Bilayer Graphene as a Platform for Bosonic Symmetry-Protected Topological States

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Bosonic symmetry protected topological (BSPT) states, the bosonic analogue of topological insulators, have attracted enormous theoretical interest in the last few years. Although BSPT states have been classified by various approaches, there is so far no successful experimental realization of any BSPT state in two or higher dimensions. In this paper, we propose that a two-dimensional BSPT state with $U(1) \times U(1)$ symmetry can be realized in bilayer graphene in a magnetic field. Here the two $U(1)$ symmetries represent total spin S^z and total charge conservation, respectively. The Coulomb interaction plays a central role in this proposal—it gaps out all the fermions at the boundary, so that only bosonic charge and spin degrees of freedom are gapless and protected at the edge. Based on the above conclusion, we propose that the bulk quantum phase transition between the BSPT and trivial phase, which can be driven by applying both magnetic and electric fields, can become a "bosonic phase transition" with interactions. That is, only bosonic modes close their gap at the transition, which is fundamentally different from all the well-known topological insulator to trivial insulator transitions that occur for free fermion systems. We discuss various experimental consequences of this proposal.

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A symmetry protected topological (SPT) state, first defined in Refs. [1,2], is the ground state of a local quantum many-body Hamiltonian whose bulk is gapped and nondegenerate, but whose boundary remains either gapless or degenerate as long as the entire system including the boundary preserves certain symmetries. Fermionic SPT states include the familiar quantum spin Hall (QSH) insulator [3,4], the three-dimensional topological insulator [5–7], and topological superconductors. Noninteracting fermionic SPT states have been fully classified and understood [8,9]. Unlike fermionic systems, bosonic SPT (BSPT) states require strong interaction to overcome the tendency to form Bose-Einstein condensates. The simplest and most well-known BSPT state is the one-dimensional Haldane phase, which can be realized in the simplest nearest-neighbor spin-1 Heisenberg chain [10,11]. However, higher dimensional generalizations of BSPT states have not been found. The only even potentially feasible experimental proposal is for a bosonic integer quantum Hall state in ultracold atoms [12], but even this seems far away, since as yet experiments with both rotating traps and artificial magnetic fields are still far from the quantum Hall regime. The exactly soluble parent Hamiltonians constructed in Refs. [1,2] in dimensions higher than 1 all involve high order multiple spin interactions, and are thus unlikely to exist in realistic materials. Up to now, all approaches to classifying and characterizing BSPT states [1,2,13–16] rely on mathematical or effective field theory descriptions, which shed little light on how to identify a realistic candidate BSPT state.

In the current paper, we hope to bridge the gap between theoretical studies and experimental realizations of BSPT states. We propose that bilayer graphene in magnetic field (with both in-plane and out-of-plane components) provides a platform of realizing and probing the two-dimensional BSPT state with $U(1)_s \times U(1)_c$ symmetry, where $U(1)_s$ and $U(1)_c$ correspond to the total spin-S^z and total electric charge conservation, respectively. Based on the formalism developed in Refs. [14,15], this state has a $\mathbb{Z}$ classification; i.e., with these symmetries there is an infinite set of nontrivial two-dimensional BSPT classes, which are indexed by an integer k. Effective field theory descriptions of these BSPT states have been given in terms of Chern-Simon field theory [14] and a nonlinear sigma model (NLSM) with a Θ-term [15,17]. The action for the latter is

$$\mathcal{S} = \int d^2x d\tau \frac{1}{g}(\partial_\mu \boldsymbol{n})^2 + \frac{i\Theta}{\Omega_3}\epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d, \qquad (1)$$

where $\boldsymbol{n} = (n_1, n_2, n_3, n_4)$ is a four component vector with unit length [15,17], and Ω_3 is the volume of a three-dimensional sphere with unit radius. In Eq. (1), the BSPT phases correspond to the strongly interacting fixed point $g \to \infty$, and $\Theta \to 2k\pi$ with nonzero integer k, while the trivial phase corresponds to the fixed point $\Theta \to 0$. The quantum phase transition between different BSPT phases is driven by tuning Θ in Eq. (1), and the critical point is at $\Theta = (2k+1)\pi$. A similar phase diagram and renormalization group flow for NLSMs in one lower dimension was studied thoroughly in Refs. [18,19].

