Nuclear Physics Around the Unitarity Limit

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(Received 18 July 2016; revised manuscript received 10 February 2017; published 15 May 2017)

We argue that many features of the structure of nuclei emerge from a strictly perturbative expansion around the unitarity limit, where the two-nucleon S waves have bound states at zero energy. In this limit, the gross features of states in the nuclear chart are correlated to only one dimensionful parameter, which is related to the breaking of scale invariance to a discrete scaling symmetry and set by the triton binding energy. Observables are moved to their physical values by small *perturbative* corrections, much like in descriptions of the fine structure of atomic spectra. We provide evidence in favor of the conjecture that light, and possibly heavier, nuclei are bound weakly enough to be insensitive to the details of the interactions but strongly enough to be insensitive to the exact size of the two-nucleon system.

DOI: 10.1103/PhysRevLett.118.202501

For the purposes of nuclear physics, QCD, the theory of strong interactions, has essentially two independent parameters, namely the up and down quark masses. Their average controls the pion mass and consequently, the range of the nuclear force $R \sim M_\pi^{-1} \simeq 1.4$ fm. Their difference, plus electromagnetism, generates small differences in masses and interactions between neutrons and protons. At the physical point, the two-nucleon (NN) scattering length in the $^3S_1$ channel is $a_t \simeq 5.4$ fm, with the deuteron as a shallow bound state ($B_D \simeq 2.224$ MeV); in the $^1S_0$ channel, $a_s \simeq -23.7$ fm, and a shallow virtual bound state exists at $B_{NN^*} \simeq 0.068$ MeV. With relatively small changes in quark masses, these states become, respectively, unbound and bound [1–4]. In the physics of cold atoms near Feshbach resonances, external magnetic fields play a role similar to the quark masses and allow the scattering length to be tuned arbitrarily [5].

Approximate correlations $B_{D,NN^*} \approx 1/(M_N a_{t,s}^2)$, with $M_N \simeq 940$ MeV the nucleon mass, hold because the size of all these scales is unnatural compared to the typical interaction range R. The NN system therefore appears close to the unitarity (or unitary) limit, where both states cross zero energy, the scattering lengths become infinite ($1/a_{t,s} = 0$), and cross sections saturate the unitarity bound. It has indeed been suggested that this happens not far from the physical point [6]. While this presumed proximity has been discussed qualitatively for a long time, it has traditionally not played any special role in constructing nuclear forces, and it is neither assessed nor exploited in order to simplify the description of nuclei. As an exception, Refs. [7,8] use potential models to map out correlations between observables in three- and four-nucleon systems as the limit is approached at fixed a_t/a_s.

Here, we argue that the typical particle binding momentum Q_A of the A-nucleon system satisfies $1/a_{s,t} < Q_A < 1/R$ so that a combined expansion in $Q_A R$ and $1/(Q_A a_{s,t})$ converges quickly and *quantitatively* reproduces the physical systems. With this, the gross features of states in the nuclear chart are determined by a very simple leading-order interaction (governed by a single parameter), whereas, much like the fine structure of atomic spectra, observables are moved to their physical values by small *perturbative* corrections. Our conjecture places nuclei in a sweet spot: bound weakly enough to be insensitive to the details of the interaction but dense enough to be insensitive to the exact values of the large two-particle scattering lengths. One might surmise that due to the absence of scales, a theory at the unitarity limit allows only for trivial observables, like bound states with zero or infinite energies. However, the nontrivial renormalization of the three-body system introduces, instead, exactly one new dimensionful parameter, which sets the scale for all few-body observables. Indeed, the energies of bosonic clusters near unitarity are determined in terms of the trimer energy [9,10].

In the following, we provide explicit evidence that our conjecture holds for the binding energies of three and four nucleons. Since the NN binding energies are small, their dynamics is dominated by large distances and small momenta, the regime of the effective range expansion (ERE) [11]. Its consequences are captured by an effective field theory (EFT) which, apart from long-range electromagnetic interactions mediated by photons (A_μ), contains only contact interactions between nonrelativistic nucleon isospin doublets $N = (pn)^T$ of proton and neutron fields. Following the notation of Ref. [12], its Lagrange density is

$$\mathcal{L} = N^\dagger \left(i\mathcal{D}_0 + \frac{\boldsymbol{\mathcal{D}}^2}{2M_N} \right) N + \sum_i C_{0,i} (N^T P_i N)^\dagger (N^T P_i N) + D_0 (N^\dagger N)^3 + \cdots, \tag{1}$$

