Experimental Demonstration of Uncertainty Relations for the Triple Components of Angular Momentum

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The uncertainty principle is considered to be one of the most striking features in quantum mechanics. In the textbook literature, uncertainty relations usually refer to the preparation uncertainty which imposes a limitation on the spread of measurement outcomes for a pair of noncommuting observables. In this work, we study the preparation uncertainty for the angular momentum, especially for spin-$1/2$. We derive uncertainty relations encompassing the triple components of angular momentum and show that, compared with the relations involving only two components, a triple constant $2/\sqrt{3}$ often arises. Intriguingly, this constant is the same for the position and momentum case. Experimental verification is carried out on a single spin in diamond, and the results confirm the triple constant in a wide range of experimental parameters.

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Introduction.—The uncertainty principle was first proposed by Heisenberg in a thought experiment showing that the measurement of an electron's position disturbs the momentum inevitably [1]. In the ensuing few years, Kennard [2], Weyl [3], Robertson [4], and Schrödinger [5] derived mathematically rigorous relations, such as the famous Heisenberg-Robertson uncertainty relation [4]

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A,B]\rangle|, \qquad (1)$$

with the standard deviation $\Delta\Omega = \sqrt{\langle\Omega^2\rangle - \langle\Omega\rangle^2}$ for the observable $\Omega = A$ or B, the angle brackets $\langle\rangle$ denoting the expectation of an operator with respect to a given state ρ, and $[A,B] = AB - BA$. The inequality imposes a trade-off between the statistical dispersions ΔA and ΔB of the pair of noncommuting observables A and B for the given quantum state ρ. This type of uncertainty, often termed as the preparation uncertainty, deals with the spread of measurement outcomes rather than Heisenberg's original idea, which investigates measurement inaccuracies [6–12]. In this Letter, we discuss only the preparation uncertainty.

The uncertainty principle can be described by various relations [13–31], but most well-known uncertainty relations deal with two observables till now. In contrast to the two-observable relation in (1), which can be derived via the Cauchy-Schwarz inequality, it is difficult to obtain a nontrivial multiobservable uncertainty relation that has a form similar to (1), although attempts to encompass three or more observables have a long history since Robertson's work in 1934 [32–45]. Recently, an uncertainty relation for three pairwise canonical observables p, q, and r satisfying $[p,q] = [q,r] = [r,p] = -i\hbar$ with $r = -p - q$ was derived by Kechrimparis and Weigert [46]. Here p and q are the momentum and position, respectively. From the inequality (1), one immediately obtains $\Delta p\Delta q \geq \hbar/2$, $\Delta q\Delta r \geq \hbar/2$, and $\Delta r\Delta p \geq \hbar/2$. By multiplying these inequalities and taking the square root, one gets $\Delta p\Delta q\Delta r \geq (\hbar/2)^{3/2}$. However, this inequality is not tight. In other words, there is no state satisfying such a lower bound. By introducing the triple constant $\tau = 2/\sqrt{3}$, the tight triple uncertainty relation $\Delta p\Delta q\Delta r \geq (\tau\hbar/2)^{3/2}$ is established.

In contrast to the couple of observables p and q, the latecomer r seems artificial and may not exhibit an explicit physical meaning. Yet the three components of angular momentum form a natural triple [47]. In this Letter, we formulate tight uncertainty relations satisfied by the triple components of angular momentum and show that the triple constant τ also arises. An experimental test is performed on a single spin in diamond.

Uncertainty relations.—In this Letter, we always set $\hbar = 1$. Let S_x, S_y, and S_z be the angular momentum operators satisfying commutation relations $[S_x, S_y] = iS_z$, $[S_z, S_x] = iS_y$, and $[S_y, S_z] = iS_x$. From the inequality (1), one obtains

$$\Delta S_x \Delta S_y \geq \frac{1}{2}|\langle S_z\rangle|, \Delta S_y \Delta S_z \geq \frac{1}{2}|\langle S_x\rangle|, \Delta S_z \Delta S_x \geq \frac{1}{2}|\langle S_y\rangle|. \tag{2}$$

