Simplifying Differential Equations for Multiscale Feynman Integrals beyond Multiple Polylogarithms

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In this Letter we exploit factorization properties of Picard-Fuchs operators to decouple differential equations for multiscale Feynman integrals. The algorithm reduces the differential equations to blocks of the size of the order of the irreducible factors of the Picard-Fuchs operator. As a side product, our method can be used to easily convert the differential equations for Feynman integrals which evaluate to multiple polylogarithms to an ε form.

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Introduction.—Precision physics with heavy particles, like the Higgs boson, the top quark, or the W and Z bosons, plays an important part in the current physics program at the LHC and will become even more important in the upcoming high luminosity runs. Precision physics requires that higher orders in perturbation theory are taken into account. There is a class of mostly massless processes, where the virtual corrections can be expressed in terms of multiple polylogarithms. Loop integrals can be tackled with differential equations [1–8]. We denote by ε the parameter of dimensional regularization. If the differential equations can be transformed to an ε form [7,8], the solution in terms of multiple polylogarithms is straightforward. However, starting at two loops, there are integrals which cannot be expressed in terms of multiple polylogarithms. The simplest example is given by the two-loop sunrise integral with internal masses [9–21], where functions related to elliptic curves occur. In the corresponding system of differential equations one faces the situation, that at order ε^0 a system of two coupled differential equations occur, which cannot be transformed away.

If we now look at realistic scattering processes at next-to-next-to-leading order (NNLO) with massive particles it is not unusual that within one topology we have several master integrals, coupled together at order ε^0 by the differential equations. We denote by N the number of master integrals within a given topology. For example, for $2 \to 2$ processes at NNLO topologies with up to 5 master integrals may occur. It would be highly prohibitive, if we had to solve at order ε^0 a coupled system of N differential equations. There are indications that topologies with three or more master integrals can be decoupled into blocks of size 2×2 at worst [16,22–25]. This raises the question, if there is a systematic method which transforms a system into an equivalent system, where at order ε^0 are the differential equations split into smaller blocks? In this Letter we will give an algorithm for this task.

The basic idea is as follows: We first reduce a multiscale problem to a single-scale problem with scale λ. In a second step we pick a master integral I and determine at order ε^0 and modulo subtopologies the maximal number of independent derivatives $I,\ (d/d\lambda)I, \ldots, (d/d\lambda)^{r-1}I$. This defines a Picard-Fuchs operator of order r. For $r < N$ the system decouples into a system of r master integrals and $(N-r)$ master integrals. Let us look at the sector with r master integrals. In a third step we factorize the Picard-Fuchs operator. This will decouple the system into blocks of the size of the order of the irreducible factors of the Picard-Fuchs operator. In a fourth step we reconstruct the multivariable transformation matrix from the single-variable one.

Although our primary interest is integrals involving elliptic sectors, it should be noted that our approach provides as a side product an algorithm to convert a multiscale system, which has a solution in terms of multiple polylogarithms to an ε form. In this respect, it complements other methods [26–33].

The method.—Let us consider a set of master integrals $I_1, \ldots, I_N$ depending on kinematic variables $x_1, \ldots, x_n$. We denote the ordered set of master integrals by the vector $\vec{I} = (I_1, \ldots, I_N)$. If the master integrals depend only on a single kinematic variable x_1, we have a single-scale problem. For two or more kinematic variables $(n \geq 2)$ we have a multiscale problem. We consider the master integrals in $D = 2m - 2\varepsilon$ space-time dimensions, with $m \in \mathbb{Z}$ and ε being the dimensional regularization parameter. Integration-by-parts identities [34,35] allow us to derive a system of differential equations of Fuchsian type

$$d\vec{I} = A\vec{I}, \tag{1}$$

where A is a matrix-valued one-form

$$A = \sum_{i=1}^{n} A_i dx_i. \tag{2}$$

The matrix-valued one-form A satisfies the integrability condition

$$dA - A \wedge A = 0. \tag{3}$$

We assume that A has an ε expansion

$$A = \sum_{j\geq 0} \varepsilon^j A^{(j)} = \sum_{i=1}^{n} \sum_{j\geq 0} \varepsilon^j A_i^{(j)} dx_i. \quad (4)$$