where $\mathcal{D}_\mu = \partial_\mu + ieA_\mu(1+\tau_3)/2$, e is the proton charge, τ_a a Pauli matrix in isospin space, and P_i projectors onto the NN S waves. $C_{0,i}$ and D_0 are "low-energy constants" (LECs), determined from QCD or experiment. This "Pionless EFT" reproduces the ERE in the NN sector [13–17] but extends it to an arbitrary number of particles and interactions with external fields. The two-body interactions with LECs $C_{0,i}$ are related to $a_{s,t}$, while higher-derivative interactions are associated with the effective ranges and higher ERE parameters, as well as higher partial waves. The organizational principle ("power counting") attributes the $C_{0,i}$ to nonperturbative leading order (LO) and higher-derivative interactions to subleading orders. These are added perturbatively and include the effects of the interaction range R in a systematic expansion in $QR \ll 1$, where Q is a typical low-momentum scale.

Stability of light nuclei results from an additional LO interaction, a single nonderivative three-nucleon ($3N$) contact interaction [18–20], with LEC D_0. Derivative corrections to this $3N$ interaction start at next-to-next-to-leading order (N^{2}LO) [19,21–24]. Little is known about the orders at which higher-body interactions appear, except that they are not LO [10,25–34]. Based on a zero-range model, Refs. [35–37] report some sensitivity of four-body energies to a four-body scale, but these results do not contradict the absence of a four-body interaction at LO in Pionless EFT. The absence of an essential four-body parameter has also been verified in the context of potential models with a short range [38–41] and of renormalization-group analyses [42–44]. As a consequence, the $3N$ LO strength parameter $\Lambda_\star$, together with the LO two-body interactions, determines the spectrum and scattering for systems with more particles [10,25–34,45,46]. This single relevant $3N$ parameter generates correlations among few-body observables such as the Phillips [47] and Tjon [48] lines.

The standard pionless formulation with finite scattering lengths as a LO input explicitly breaks two important symmetries: first, the $\mathrm{SU}(4)_W$ Wigner symmetry of combined spin and isospin transformations [49] is broken in the two-body sector for $a_t \neq a_s$ [50], while it is obeyed by the $3N$ interaction [20]. Second, discrete scale invariance leads to the log-periodic shape of the running coupling D_0 [18–20] and to an infinite geometric tower of Efimov states in the three-body system [51], both determined by the $3N$ LO strength parameter $\Lambda_\star$ [see Eq. (5) below]. It too is broken for $a_{s,t} \neq \infty$. (Note that in contrast to Refs. [5,51,52], scattering lengths are not rescaled with the discrete scaling factor here.)

The unitarity limit manifestly respects both symmetries and has $\Lambda_\star$ left as a single parameter. In our calculation, this is fixed at LO to reproduce the physical triton (degenerate with ^{3}He at this order) as one of the Efimov states. In fact, the $3N$ and $4N$ systems in the unitarity limit decouple into a symmetric piece, identical to a formulation of three- and four-boson systems and an antisymmetric piece. At unitarity, a three-boson Efimov state with binding energy B_3 is associated with two four-boson states [28]—one relatively deep at $B_4/B_3 \simeq 4.611$ and one barely below the particle-trimer threshold, $B_{4^*}/B_3 \simeq 1.002$ [40]. For the α particle, the ground state is at $B_\alpha/B_H \simeq 3.66$ and the excited state at $B_{\alpha^*}/B_H \simeq 1.05$, where $B_H \simeq 7.72$ MeV is the ^{3}He binding energy. In addition, models (see, e.g., Ref. [53]) indicate the existence of a virtual $3N$ state at $B_{T^*} \lesssim 0.5$ MeV for physical scattering lengths, which becomes the second Efimov state as B_D is decreased. Thus, the $3N$ and $4N$ spectra are consistent with mildly broken discrete scale invariance, suggesting a perturbative treatment, described below.