By multiplying the above inequalities and taking the square root, one directly gets a trivial relation $\Delta S_x \Delta S_y \Delta S_z \geq |\frac{1}{8}\langle S_x\rangle\langle S_y\rangle\langle S_z\rangle|^{1/2}$, where the equality holds only when both sides are zero. We tighten the lower bound for spin-1/2 by introducing the triple constant τ (see a sketch of the proof in Supplemental Material [48]), i.e.,

$$\Delta S_x \Delta S_y \Delta S_z \geq \left|\frac{\tau^3}{8}\langle S_x\rangle\langle S_y\rangle\langle S_z\rangle\right|^{1/2}. \tag{3}$$

The equality in (3) holds when $|r_x| = |r_y| = |r_z| = 1/\sqrt{3}$ or $(|r_x|-1)(|r_y|-1)(|r_z|-1) = 0$. Here r_x, r_y, and r_z are the components of the Bloch vector $\boldsymbol{r}$ of a qubit state with the density matrix $\rho = (\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})/2$.

Besides multiplicative form uncertainty relations, one may also tighten additive form relations. The inequalities in (2) entail

$$\begin{aligned}(\Delta S_x)^2 + (\Delta S_y)^2 &\geq |\langle S_z\rangle|,\\ (\Delta S_y)^2 + (\Delta S_z)^2 &\geq |\langle S_x\rangle|,\\ (\Delta S_z)^2 + (\Delta S_x)^2 &\geq |\langle S_y\rangle|.\end{aligned} \tag{4}$$

From the above inequalities, one immediately gets $(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq \frac{1}{2}(|\langle S_x\rangle| + |\langle S_y\rangle| + |\langle S_z\rangle|)$, which is again not tight. We also tighten the lower bound by introducing the triple constant τ (see a sketch of the proof in Supplemental Material [48]), i.e.,

$$(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq \frac{\tau}{2}(|\langle S_x\rangle| + |\langle S_y\rangle| + |\langle S_z\rangle|). \tag{5}$$

The equality in (5) is attained if and only if $|r_x| = |r_y| = |r_z| = 1/\sqrt{3}$ for spin-1/2.

Interestingly, the uncertainty relations (2)–(5) are analogous to the geometric relations of an equilateral triangle, as depicted in Fig. 1 (see proofs of these geometric relations in Supplemental Material [48]). We leave the experimental demonstrations of the uncertainty relations (3) and (5) to the next section.

The uncertainty relations (3) and (5) have state-dependent lower bounds. It is similar for uncertainty relations with state-independent lower bounds. The pairwise inequalities for spin-1/2 [8], namely, $(\Delta S_x)^2 + (\Delta S_y)^2 \geq 1/4$, $(\Delta S_y)^2 + (\Delta S_z)^2 \geq 1/4$, and $(\Delta S_z)^2 + (\Delta S_x)^2 \geq 1/4$, immediately yield the inequality $(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq 3/8$, which is not tight. A tight lower bound needs an additional factor τ^2 [43,50], i.e.,

$$(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq \frac{1}{2} = \frac{3}{8}\tau^2. \tag{6}$$

The equality is attained if and only if $|\boldsymbol{r}| = 1$; i.e., the qubit is in a pure state. The relation (6) is also supported by our experiment.

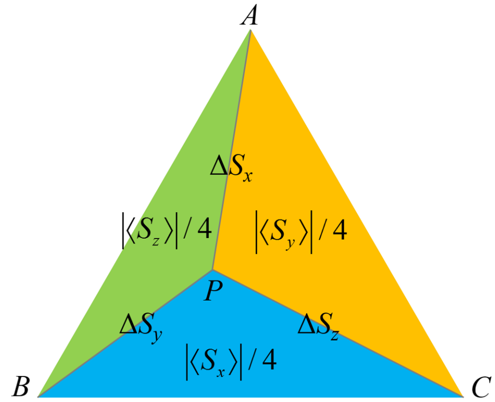

FIG. 1. Geometric analog of uncertainty relations (2)–(5). The equilateral triangle has vertices A, B, C and a point P inside. The lengths of the line segments PA, PB, and PC are denoted by $|PA|$, $|PB|$, and $|PC|$, respectively. The areas of the triangles PAB, PBC, and PCA are denoted by $|\triangle PAB|$, $|\triangle PBC|$, and $|\triangle PCA|$, respectively. These geometric quantities have the same relations as the inequalities (2)–(5) under the correspondence $|PA| \leftrightarrow \Delta S_x$, $|PB| \leftrightarrow \Delta S_y$, $|PC| \leftrightarrow \Delta S_z$ and $|\triangle PAB| \leftrightarrow |\langle S_z\rangle|/4$, $|\triangle PBC| \leftrightarrow |\langle S_x\rangle|/4$, $|\triangle PCA| \leftrightarrow |\langle S_y\rangle|/4$.