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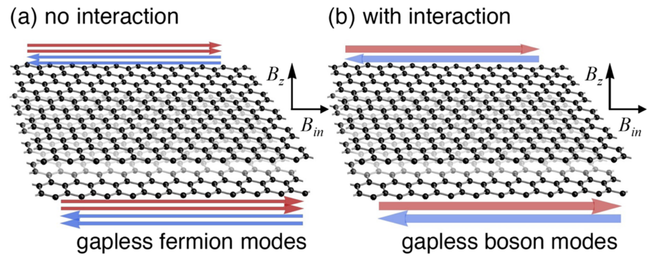

FIG. 1. Schematic of bilayer graphene in the presence of a magnetic field with both in-plane and out-of-plane components. (a) Without interactions, the boundary hosts two channels of fermionic edge states with total central charge $c = 2$. (b) Including the Coulomb interactions, there is only one gapless channel of bosonic edge state with $c = 1$.

Let us elaborate on our claim. It was proposed that an out-of-plane magnetic field drives undoped graphene into a "quantum spin Hall insulator" [20] (it is also called the ferromagnetic quantum Hall state, since the bulk is fully spin polarized. In order to avoid a canted antiferromagnetic (CAF) phase, one also needs an in-plane magnetic field to increase the Zeeman coupling [21,22], which is discussed in detail in Supplemental Material [23]). In a bilayer, this possesses at the Hartree-Fock level two channels of counterpropagating spin-filtered helical fermionic edge states [22,30]. However, when the Coulomb interaction is included, we demonstrate that (as illustrated in Fig. 1), the behavior is qualitatively modified to correspond precisely to that of the BSPT theories, Eq. (1) with $k = 1$, so that, although it is built from electrons, it is a proper BSPT state in the following senses: 1. The Coulomb interaction, which is expected to play an important role in this system, induces a gap for all fermionic excitations at the boundary, while bosonic charge and spin excitations remain gapless and protected by the two $U(1)$ symmetries [Fig. 1(b)]. 2. Using the Chalker-Coddington picture [31], the bulk quantum phase transition between the nontrivial SPT phase ($k = 1$) and trivial ($k = 0$) phase (hereafter phrased as topological to trivial transition) can be described by percolation of domains and the corresponding network of interface or boundary states. Because the boundary only has gapless bosonic modes, such a topological quantum phase transition can occur while preserving the bulk gap for fermionic quasiparticles. The topological to trivial transition can be driven by varying competing magnetic and electric fields, and we propose that the bosonic scenario for this quantum phase transition occurs with sufficiently strong interactions. This is a qualitatively different situation from the well-known topological to trivial transitions in weakly correlated systems, such as the plateau transition between integer quantum Hall states, or the transition between normal and topological band insulators—these transitions have a free fermion description that involves the fermion gap closing in the bulk. The above statement is supported by recent numerical studies of a similar model on the bilayer honeycomb lattice [32,33].

We now proceed to an exposition of these results. For noninteracting bilayer graphene, there are two channels of helical edge states, described by the Hamiltonian

$$H_0 = \int dx \sum_{l=1}^{2} \psi^\dagger_{l,L} iv\partial_x \psi_{l,L} - \psi^\dagger_{l,R} iv\partial_x \psi_{l,R}, \qquad (2)$$

where $l = 1, 2$ labels the channels, L, R denote the left- and right-moving fermions, respectively, which also correspond to electrons with spin up and down, and v is the Fermi velocity [34]. The presence of some counterpropagating edge states was deduced experimentally from nonlocal transport signatures [22]. When the Coulomb interaction is ignored, the boundary is a free fermion conformal field theory (CFT) with central charge $c = 2$.

The free fermion edge states can be bosonized into two flavors of free bosons,

$$H_0 = \int dx \sum_{l=1}^{2} \frac{v}{2K} (\partial_x \theta_l)^2 + \frac{vK}{2} (\partial_x \phi_l)^2, \qquad (3)$$

where $[\theta_l(x), \partial_{x'} \phi_{l'}(x')] = i\delta(x - x')\delta_{ll'}$, and $\psi_{l,L/R} \sim e^{i\theta_l \pm i\pi\phi_l}$. For free one-dimensional fermions without interaction, the Luttinger parameter $K = \pi$.