For the calculation, the LO two-body potentials derived from Eq. (1) are written as

$$V_2^{(0)} = \sum_i C_{0,i}^{(0)} |i\rangle |g\rangle \langle g| \langle i|, \tag{2}$$

where $|i\rangle$ collects the spin-isospin structure and $|g\rangle$ implements a separable regularization. With $\mathbf{p}$ the momentum corresponding to the kinetic energy E in the NN center-of-mass system, $g(p) \equiv \langle \mathbf{p}|g\rangle$ satisfies $g(0) = 1$ and $g(p \gg \Lambda) \ll 1$ for arbitrary cutoff Λ. The results shown below have been obtained with two different implementations of the theory. For the ^{3}He calculation, we follow Ref. [12], which uses a sharp cutoff (step function) regulator $g_s(p) = \theta(\Lambda - p)$ and includes the two-body interactions through dibaryon auxiliary fields, in lieu of Eq. (2). Our new Faddeev (-Yakubovsky) calculations use a convenient Gaussian regulator instead, $g_G(p) = \exp(-p^2/\Lambda^2)$. Since only the LO calculation mandates a nonperturbative treatment, we use (distorted-wave) perturbation theory for higher orders; i.e., NLO results depend only linearly on NLO contributions. These are (i) terms as in Eq. (2) accounting for LEC shifts $C_{0,i}^{(0)} \to C_{0,i}^{(1)}$, corresponding to the expansion $C_{0,i} = C_{0,i}^{(0)} + C_{0,i}^{(1)} + \cdots$ and shifting to the physical values of $a_{s,t}$, (ii) one Coulomb interaction $\sim e^2 M_N/(4\pi Q)$ in the pp channel, (iii) one isospin-breaking contact interaction in the pp channel as in Eq. (2), with $C_{0,s}^{(0)} \to \Delta C_{0,s,pp}^{(1)}$, required for proper renormalization of Coulomb effects, and (iv) one range correction per NN S wave.

In order to treat Coulomb effects in the $3N$ sector perturbatively, Ref. [12] expanded the $^1S_0$ channel around the unitary limit. Here, we also expand the $^3S_1$ channel in $1/(Q_3 a_t)$, which is a significantly more radical simplification, given that a_t is not nearly as large as a_s. The two-body amplitude is a geometric series that can be resummed analytically for a separable regulator. We remove the arbitrary Λ dependence from observables by demanding that the two inverse scattering lengths vanish at LO and enter linearly at NLO. Renormalization is achieved if

$$C_{0,i}^{(0)} = \frac{-2\pi^2}{M_N\Lambda}\theta^{-1}, \qquad C_{0,i}^{(1)} = \frac{M_N}{4\pi a_i} C_{0,i}^{(0)2}, \tag{3}$$

where $\theta = \int dq g^2(q)/\Lambda$ is a regulator-dependent pure number. The LO amplitude takes then the scale invariant form $T_i^{(0)}(E) \propto 1/\sqrt{-M_N E - i\varepsilon}$, with NLO corrections proportional to $C_{0,i}^{(1)}$. The deuteron binding energy vanishes up to NLO, but $(B_D)^{\mathrm{N^2LO}} = 1/(M_N a_t^2) \simeq 1.41$ MeV coincides with the standard zero-range value. For more details, see Ref. [54].

A $3N$ potential is needed at LO for renormalizability, i.e., to avoid the Thomas collapse [55] and ensure that three-body observables have a well-defined limit for $\Lambda \gg 1/R$ [18–20]. We choose a separable form

$$V_3^{(0)} = D_0^{(0)} |^3\mathrm{H}\rangle |\xi\rangle\langle\xi| \langle^3\mathrm{H}|, \tag{4}$$

where $|^3\mathrm{H}\rangle$ denotes the projector onto the $J = S = T = 1/2$ $3N$ state and $|\xi\rangle$ the regulator, either sharp or $\langle u_1 u_2|\xi\rangle = g_G(\sqrt{u_1^2 + 3u_2^2/4})$ for Jacobi momenta $u_{1,2}$. We take the triton binding energy as the one observable needed to fix $D_0^{(0)}(\Lambda)$.