Here it should be noted that the inequality (5) is, in fact, valid for any spin quantum number. The relation (6) turns into

$$(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq s \tag{7}$$

for the spin quantum number s [50], and the factor τ^2 does not hold for $s \geq 1$ (see the explanation in Supplemental Material [48]). It should also be noted that, although the state-dependent lower bound in the inequality (3) vanishes in some cases, this uncertainty relation is not covered by the prominent relation (7) and is valuable in its own right.

Experimental demonstration.—To verify the uncertainty relations (3) and (5), we carry out the experiment on a negatively charged nitrogen-vacancy (NV) center in diamond. The single spins in NV centers are convenient to initialize and read out, have long coherence times, and can be manipulated with high precision. These advantages enable NV centers to be widely applied in nanoscale sensing, quantum information, and fundamental physics [51–54].

The diamond we use is a bulk sample with the ^{13}C nuclide at the natural abundance of about 1.1% and the nitrogen impurity less than 5 ppb. The NV center is composed of one substitutional nitrogen atom and an adjacent vacancy as shown in Fig. 2(a). The electronic ground state $^3A_2$ is a triplet state and has a zero-field splitting of about 2.87 GHz. With a static magnetic field of around 510 G applied along the NV axis, both the electron spin and the host nitrogen nuclear spin are polarized by optical pumping [55,56]. The two levels $|m_s = 0\rangle$ and $|m_s = -1\rangle$ act as a spin-1/2 system or qubit which is manipulated by resonant microwave (MW) pulses. The spin state can be read out by optical excitation and red fluorescence detection. To enhance the fluorescence

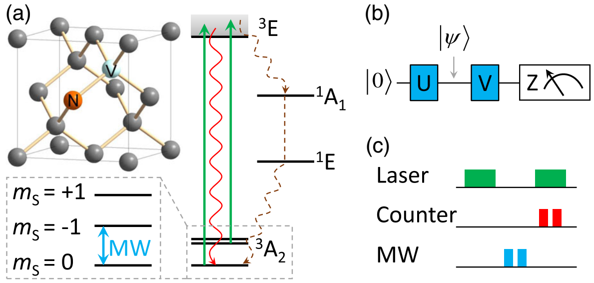

FIG. 2. Experimental system and method. (a) Negatively charged NV center in diamond and the electronic energy level structure. The NV center consists of a substitutional nitrogen atom and a neighboring vacancy. The electronic ground state $^3A_2$ is a triplet state, where two levels are encoded as a qubit and exploited in the experiment. (b) Quantum circuit for qubit control and measurement. (c) Pulse sequence for implementing the quantum circuit. In the experiment, such a process is repeated four million times.