The differential equations are usually solved order by order in ε. A crucial role for solving the system is played by the first term $A^{(0)}$. The higher terms $A^{(j)}$ (with $j \geq 1$) only give additional integrations over expressions of lower order. We therefore seek transformations, which simplify $A^{(0)}$. Under a change of basis

$$\vec{J} = U\vec{I}, \quad (5)$$

one obtains

$$d\vec{J} = \tilde{A}\,\vec{J}, \quad (6)$$

where the matrix $\tilde{A}$ is related to A by

$$\tilde{A} = UAU^{-1} - UdU^{-1}. \quad (7)$$

The master integrals can be expressed in terms of multiple polylogarithms if there is a transformation U such that $\tilde{A}^{(0)} = 0$.

The matrix A has a natural lower block triangular form, which derives from the top topology and its subtopologies, obtained by pinching of propagators. In the following we consider the top topology and we work modulo subtopologies. The inclusion of subtopologies leads only to integrations over already determined terms. Let us assume that the top topology has N master integrals. We are in particular interested in the case where no transformation U exists, such that $\tilde{A}^{(0)} = 0$. Although it might seem at first sight that we face in this situation, at order ε^0, a coupled system of N differential equations, it is very often the case that the system decouples into blocks of smaller size. In this Letter we give a systematic method to decouple the system.

We first reduce the multiscale problem to a single-scale problem. Let $\alpha = [\alpha_1 : \ldots : \alpha_n] \in \mathbb{CP}^{n-1}$ be a point in projective space. Without loss of generality we work in the chart $\alpha_n = 1$. Following Ref. [24], we consider a path $\gamma_\alpha : [0, 1] \to \mathbb{C}^n$, indexed by α and parametrized by a variable λ. Explicitly, we have

$$x_i(\lambda) = \alpha_i \lambda, \qquad 1 \leq i \leq n. \quad (8)$$

We then view the master integrals as functions of λ. In other words, we look at the variation of the master integrals in the direction specified by α. For the derivative with respect to λ we have

$$\frac{d}{d\lambda}\vec{I} = B\vec{I}, \qquad B = \sum_{i=1}^{n} \alpha_i A_i. \quad (9)$$

The matrix B has again a Taylor expansion in ε:

$$B = B^{(0)} + \sum_{j>0} \varepsilon^j B^{(j)}. \quad (10)$$

Let I be one of the master integrals $\{I_1, \ldots, I_N\}$. Equation (9) allows us to express the kth derivative of I with respect to λ as a linear combination of the original master integrals. We recall that we work modulo subtopologies. We may even work modulo ε corrections by using $B^{(0)}$ instead of the full matrix B. We then determine the largest number r, such that the matrix which expresses I, $(d/d\lambda)I, \ldots, (d/d\lambda)^{r-1}I$, in terms of the original set $\{I_1, \ldots, I_N\}$ has full rank. Obviously, we have $r \leq N$. In the case $r < N$ we complement the set $I, (d/d\lambda)I, \ldots, (d/d\lambda)^{r-1}I$ by $(N - r)$ elements $I_{\sigma_{r+1}}, \ldots, I_{\sigma_N} \in \{I_1, \ldots, I_N\}$ such that the transformation matrix has rank N. The elements $I_{\sigma_{r+1}}, \ldots, I_{\sigma_N}$ must exist, since we assumed that the set $\{I_1, \ldots, I_N\}$ forms a basis of master integrals for this topology. The basis $\{I, (d/d\lambda)I, \ldots, (d/d\lambda)^{r-1}I, I_{\sigma_{r+1}}, \ldots, I_{\sigma_N}\}$ decouples the system into a block of size r, which is closed under differentiation at order ε^0 modulo subtopologies and a remaining sector of size $(N - r)$.