An *a priori* estimate of the typical A-body scale equidistributes the total binding energy amongst its constituents, i.e., $Q_A \sim \sqrt{2M_N B_A/A}$. For three nucleons, $Q_3 \approx 70$ MeV appears indeed in the sweet spot, namely larger than $1/a_{s,t} \lesssim 45$ MeV but smaller than the breakdown scale of Pionless EFT (expected to be about 140 MeV). We thus compare the running of $D_0^{(0)}$ at full-unitarity LO with the result of standard pionless LO (scattering lengths at their physical values) and of $^1S_0$ unitarity [12]. Taking the same sharp-momentum regulator, all three cases agree well with the log-periodic form [18–20]

$$D^{(0)}(\Lambda) \propto \frac{1}{\Lambda^4} \frac{\sin(s_0 \log(\Lambda/\Lambda_*) - \arctan s_0^{-1})}{\sin(s_0 \log(\Lambda/\Lambda_*) + \arctan s_0^{-1})}, \tag{5}$$

where $s_0 \simeq 1.00624$. The proportionality factor is scheme and regulator dependent. We find $\Lambda_* = 175, 168, 234$ MeV for standard Pionless EFT, $^1S_0$ unitarity, and full unitarity, respectively. The changes at $^1S_0$ and full unitarity go in opposite directions since the $^1S_0$ interaction is more attractive at unitarity than for the physical scattering length as the LO input, while the $^3S_1$ interaction is less attractive.

At NLO, the $3N$ interaction has the same form as in Eq. (4), with $D_0^{(0)} \to D_0^{(1)}$ chosen to keep the triton energy unchanged. As a nontrivial check, we repeat the ^{3}He calculation of Ref. [12] with a full-unitarity LO and add finite $1/a_{s,t}$ plus one-photon exchange and its counterterm as NLO corrections. Figure 1 shows excellent agreement, with only small differences to $^1S_0$ unitarity. Convergence with the cutoff is evident. We predict a triton-helion splitting $(B_T - B_H)^{\mathrm{NLO}} \simeq (0.92 \pm 0.18)$ MeV, compared to the experimental value $(B_T - B_H)^{\mathrm{exp}} \simeq 0.764$ MeV. Our 20% error estimate follows Ref. [12] and is larger than the additional $\mathcal{O}(1/(Q_3 a_t)^2)$ from the new expansion.

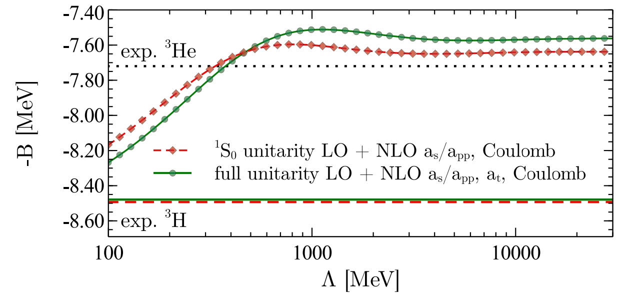

FIG. 1. Three-nucleon binding energies at NLO as a function of the $3N$ sharp cutoff. (Red) dashed curves: results of Ref. [12], keeping the physical a_t at LO. (Green) solid curves: effect of taking both S waves to the unitarity limit at LO and then including the physical values perturbatively at NLO, along with one-photon exchange and its counterterm. Horizontal lines: experimental values. Top (bottom) lines: ^{3}He (^{3}H, fitted).

At full unitarity, the LO spectrum consists of a series of states spaced by a factor of $\exp(2\pi/s_0) \approx 515$ [51]. An infinite number of states shallower than the triton/helion accumulates at zero energy. These lie outside the range of applicability of our expansion since their typical momenta are not large compared to $1/|a_{s,t}|$. Nevertheless, looking at their perturbative shifts, we find that at NLO they remain an order of magnitude shallower than the N^{2}LO deuteron, and generally, we expect them to disappear above threshold at N^{2}LO. In addition, with increasing cutoff, deeper $3N$ states enter the spectrum with binding momenta well above $1/R$. These are well outside the range of validity of the EFT and thus not a fundamental issue, but they complicate the numerical solution of the $4N$ problem. For now, we restrict our $4N$ calculations to a Λ range in which these are absent. For simplicity, we neglect electromagnetic and range corrections at NLO, focusing on the $1/(Q_4 a_t)$ expansion.

Our $4N$ calculation follows Refs. [25–27] (based, in turn, on Ref. [56]), with an independently developed numerical implementation. We include a sufficient number of angular components to ensure numerical convergence; see a subsequent publication for details [57].

Figure 2 shows results for the α-particle binding energy $B_\alpha(\Lambda)$. They are well fitted with a quadratic polynomial in $1/\Lambda$ for large Λ, which we therefore use to extrapolate $\Lambda \to \infty$. For standard Pionless EFT at LO, they are consistent with Refs. [25–27]. Implementing the unitarity limit at LO leads to about 10 MeV more binding, as expected from a more attractive $3N$ interaction. We find a bound excited state just below the proton-triton breakup threshold, in contrast to 0.3 MeV above, as indicated by experiment. The LO results $(B_\alpha/B_T)^{\mathrm{LO}} = 4.66$ for the ground and $(B_{\alpha^*}/B_T)^{\mathrm{LO}} = 1.002$ for the excited state of ^{4}He agree with four-boson unitarity [27,28,40].