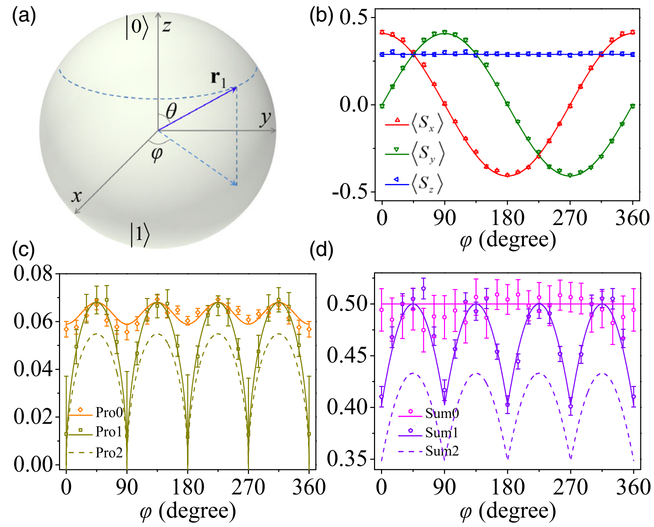

FIG. 3. Experimental results. (a) Bloch sphere and the vector $\mathbf{r}_1$ with $\theta = \arctan\sqrt{2}$. (b) Expectations of S_x, S_y, and S_z. The red, olive, and blue curves, in turn, represent the theoretical values of $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$. The corresponding scattered points represent the experimental values. (c) Experimental demonstration for the uncertainty relation of the multiplicative form. The solid orange, solid green, and dashed green curves, in turn, represent the theoretical values of Pro0, Pro1, and Pro2, where $\mathrm{Pro0} = \Delta S_x \Delta S_y \Delta S_z$, $\mathrm{Pro1} = |\tau^3\langle S_x\rangle\langle S_y\rangle\langle S_z\rangle/8|^{1/2}$, and $\mathrm{Pro2} = |\langle S_x\rangle\langle S_y\rangle\langle S_z\rangle/8|^{1/2}$. The orange and green scattered points represent the experimental value of Pro0 and Pro1, respectively. (d) Experimental demonstration for the uncertainty relations of the additive form. The solid magenta, solid violet, and dashed violet curves, in turn, represent the theoretical values of Sum0, Sum1, and Sum2, where $\mathrm{Sum0} = (\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2$, $\mathrm{Sum1} = \tau(|\langle S_x\rangle| + |\langle S_y\rangle| + |\langle S_z\rangle|)/2$, and $\mathrm{Sum2} = (|\langle S_x\rangle| + |\langle S_y\rangle| + |\langle S_z\rangle|)/2$. The magenta and violet scattered points represent the experimental value of Sum0 and Sum1, respectively. Error bars represent ± 1 s.d.

collection, a solid immersion lens is fabricated on the diamond above the NV center [57,58]. In the following, the rotating frame determined by the resonant MW is adopted, and, in this rotating frame, the two levels $|m_s = 0\rangle$ and $|m_s = -1\rangle$ are labeled by $|0\rangle$ and $|1\rangle$, respectively.

At first, the qubit is initialized to the state $|0\rangle$, and then the desired state $|\psi\rangle$ is prepared by an operation U. After that, an operation V is applied. Finally, the measurement in the $\{|0\rangle, |1\rangle\}$ basis, or, equivalently, the measurement of the observable S_z, is performed. The combined effect of the operation V and the measurement of S_z amounts to the measurement of $V^\dagger S_z V$. In our experiment, the process from initialization to measurement is repeated four million times to acquire the expectation of $V^\dagger S_z V$ associated with the state $|\psi\rangle$. The standard deviation can then be calculated as $\Delta(V^\dagger S_z V) = \sqrt{1/4 - \langle V^\dagger S_z V\rangle^2}$. Different observables, including S_x, S_y, and S_z, can be constructed by adjusting V.

We select two series of pure states $|\psi\rangle$ with Bloch vectors $\mathbf{r}_1 = (\sqrt{2/3}\cos\varphi, \sqrt{2/3}\sin\varphi, 1/\sqrt{3})$ and $\mathbf{r}_2 = (\sin\theta/\sqrt{2}, \sin\theta/\sqrt{2}, \cos\theta)$ as illustrated in Figs. 3(a) and 4(a). The expectations of S_x, S_y, and S_z are illustrated in Figs. 3(b) and 4(b). The verification of the multiplicative uncertainty relation in (3) is shown in Figs. 3(c) and 4(c). The results demonstrate that, for the product $\Delta S_x \Delta S_y \Delta S_z$, the lower bound with the triple constant τ, namely, $|\tau^3\langle S_x\rangle\langle S_y\rangle\langle S_z\rangle/8|^{1/2}$, outperforms the naive lower bound without the triple constant, namely, $|\langle S_x\rangle\langle S_y\rangle\langle S_z\rangle/8|^{1/2}$. The results for the sum $(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2$ are shown in Figs. 3(d) and 4(d). The lower bound with the triple constant τ, namely, $\tau(|\langle S_x\rangle| + |\langle S_y\rangle| + |\langle S_z\rangle|)/2$, outperforms the naive one without the triple constant, namely, $(|\langle S_x\rangle| + |\langle S_y\rangle| + |\langle S_z\rangle|)/2$. The same results also support the uncertainty relation in (6) which was derived previously [43,50]. Therefore, the triple uncertainty relations (3), (5), and (6) are confirmed by the experimental results.

Discussions.—The experimental errors mainly come from the imperfection of microwave pulses and the fluctuation of photon counts. The error bars are smaller than the data markers in Figs. 3(b) and 4(b) but are much larger in Figs. 3(c), 3(d), 4(c), and 4(d). The enlargement of errors is due to error propagation.