Let us now investigate under which conditions the block of size r can be decomposed further. We recall that r is the largest number such that $I, (d/d\lambda)I, \ldots, (d/d\lambda)^{r-1}I$ are independent. It follows that $(d/d\lambda)^r I$ can be written as a linear combination of $I, (d/d\lambda)I, \ldots, (d/d\lambda)^{r-1}I$. This defines the Picard-Fuchs operator L_r for the master integral I with respect to λ:

$$L_r I = 0, \qquad L_r = \sum_{k=0}^{r} R_k \frac{d^k}{d\lambda^k}, \quad (11)$$

where the coefficients R_k are rational functions in λ and we use the normalization $R_r = 1$. Note that the zero on the right-hand side of Eq. (11) is understood modulo subtopologies and modulo terms of order ε. Using always $B^{(0)}$ instead of B ensures that L_r is independent of ε. The Picard-Fuchs operator is easily obtained by a transformation to the basis $I, (d/d\lambda)I, \ldots, (d/d\lambda)^{r-1}I$. In this basis the $r \times r$ matrix $\tilde{B}$ has the form

$$\begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & 0 & 1 \\ -R_0 & -R_1 & \ldots & -R_{r-2} & -R_{r-1} \end{pmatrix}. \quad (12)$$

It is very often the case that the operator L_r factorizes [6]:

$$L_r = L_{1,r_1} L_{2,r_2}, \ldots, L_{s,r_s}, \quad (13)$$

where L_{i,r_i} denotes a differential operator of order r_i. Clearly, we have $r_1 + \cdots + r_s = r$. The factorization in Eq. (13) can be obtained with the help of standard computer algebra systems. For example, MAPLE offers the command "DFactor." The factorization in Eq. (13) can be used to convert the system of differential equations at order ε^0 into a block triangular form with blocks of size $r_1, r_2, \ldots, r_s$. A basis for block i is given by

$$J_{i,j} = \frac{d^{j-1}}{d\lambda^{j-1}} L_{i+1,r_{i+1}}, \ldots, L_{s,r_s} I, \qquad 1 \leq j \leq r_i. \tag{14}$$

Let us denote $\vec{J} = (J_{1,1}, \ldots, J_{1,r_1}, J_{2,1}, \ldots, J_{s,r_s})$. Expressing the elements of $\vec{J}$ in terms of the original integrals $\vec{I}$ defines a transformation matrix

$$\vec{J} = V\vec{I}. \tag{15}$$

V is a function of the parameters α and of λ:

$$V = V(\alpha_1, \ldots, \alpha_{n-1}, \lambda). \tag{16}$$

We recall that we work in the chart $\alpha_n = 1$. Setting

$$U = V\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, x_n\right) \tag{17}$$

gives the transformation in terms of the original variables $x_1, \ldots, x_n$. Let us mention that there might be terms in the original A, which map to zero in B for the class of paths considered in Eq. (8). These terms are derivatives of functions being constant on lines through the origin. An example is given by

$$d \ln Z(x_1, \ldots, x_n), \tag{18}$$

where $Z(x_1, \ldots, x_n)$ is a rational function in $(x_1, \ldots, x_n)$ and homogeneous of degree zero in $(x_1, \ldots, x_n)$. On the one hand, these terms don't contribute if we integrate the differential equation along the paths of Eq. (8). On the other hand, these terms are in many cases easily removed by a subsequent transformation.

The case of linear factors.—Let us consider the special case, where the Picard-Fuchs operator L_r factorizes completely into linear factors:

$$L_r = L_{1,1} L_{2,1}, \ldots, L_{r,1}, \tag{19}$$

with

$$L_{i,1} = \frac{d}{d\lambda} + R_{i,0}. \tag{20}$$

$R_{i,0}$ is a rational function in λ. Then it is possible to construct a transformation such that $\tilde{A}^{(0)} = 0$. We first set

$$J_{i,1} = \exp\left(\int^\lambda d\tilde{\lambda} R_{i,0}\right) L_{i+1,1}, \ldots, L_{r,1} I. \tag{21}$$