Remarkably, the first $1/(Q_4 a_t)$ correction brings us very close to the standard Pionless EFT result. That is perhaps accidental due to the highly symmetrical nature of the α

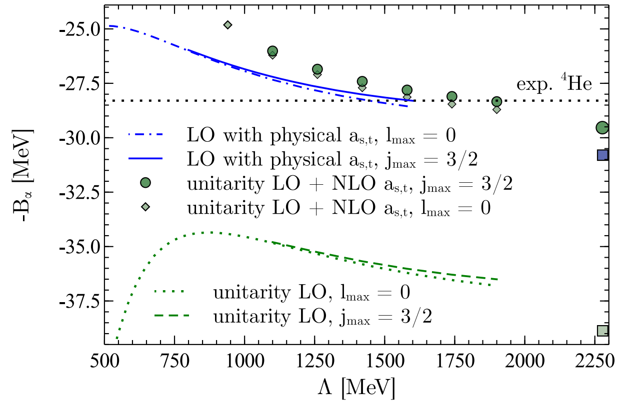

FIG. 2. ^{4}He binding energy as function of the Gaussian cutoff. (Blue) solid and (green) dashed lines: standard Pionless EFT and full unitarity at LO, respectively, with partial waves up to $j_{\max} = 3/2$. (Blue) dot-dashed and (green) dotted line: same for S waves only ($l_{\max} = 0$). (Green) diamonds ($l_{\max} = 0$) and circles ($j_{\max} = 3/2$): first-order corrections in $1/a_{s,t}$ are added. Results for $j_{\max} = 2$ are almost identical to $j_{\max} = 3/2$ and not shown. Large symbols on right edge: $\Lambda \to \infty$ extrapolation (see text).

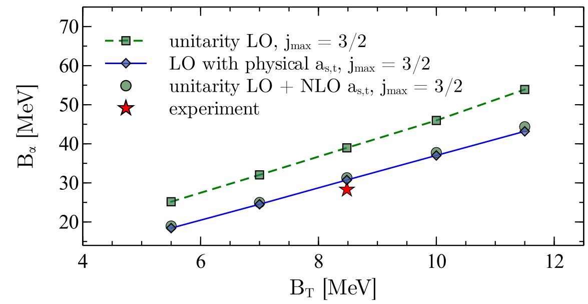

FIG. 3. Tjon line: correlation between the ^{4}He and ^{3}H binding energies. (Blue) solid curve: standard pionless LO result; (green) dashed upper curve: unitarity limit at LO. Additional points nearly on top of the blue curve: inverse scattering lengths as first-order perturbation. Star: experimental point.

particle, the level of agreement in the helion being perhaps more representative. All results in Fig. 2 are uncorrected for electromagnetic and range effects. At present, no calculation of these effects in Pionless EFT exists, except for Ref. [46], where higher orders were, however, not treated perturbatively. With the uncertainty expected to be dominated by range corrections $\mathcal{O}(r_{s,t}/a_{s,t}) \simeq 30\%$, we obtain $(B_\alpha)^{\mathrm{NLO}(r=0)} = 29.5 \pm 8.7$ MeV with zero effective ranges. The ratio $(B_\alpha/B_T)^{\mathrm{NLO}(r=0)} \approx 3.48$ is in good agreement with $(B_\alpha/B_T)^{\exp} = 3.34$.

While slow convergence for the excited-state wave function puts its full NLO calculation beyond our current computational resources, we used a four-boson model tuned such that its LO is similar to the ^{4}He system, namely choosing $B_3 = 8.5$ MeV. Since the four-boson and $4N$ systems are identical for exact $\mathrm{SU}(4)_W$ symmetry, this is an adequate rendering of the more complex physical world. Calculations with $a_2 = 20$ fm and $a_2 = 5$ fm (covering the range of typical nuclear scales) indicate that first-order perturbations in $1/a_2$ indeed push the state at $B_{4^*} \gtrsim 8.5$ MeV above threshold by about 0.2 and 0.5 MeV for $a_2 = 20$ and 5 fm, respectively. That corresponds to $(B_{4^*}/B_3)^{\mathrm{NLO}(r=0)}$ in the range 0.94…0.98, compared to $(B_{\alpha^*}/B_T)^{\exp} = 0.96$. The four-body excited state and the particle-trimer threshold are close, but both are still far from the four-particle threshold; it is reassuring that we can improve the description of the excited state at NLO.