Coulomb interactions $H_{\rm int}$ are conveniently handled in the bosonization framework. Using the representation of the fermion density $n_l \sim \partial_x \phi_l$, one obtains

$$H_{\rm int} = \int dx \sum_{l=1}^{2} \frac{U_{\rm intra}}{2} (\partial_x \phi_l)^2 + U_{\rm inter} \partial_x \phi_1 \partial_x \phi_2 + H_v, \qquad (4)$$

where $U_{\rm intra}$ and $U_{\rm inter}$ represent intralayer and interlayer forward-scattering interactions, respectively. H_v is an anharmonic vertex term, and plays a central role here [35],

$$H_v \sim \alpha \cos(2\pi\phi_1 - 2\pi\phi_2). \qquad (5)$$

Here we have assumed that the long range Coulomb interaction is screened to a short range one, but this is not essential. Physically H_v describes the backscattering between two channels of edge states, $H_v \sim \psi^\dagger_{1,L} \psi_{1,R} \psi^\dagger_{2,R} \psi_{2,L}$. The anharmonic H_v is relevant in the renormalization group sense, as long as $U_{\rm intra} > U_{\rm inter}$. This condition is naturally satisfied because $U_{\rm inter}$ is suppressed by the square of the wave function overlap between the two channels of edge states.

When it is relevant, H_v "pins" the bosonic mode $\phi_- = (\phi_1 - \phi_2)/2$, causing large fluctuations of $\theta_- = \theta_1 - \theta_2$, leading to a gap in this antisymmetric sector, and also a gap for all fermions at the boundary. The symmetric edge modes $\phi = (\phi_1 + \phi_2)/2$ and $\theta = \theta_1 + \theta_2$, however, remain gapless, because θ transforms under symmetry $U(1)_c$, while ϕ transforms under $U(1)_s$. It is straightforward to show—see below—that only physical operators that create bosonic excitations can be built from the gapless ϕ, θ fields, consistent with the statement that the boundary has symmetry-protected gapless bosonic modes. The size of the fermion gap at the boundary state is estimated in detail in Supplemental Material.

The effective low energy theory that describes the canonical conjugate modes ϕ and θ reads

$$\tilde{H} = \int dx \frac{\tilde{v}}{2\tilde{K}} (\partial_x \theta)^2 + \frac{\tilde{v}\tilde{K}}{2} (\partial_x \phi)^2. \qquad (6)$$

Hence interaction reduces the central charge of the system from $c = 2$ to $c = 1$. Because θ and ϕ transform nontrivially [i.e., shift under $U(1)_c$ and $U(1)_s$ symmetries, respectively], there are no anharmonic vertex operators allowed by symmetry in Eq. (6). Because θ and ϕ are "dual" to each other, a unit soliton of ϕ at the one-dimensional boundary carries charge $2e$, and a unit soliton of θ carries spin $S^z = 1$. The gaplessness of the boundary state is protected by the $U(1)_c \times U(1)_s$ symmetry alone: even if the translation symmetry of the boundary is broken by disorder (which is inevitable in any real system), as long as the $U(1)_c \times U(1)_s$ symmetry is preserved, the boundary must still remain gapless. The edge state in our system is also very different from the cases studied in Refs. [36,37], since in those systems the states localized at the domain wall are unstable to disorder.

Here we note that although the bosonization of the edge states of bilayer graphene in a magnetic field was also studied in Refs. [38,39], in these works only the spin symmetry was considered in the bosonization, and the conclusion of Refs. [38,39] was that the system is equivalent to a one-dimensional spin model. Here we stress that, both the $U(1)_s$ and $U(1)_c$ symmetries are crucial to define the BSPT state: i.e., if either of the $U(1)$ symmetries is broken (for example if the bulk forms a canted antiferromagnetic order), the system becomes a trivial state. With both $U(1)$ symmetries in our system, the boundary theory Eq. (6) must remain gapless, and it can never be realized as a one-dimensional system, but rather only as the boundary of a two-dimensional system, which is an essential property of all SPT states.

Let us discuss the operator content further. Assuming ϕ_- is pinned and θ_- fluctuates strongly, one can obtain the low energy components of the four component vector $\boldsymbol{n}$ in Eq. (1),

$$\begin{aligned} n_1 + i n_2 &\sim \epsilon_{\alpha\beta} \psi_{1,\alpha} \psi_{2,\beta} \sim e^{i\theta}, \\ n_3 + i n_4 &\sim \sum_l (-1)^l \psi_l^\dagger \sigma^+ \psi_l \sim e^{i2\pi\phi}. \end{aligned} \qquad (7)$$