In addition to the triple uncertainty relations (3), (5), and (6), some other uncertainty relations exhibit similar behavior. For instance, from the inequality $(\Delta A)^2 + (\Delta B)^2 \geq [\Delta(A+B)]^2/2$ [28], one has

$$(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq \frac{1}{4}\{[\Delta(S_x + S_y)]^2 + [\Delta(S_y + S_z)]^2 + [\Delta(S_z + S_x)]^2\}.$$

By tightening the above inequality for spin-1/2, we have

$$(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq \frac{2}{5}\{[\Delta(S_x + S_y)]^2 + [\Delta(S_y + S_z)]^2 + [\Delta(S_z + S_x)]^2\}. \quad (8)$$

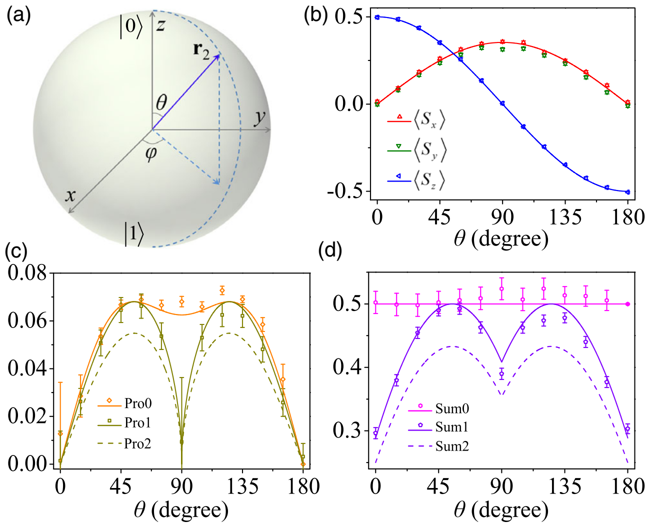

FIG. 4. Experimental results. (a) Bloch sphere and the vector $\mathbf{r}_2$ with $\varphi = \pi/4$. (b) Expectations of S_x, S_y, and S_z. (c) Experimental demonstration for the uncertainty relation of the multiplicative form. (d) Experimental demonstration for the uncertainty relations of the additive form. All the symbols have the same meaning as those in Fig. 3.

The equality holds if and only if the spin-1/2 system is in a pure state, which should also satisfy $r_x + r_y + r_z = 0$.

One can also derive tight entropic uncertainty relations for spin-1/2. The typical entropic uncertainty relation derived by Maassen and Uffink is given by $H(A|\rho) + H(B|\rho) \geq -2\log c(A,B)$, where $H(A|\rho)$ and $H(B|\rho)$ are the Shannon entropy of the measurement outcomes of A and B with a given state ρ and the number $c(A,B) = \max_{i,j}|\langle a_i|b_j\rangle|$ with $|a_i\rangle$ and $|b_j\rangle$ being the eigenstates of A and B, respectively [18]. From the pairwise relations $H(S_x) + H(S_y) \geq \log 2$, $H(S_y) + H(S_z) \geq \log 2$, and $H(S_z) + H(S_x) \geq \log 2$, one has $H(S_x) + H(S_y) + H(S_z) \geq \frac{3}{2}\log 2$, which is again not tight. A tight lower bound needs an additional factor τ^2 [34,35,39], i.e.,

$$H(S_x) + H(S_y) + H(S_z) \geq \log 4 = \frac{3\tau^2}{2}\log 2. \quad (9)$$

The equality holds if and only if the qubit is in one of the eigenstates of S_x, S_y, or S_z.

Additionally, we conjecture that the relation (3) is also valid for any spin quantum number s. The equality holds when $\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = s(s+1)/3$ and $|\langle S_x\rangle| = |\langle S_y\rangle| = |\langle S_z\rangle| = s/\sqrt{3}$ and also holds when $(|\langle S_x\rangle| - s)(|\langle S_y\rangle| - s)(|\langle S_y\rangle| - s) = 0$.

Conclusion.—We have derived tight uncertainty relations for the triple components of angular momentum in the spin-1/2 representation. The triple constant τ exhibits its universality to some extent. The experimental demonstration with the single spin of an NV center consistently supports the theoretical results. Our work enriches the uncertainty relations of more than two observables.

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