This transforms the system to a form

$$\frac{d}{d\lambda}\vec{J} = \tilde{B}\vec{J}, \tag{22}$$

where the ε^0-term $\tilde{B}^{(0)}$ is lower triangular with zeros on the diagonal. The possible nonzero entries in the lower triangle of $\tilde{B}^{(0)}$ are easily removed. It is sufficient to discuss the case of a 2×2 matrix: For

$$\tilde{B}^{(0)} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} \tag{23}$$

the transformation

$$\tilde{V}^{(0)} = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \tag{24}$$

with

$$f = -\int^\lambda d\tilde{\lambda} N \tag{25}$$

removes the term in the lower left corner. The function N has a partial fraction decomposition in λ and we use this technique to remove all terms which are polynomials in λ or poles of order 2 and higher. Single poles integrate to logarithms and indicate that our basis elements have nonuniform weight. These are removed by rescaling the master integrals by ε-dependent prefactors, such that the master integrals have uniform weight. In summary, this gives an easy method to convert a system, where every Picard-Fuchs operator factorizes into linear factors, into ε form.

Examples.—Let us now look at a few examples. As a first example we consider a two-loop four-point integral with six propagators, shown in Fig. 1. Internal solid lines correspond to a mass m, dashed lines to mass zero. The external momenta are on-shell, $p_1^2 = p_2^2 = 0$ and $p_3^2 = p_4^2 = m^2$. The Mandelstam variables are defined by $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$. The integral depends on two dimensionless variables x_1 and x_2, which we may choose as [36–39]

$$s = -m^2 \frac{(1 - x_1)^2}{x_1}, \qquad t = -m^2 x_2. \tag{26}$$

The internal propagators are labeled by numbers as shown in Fig. 1. We denote by $I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}$ the scalar integral, where propagator j occurs to the power ν_j. The seven indices refer to a double-box integral with seven propagators. We use the convention that a scalar propagator is given by $1/(-q_j^2 + m_j^2)$. We use the program "Reduze" [40,41] to obtain an initial set of master integrals together with the corresponding system of differential equations. The integral

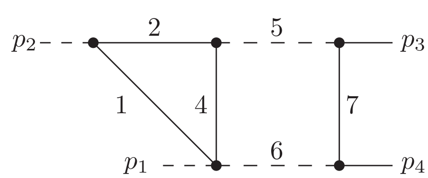

FIG. 1. A two-loop four-point integral with six propagators.

in Fig. 1 has an elliptic subtopology, obtained by pinching propagators 2, 5, and 6. The topology in Fig. 1 has two master integrals, which we may take as

$$\vec{I} = [(1+2\varepsilon)I_{1101111}, I_{1101211}]. \quad (27)$$

The prefactor $(1+2\varepsilon)$ in front of the first master integrals ensures that only $A^{(0)}$ and $A^{(1)}$ appear in the ε expansion of A. The Picard-Fuchs operator for $I = (1+2\varepsilon)I_{1101111}$ is of order 2 and factorizes into linear factors:

$$L_2 = \left(\frac{d}{d\lambda} + \frac{1}{\lambda+1} + \frac{2\alpha_1}{\alpha_1\lambda - 1} + \frac{2\alpha_1\lambda}{\alpha_1\lambda^2+1} - \frac{2\alpha_1\lambda}{\alpha_1\lambda^2-1}\right) \times \left(\frac{d}{d\lambda} - \frac{1}{\lambda} + \frac{1}{\lambda+1} + \frac{2\alpha_1}{\alpha_1\lambda-1}\right). \quad (28)$$

We may therefore transform to a basis, where $\tilde{B}^{(0)}$ is lower triangular with zeros on the diagonal. The entry on the lower left corner of $\tilde{B}^{(0)}$ has only single poles and is removed by rescaling the first master integral with $(1+2\varepsilon)$ and the second master integral by ε. This converts the system with respect to the variable λ to ε form. Going back to the original variables we find

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} d\ln\left(\frac{x_1}{x_2}\right) + \varepsilon\tilde{A}^{(1)}. \quad (29)$$