Here $n_1 + i n_2$ corresponds to an interlayer spin-singlet ($S^z = 0$) Cooper pair, while n_3 and n_4 correspond to in-plane magnetic order. All components of the vector $\boldsymbol{n}$ have power-law correlations at the boundary, and their scaling dimensions are $\Delta[\epsilon_{\alpha\beta}\psi_{1,\alpha}\psi_{2,\beta}] = (\tilde{K}/4\pi)$, $\Delta[\sum_l (-1)^l \psi_l^\dagger \sigma^+ \psi_l] = (\pi/\tilde{K})$. Thus we see that indeed the low energy correlations at the edge all correspond to bosonic fields, which could be built from elementary bosons of even charge and integer spin. The presence of four distinct "primary fields" is characteristic of the Wess-Zumino-Witten (WZW) $\mathrm{SU}(2)_1$ CFT, which is well known to be expressible in terms of a single gapless boson and has $c = 1$ [40,41]. The model in Eq. (6) is a deformation of the usual $\mathrm{SU}(2)_1$ theory that reduces the symmetry to $U(1)_c \times U(1)_s$. It is also equivalent to a (deformed) $O(4)$ NLSM with a $k = 1$ WZW term—see, e.g., Ref. [42].

Equation (7) identified the effective bosonic degrees of freedom that form a bosonic SPT state in the bulk. There are two flavors of bosons carrying charge and spin quantum numbers, respectively. Following the method of Ref. [43], we can derive the wave function of the bosons in the bulk, by calculating the following correlation function of the boundary conformal field theory,

$$\Psi(z_1, z_2 \cdots w_1, w_2 \cdots) \sim \left\langle \prod_j e^{i\theta(z_j)} \prod_k e^{2\pi i \phi(w_k)} \mathcal{O}_{bg} \right\rangle, \qquad (8)$$

where z_j and w_k are the complex coordinates in the two-dimensional plane for the two flavors of bosons. This is equivalent to calculating the partition function of a two-dimensional Coulomb gas with both electric and magnetic charges [44,45], and $\mathcal{O}_{bg}$ represents a neutralizing background charge operator. The correlation function in Eq. (8) can be evaluated with either Eq. (3) or Eq. (6), and the result will be qualitatively the same,

$$\Psi(z_1, z_2 \cdots w_1, w_2 \cdots) \sim \mathrm{Norm}(z_j, w_k) \prod_{j,k} (z_j - w_k), \qquad (9)$$

where $\mathrm{Norm}(z_j, w_k)$ only depends on the norm of $z_j - w_k$, $z_i - z_j$ and $w_i - w_j$, and contains all the dependence upon the Luttinger parameters in Eqs. (3) and (6). This wave function indeed represents a bosonic SPT state: it is symmetric under interchange of identical z_i or w_j bosons, and the two flavors of bosons view each other as a 2π-flux. This mutual "flux attachment" picture is the very essence of the BSPT state [46].

Knowing the effective field theory at the boundary is the $(1+1)$-dimensional NLSM for $\boldsymbol{n}$ with a Wess-Zumino-Witten term at level $k = 1$; the bulk theory can be constructed with the Chalker-Coddington network model [31], and as was shown in Refs. [13,47], the bulk theory obtained by this construction is precisely Eq. (1) with $\Theta = 2\pi$. The physical meaning of this topological Θ-term is that a vortex of (n_1, n_2), i.e., a vortex of the superconductor order parameter, which traps magnetic flux

$(hc/2e)$, carries spin $S^z = 1$, which is perfectly consistent with the physics of the bilayer QSH state.

It is worth contrasting with the case of a single layer QSH insulator, in which the boundary cannot be driven into a state with gapped fermions but gapless bosonic modes, as long as the $U(1)_c$ and time-reversal [or $U(1)_s$] symmetry of the QSH insulator are preserved [48,49]. The mapping between the fermionic QSH insulator and BSPT is only valid for two copies of QSH insulators (which mathematically is equivalent to four copies of $p \pm ip$ topological superconductors), as was shown in Ref. [50].

