}^1. \quad (36)$$

This can be seen as a consequence of the formula for the Berry curvature (33) following the arguments of [16]. In addition to $\nu^{\text{PH}} = 1 - \nu$ it implies

$$\mathcal{S}^{\text{PH}} = \frac{1 - \nu S}{1 - \nu}, \quad \text{var}(s)^{\text{PH}} = \frac{\nu}{\nu - 1} \left(\frac{(1 - S)^2}{4(1 - \nu)} + \text{var}(s) \right). \quad (37)$$

PHD also transforms the chiral central charge according to $c_-^{\text{PH}} = 1 - c_-$, and $\tilde{c}^{\text{PH}} = c_-^{\text{PH}} - 12\nu^{\text{PH}} \text{var}(s)^{\text{PH}}$. Curiously,

if the initial state had $\text{var}(s) = 0$, then $\text{var}(s)^{\text{PH}} \neq 0$, unless $S = 1$.

As another example we provide an explicit formula for the first two terms in the long-wave expansion of the projected SSF of a PH-dual state

$$\bar{s}_{1-\nu}(q) = \frac{\nu(S-1)}{8(1-\nu)}q^4 + \frac{(-6b+5\nu-3\nu S)}{48(1-\nu)}q^6 + \dots, \quad (38)$$

where b is given by (10) and all of the topological quantum numbers are known for a large variety of states [14]. Equation (38) follows from using (13) in (27), and is another new result of the present Letter. For example, it can be readily used to determine the projected SSF for the $\nu = (q-1)/q$ states, related to the Laughlin states by the PHD. Note that all of the quantum numbers are taken from the state at filling ν . We are not aware of this type of general result in the literature.

Finally, we comment on the topological quantum numbers for a *self-dual* state which satisfies $\Omega_\nu = \Omega_{1-\nu}$. This state was discussed by Son [3] and is known as the PH Pfaffian. We find that it must have $\nu = 1/2$, $S = 1$, $c_- = 1/2$ and $\text{var}(s) = 0$.

Jain states.—We apply our relations to the PH duals of $\nu = N/(2N+1)$ Jain states with the relevant topological quantum numbers given by

$$S = N + 2, \quad c_- = N, \quad \nu \text{var}(s) = \frac{N(N^2 - 1)}{12}. \quad (39)$$

Then the projected SSF of the PH-dual state is given by

$$\bar{s}_{\frac{N+1}{2N+1}}(q) = \frac{N}{8}q^4 + \frac{N^4 + 2N^3 - 2N^2 - 2N}{48(N+1)}q^6 + \dots \quad (40)$$

To the best of our knowledge (40) is a new result.

Hall conductivity of Jain's state at the filling factor $\nu = N/(2N+1)$ and its PH-dual state at the filling factor $(1-\nu) = [(N+1)/(2N+1)]$ can be calculated exactly, in the large N limit using the Dirac composite fermion theory and the results of [46]. In the regime $z = q(2N+1) \sim 1$ [47] the result is

$$\sigma_\nu^H(q) = \frac{((4N+2)^2 - z^2)(8N + \frac{2zJ_2(z)}{J_1(z)})}{64\pi(2N+1)^3}, \quad (41)$$

$$\sigma_{1-\nu}^H(q) = \frac{((4N+2)^2 - z^2)(8N + 8 - \frac{2zJ_2(z)}{J_1(z)})}{64\pi(2N+1)^3}, \quad (42)$$

where $J_\alpha(z)$ is the Bessel function. The correction to this is order $O(N^{-4})$. We have also assumed absence of long-range interactions.

The projected SSF can also be derived exactly using Dirac composite fermion theory [3]

$$\bar{s}_\nu(q) = \frac{z^3((4N+2)^2 - z^2)J_2(z)}{32N(2N+1)^4J_1(z)}, \quad (43)$$

$$\bar{s}_{1-\nu}(q) = \frac{z^3((4N+2)^2 - z^2)J_2(z)}{32(N+1)(2N+1)^4J_1(z)}. \quad (44)$$

With these expressions at hand we can check that (21) holds up to order N^{-2} . It also follows from (41)–(44) that (8) and (28) hold in the large N limit at leading and subleading orders in N . We emphasize that these are quite nontrivial checks that probe the relations we derived in *all* orders in the momentum expansion.

Conclusion.—We have presented arguments for the particle-hole duality in the lowest Landau level. This duality implies several exact, nonperturbative relations between the observables in the pair of PH-dual states such as static structure factor, Hall conductivity, and response of the electron density to curvature. Our results for the coefficients of the small momentum expansion of these response functions likely do not hold for PH-dual states which are both nonchiral (as defined in [48], see also [8])—a notable example of such a state is the PH Pfaffian. However, Eq. (27) should be applicable to such states as well. We leave the investigation of general nonchiral states (and in particular the PH Pfaffian) to future work.

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