The ε^0 term is easily removed by multiplying both master integrals by x_2/x_1. In summary we find that with

$$U = \begin{pmatrix} U_{11} & -\frac{(1+2\varepsilon)(x_1-1)^3(x_2+1)^2}{2x_1(x_1+1)} \\ \frac{\varepsilon(x_2+1)(x_1-1)^2}{x_1} & 0 \end{pmatrix},$$
$$U_{11} = \frac{(1+2\varepsilon)(x_1-1)(x_2^2x_1 + x_2x_1^2 + x_2 - x_1^2 + 3x_1 - 1)}{2x_1(x_1+1)} \quad (30)$$

the transformed system is given by

$$\tilde{A} = \varepsilon\left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} d\ln(x_1+1) - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} d\ln(x_1-1) - \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} d\ln(x_2+1) + \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} d\ln(x_1+x_2) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} d\ln(x_1x_2+1)\right]. \quad (31)$$

We see that this topology can be transformed to ε form and does not introduce new elliptic integrations.

Let us now look at a more involved example. We consider the two-loop four-point integral with five propagators shown in Fig. 2. The kinematics is as in our first

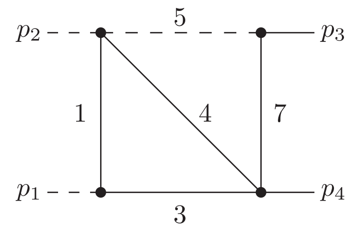

FIG. 2. A two-loop four-point integral with five propagators.

example. This topology has five master integrals. As our initial basis we take

$$\vec{I} = (\varepsilon I_{1011101}, I_{2011101}, I_{1021101}, I_{1012101}, I_{1011201}). \quad (32)$$

Multiplying the first master integral by ε ensures that only $A^{(0)}$ and $A^{(1)}$ appear in the ε expansion of A. For the specific system under consideration this also decouples the first master integral $I_1 = \varepsilon I_{1011101}$ at order ε^0 from the remaining ones. We therefore have to consider only a 4×4 system. Let us pick $I_2 = I_{2011101}$. Working modulo ε terms, we find that already the third derivative of I_2 can be expressed as a linear combination of the lower ones. Adding $I_5 = I_{1011201}$ to $I_2, (d/d\lambda)I_2, (d/d\lambda)^2 I_2$ will give a transformation matrix of full rank. This decouples I_5 from the 3×3 system formed by $I_2, (d/d\lambda)I_2, (d/d\lambda)^2 I_2$. The Picard-Fuchs operator for I_2 is therefore of order 3. It factorizes into a second-order operator and a first-order operator:

$$L_3 = L_2 L_1. \quad (33)$$

This decouples the 3×3 system into a 2×2 system and a 1×1 system. The 2×2 system is irreducible. When lifting the result from the single-scale case to the multiscale case with the variables $\{x_1, x_2\}$, we again perform an additional transformation, which removes $d\ln(x_1/x_2)$ terms. In summary, we are able to decompose the five master integrals for this topology at order ε^0 in blocks of size

$$1, 2, 1, 1. \quad (34)$$

The explicit expressions are longer; however, we may display the structure of $\tilde{A}$. We have

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \end{pmatrix} + \varepsilon\tilde{A}^{(1)}, \quad (35)$$

where $*$ indicates a nonzero entry. In this example we see that $A^{(0)}$ cannot be transformed to zero. We find an irreducible 2×2 system at order ε^0. However, we achieved a simplification of the original 5×5 system to smaller blocks.

In addition we have applied our method successfully to all sectors of the seven-propagator double-box integral, including the top sector with seven propagators. This sector has five master integrals and decouples into blocks of size 1, 2, 1, and 1.

Conclusions.—In this Letter we presented an algorithm to simplify differential equations for multiscale Feynman integrals. We first reduced the problem to a single-scale problem and then exploited factorization properties of the Picard-Fuchs operator. This allows us to decouple the system at order ε^0 into blocks of the sizes of the irreducible factors of the Picard-Fuchs operator. We expect this technique to be useful for precision calculations. A particular special case is given when all Picard-Fuchs operators factorize into linear factors. In this case, our method provides an easy algorithm to convert a multiscale differential system into ε form.

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