By varying competing electric and magnetic fields normal to the layer, a quantum phase transition can occur between the BSPT and the trivial state in the two-dimensional bulk. Using the Chalker-Coddington network picture, one may construct a theory for the two-dimensional bulk phase transition that involves only gapless bosonic modes and retains the single-fermion gap. In the field theory Eq. (1) this transition occurs when Θ is tuned to π. Although directly analyzing the bulk field theory at $\Theta = \pi$ is difficult, recent unbiased determinant quantum Monte Carlo simulation on a similar bilayer honeycomb lattice interacting fermion model confirms that this purely bosonic topological-trivial quantum phase transition can indeed happen [32,33], which is fundamentally different from the ordinary topological to trivial transition in any free fermion system. Maintaining the single particle gap requires strong interactions, and other less interesting possibilities are possible in experiment if correlations are not sufficiently strong, such as intermediate phases between the BSPT phase and the trivial phase. Nevertheless, a direct second order bosonic transition like the one found in Refs. [32,33] seems allowed and a quite interesting prospect.

Experimental implications.—The central prediction of our theory is that in a bilayer graphene in the quantum spin Hall phase [22], the gapless boundary modes are bosonic rather than fermionic. The low energy charge carriers on the edge are Cooper pairs $\epsilon_{\alpha\beta}c_{1,\alpha}c_{2,\beta}$, with charge $2e$. Tunneling from a normal metal electrode or tip is predicted to show a hard gap, despite ballistic, dissipationless in-plane resistance. Conversely, tunneling from a superconducting tip should show zero gap.

A purely transport measurement is also possible using shot noise, which has previously been used to probe fractional charges in quantum Hall edge states [51–54]. By introducing a quantum point contact, either using electrostatic gates or a nanoconstriction, edge-to-edge backscattering is possible at that contact, with a finite transmission probability [54]. Individual tunneling events carry charge $\pm 2e$, which is directly observable in the noise spectrum. The detailed calculation about the shot noise in a quantum point contact geometry has been presented in a follow-up paper by some of the current authors [55].

Here we propose a different method to measure the carrier charge at the boundary. Compared with the point-contact geometry, our current proposal is easier to implement experimentally, and more convenient to analyze theoretically, as it only involves one edge instead of two opposite edges. Our proposal is based on the dual-gated geometry that has been used in experiments Ref. [22]. The screened Coulomb interaction in our system can be tuned by its distance d to the gates due to screening. The competition between interaction and the Zeeman energy can lead to a rich phase diagram, and when the interaction is dominant, the system develops a CAF order [22]. The size of the fermion gap at the boundary, as well as the magnetic field required to realize the BSPT state in this setup, is discussed in detail in Supplemental Material.

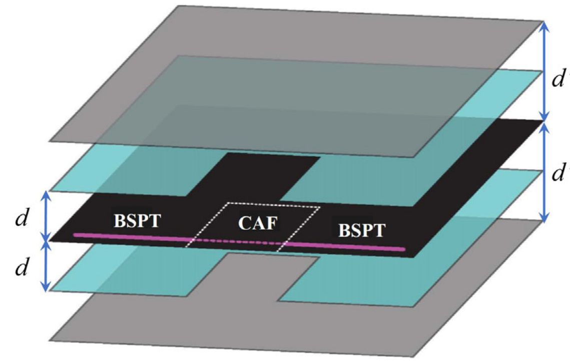

FIG. 2. Our proposed setup for measuring the carrier charge at the boundary of our system. Most of the sample is screened by the inner symmetric gates, while the unscreened region has a stronger interaction that leads to a CAF order, and induces backscattering of the edge states. We also add a pair of outer gates to control the strength of interaction in the CAF region.

The stability of the edge states of our system relies on the conservation of S^z, and if locally the S^z conservation is broken, the edge modes encounter backscattering, and hence lead to noise of the current. We propose to screen the Coulomb interaction for most of the sample, while leaving a region close to the edge unscreened, in order to develop a local CAF order, which serves as a local "magnetic impurity" that breaks the S^z conservation. We calculate the quantum shot noise in Supplemental Material with the proposed setup Fig. 2, and recover the expected result,

$$\tilde{S}(\omega = 0) = 2e^* \langle I \rangle \coth \frac{e^* V}{2k_B T}. \tag{10}$$

$e^* = 2e$ is the smoking gun signature of the BSPT state proposed in our work.

If a direct second order quantum phase transition between the BSPT and trivial phase found in Refs. [32,33] indeed happens in a real system, then at the transition, which corresponds to a $(2+1)$-dimensional CFT, the bulk conductivity should be a universal value $\sigma = De^2/h$, where D is an order-1 universal constant [56,57]. Moreover the transition should be accompanied by a closing of the spin gap, with observable consequences for spin susceptibility as well as thermal transport measurements.

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The first two authors made equal contributions to this work.

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