Mellin Amplitudes for Supergravity on $AdS_5 \times S^5$

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(Received 29 November 2016; published 1 March 2017)

We revisit the calculation of holographic correlation functions in type-IIB supergravity on $AdS_5 \times S^5$. Results for four-point functions simplify drastically when expressed in Mellin space. We conjecture a compact formula for the four-point functions of one-half Bogomol'nyi-Prasad-Sommerfield single-trace operators of arbitrary weight. Our methods rely on general consistency conditions and eschew detailed knowledge of the supergravity effective action.

DOI: 10.1103/PhysRevLett.118.091602

Introduction.—Despite almost two decades of relentless efforts, we are still far from harnessing the full computational power of the AdS/CFT correspondence. In the canonical duality [1–3] between $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and type-IIB string theory on $AdS_5 \times S^5$, the bulk description is most tractable in the classical supergravity regime, which describes planar SYM theory at large 't Hooft coupling. Supergravity is, however, still a complicated nonlinear theory, and only the simplest observables have been computed so far. In this Letter we revisit the holographic calculation of four-point correlation functions of one-half Bogomol'nyi-Prasad-Sommerfield (BPS) single-trace operators [4]. In the supergravity limit, there is a straightforward algorithm that computes them as a sum of tree-level Witten diagrams, whose vertices are encoded in the AdS_5 effective action [8] obtained by Kaluza-Klein (KK) reduction of type-IIB supergravity on S^5.

The difficulty of the calculation grows quickly with the KK level and complete results are only available for a handful of four-point correlators. There are some hints that final answers are simpler than the intermediate calculations. For example, evaluating the four-point function of the lowest KK mode (corresponding to the stress-tensor supermultiplet) is a nontrivial task [9,10], but the result can be written as a single quartic Witten diagram [11]. One is tempted to draw an analogy with tree-level gluon scattering amplitudes in $4d$ Yang-Mills theory, where the traditional Feynman diagram expansion hides the true simplicity of the on-shell answer [12]. Moreover, it is our belief that holographic n-point functions of arbitrary KK modes must be completely fixed by general consistency requirements such as superconfomal symmetry and crossing—this is a restatement of uniqueness of the two-derivative action of $10d$ type-IIB supergravity (up to field redefinitions). It must then be possible to bypass the diagrammatic expansion altogether and directly *bootstrap* the holographic correlators. The natural language for such an approach is the Mellin representation of conformal field theory (CFT) correlators, initiated by Mack [14] and developed in Refs. [15–20]. In Mellin space, tree-level AdS_5 correlators are rational functions of Mandelstam-like invariants, with poles and residues controlled by factorization, in direct analogy with tree-level scattering amplitudes in flat space.

In this Letter we report an elegant formula for the four-point function of arbitrary single-trace one-half BPS operators in the supergravity limit. We have discovered a simple expression that satisfies all consistency conditions and reproduces all explicitly calculated examples [10,21–24]. We believe that this is the unique solution of our bootstrap problem, but a complete proof of uniqueness is presently lacking.

Superconformal symmetry.—Let us first review the constraints of superconformal invariance. We focus on one-half BPS local operators, $\mathcal{O}_p^{I_1\ldots I_p}(x)=\mathrm{Tr}X^{\{I_1}\ldots X^{I_p\}}(x)$, $I_k=1,\ldots,6$, in the symmetric-traceless representation of the $SO(6)$ R symmetry. It is convenient to keep track of the R-symmetry structure by contracting the $SO(6)$ indices with a null vector:

$$O_p(x,t) = t_{I_1}\ldots t_{I_p} O_p^{I_1\ldots I_p}(x), \qquad t\cdot t = 0. \tag{1}$$

The four-point correlator

$$G_{p_1p_2p_3p_4} = \langle O_{p_1}O_{p_2}O_{p_3}O_{p_4}\rangle \tag{2}$$

is then a function of the four spacetime coordinates x_i and of the four "internal" coordinates t_i. Invariance under the conformal group $SO(4,2)$ and R-symmetry group $SO(6)$ implies that it is really a function of conformal cross ratios U and V and of R-symmetry cross rations σ and τ, up to a kinematic prefactor [25]:

$$G(x_i,t_i) = \prod_{i<j}\left(\frac{t_{ij}}{x_{ij}^2}\right)^{\gamma_{ij}^0}\left(\frac{t_{12}t_{34}}{x_{12}^2x_{34}^2}\right)^L \mathcal{G}(U,V;\sigma,\tau), \tag{3}$$

where $x_{ij} = x_i - x_j$, $t_{ij} = t_i\cdot t_j$ and

$$U = \frac{(x_{12})^2(x_{34})^2}{(x_{13})^2(x_{24})^2}, \qquad V = \frac{(x_{14})^2(x_{23})^2}{(x_{13})^2(x_{24})^2},$$
$$\sigma = \frac{t_{13}t_{24}}{t_{12}t_{34}}, \qquad \tau = \frac{t_{14}t_{23}}{t_{12}t_{34}}. \tag{4}$$