As a final test, Fig. 3 shows the Tjon line, i.e., the correlation between $3N$ and $4N$ binding energies, obtained by varying Λ_*. All results are calculated with $j_{\max} = 3/2$ in the $4N$ system and use the $\Lambda \to \infty$ extrapolation discussed above. The extrapolation uncertainty is negligible compared to the 30% estimated EFT truncation error. The remarkable agreement in Fig. 2 persists off the physical point, providing further evidence of the power of a perturbative expansion around the unitarity limit. The relation between triton and ^{4}He binding energies is still nearly perfectly linear at NLO.

Our results suggest good convergence of the expansion around unitarity, both order by order and to real-world values. While the condition $Q_A R < 1$ is better satisfied for lighter systems, it may provide at least qualitative insight into the binding mechanisms of even the heaviest nuclei ($B_A/A \approx 8$ MeV). Indeed, the rate of convergence in observables provides ample evidence that $Q_3 R$ is much smaller than its *a priori* estimate, see, e.g., Refs. [12,21,23,58–61], and is suggestive that $Q_4 R$ is smaller, too [29,32,46]. There is also circumstantial evidence that this may hold for $A > 4$ [10].

A recent expansion around the $\mathrm{SU}(4)_W$ limit, with the averaged physical values of the ERE parameters at LO, found good convergence for $3N$ binding energies and radii [62]. Together with the fact that $\mathrm{SU}(4)_W$ has some success in heavier nuclei (see, e.g., Refs. [63,64] and references therein), this adds further credibility to our conjecture. In the future, we will investigate our expansion in heavier nuclei, such as the isoscalars ^{16}O [34] and ^{6}Li [45], including observables beyond binding energies, for example, electromagnetic form factors. For nuclear matter, saturation energies and densities are correlated (Coester line) [65] and conjectured to be correlated to the $3N$ binding energy [66]. In the unitarity limit, this can be understood from discrete scale invariance [67].

In summary, we demonstrated that the physics of $A \leq 4$ nucleons is governed to a good first approximation by a single parameter Λ_*, with controlled corrections stemming from deviations from unitarity, the interaction range, and isospin-breaking effects. We conjecture that it also converges for other light nuclei and speculate about its relevance for heavy nuclei and nuclear matter. It may not be a surprise that our results in the unitarity limit are perturbatively close to those where the physical scattering lengths are used at LO. Surprising is, however, how well the expansion appears to converge. Our expansion turns the focus away from details of the two-body system, which has traditionally been taken as a starting point to the structure of higher-body bound states, and shifts it to a three-body interaction that underlies systems around unitarity [5,9,10]. That adds the intriguing possibility that the approach developed here for nucleons may prove successful also in atomic and molecular physics, where finite scattering lengths are currently treated nonperturbatively.

We thank J. Carbonell, R. Furnstahl, A. Gezerlis, and D. R. Phillips for discussions; the Institute for Nuclear Theory at the University of Washington for hospitality during program INT 16-1 Nuclear Physics from Lattice QCD; and the participants of the EMMI RRTF workshop ER15-02: Systematic Treatment of the Coulomb Interaction in Few-Body Systems at GSI as well as of the workshop Lattice Nuclei, Nuclear Physics and QCD—Bridging the Gap at the ECT*. This work is supported in part by the NSF under award PHY–1306250 (SK), by the DOE, NUCLEI SciDAC Collaboration award DE-SC0008533 (SK), by the DOE, Office of Science, Office of Nuclear Physics, under Award Nos. DE-SC0015393 (HWG) and DE-FG02-04ER41338 (UvK), by the BMBF under Contract No. 05P15RDFN1 (HWH), by the DFG through SFB 1245 (HWH), by the Dean's Research Chair Programme of the Columbian College of Arts and Sciences of The George Washington University (HWG), as well as by the ERC Grant No. 307986 STRONGINT (SK). Some results reported here were obtained using computing time made available by the State of Hesse on the Lichtenberg High-Performance Computer at TU Darmstadt.

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