The exponents γ_{ij}^0 are given by

$$\gamma_{12}^0 = \frac{p_1 + p_2 - p_3 - p_4}{2}, \qquad \gamma_{13}^0 = \frac{p_1 + p_3 - p_2 - p_4}{2},$$
$$\gamma_{34}^0 = \gamma_{24}^0 = 0, \qquad \gamma_{14}^0 = p_4 - L,$$
$$\gamma_{23}^0 = p_4 - L - \frac{p_1 + p_4 - p_2 - p_3}{2}. \tag{5}$$

Finally, the exponent L is defined as follows. Assuming without loss of generality $p_1 \geq p_2 \geq p_3 \geq p_4$, we distinguish two cases: $p_1 + p_4 \leq p_2 + p_3$ (case I) and $p_1 + p_4 > p_2 + p_3$ (case II). Then [26],

$$L = p_4 \qquad (\text{case I}),$$
$$L = \frac{p_2 + p_3 + p_4 - p_1}{2} \qquad (\text{case II}). \tag{6}$$

It immediately follows from these definitions that $\mathcal{G}(U, V; \sigma, \tau)$ is a degree L polynomial in σ and τ:

$$\mathcal{G}(U, V; \sigma, \tau) = \sum_{0 \leq m+n \leq L} \sigma^m \tau^n \mathcal{G}^{(m,n)}(U, V). \tag{7}$$

Invariance under the full superconformal symmetry $PSU(2,2|4)$ further implies the Ward identity [27,28],

$$\partial_{\bar{z}}[\mathcal{G}(z\bar{z}, (1-z)(1-\bar{z}); \alpha\bar{\alpha}, (1-\alpha)(1-\bar{\alpha}))|_{\bar{\alpha} \to 1/\bar{z}}] = 0, \tag{8}$$

where we have performed the useful change of variables $U = z\bar{z}$, $V = (1-z)(1-\bar{z})$, $\sigma = \alpha\bar{\alpha}$, $\tau = (1-\alpha)(1-\bar{\alpha})$. Its solution can be written as [27,28]

$$\mathcal{G}(U, V; \sigma, \tau) = \mathcal{G}_{\text{free}}(U, V; \sigma, \tau) + R\mathcal{H}(U, V; \sigma, \tau), \tag{9}$$

where $\mathcal{G}_{\text{free}}$ is the answer in free SYM theory and

$$R = \tau 1 + (1 - \sigma - \tau)V + (-\tau - \sigma\tau + \tau^2)U$$
$$+ (\sigma^2 - \sigma - \sigma\tau)UV + \sigma V^2 + \sigma\tau U^2$$
$$= (1 - z\alpha)(1 - \bar{z}\alpha)(1 - z\bar{\alpha})(1 - \bar{z}\,\bar{\alpha}). \tag{10}$$

All dynamical information is contained in the *a priori* unknown function $\mathcal{H}(U, V; \sigma, \tau)$.

Mellin.—The Mellin amplitude $\mathcal{M}$ is defined as [14]

$$\mathcal{M}(s, t; \sigma, \tau) = \frac{M(s, t; \sigma, \tau)}{\Gamma_{p_1 p_2 p_3 p_4}}, \tag{11}$$

where

$$M(s, t; \sigma, \tau) = \int_0^\infty dV V^{-t/2 + (\min\{p_1+p_4, p_2+p_3\}/2) - 1}$$
$$\times \int_0^\infty dU U^{-s/2 + (p_3+p_4)/2 - L - 1}$$
$$\times \mathcal{G}_{\text{conn}}(U, V; \sigma, \tau) \tag{12}$$

is an integral transform of the *connected* four-point function with respect to the conformal cross ratios, and

$$\Gamma_{p_1 p_2 p_3 p_4} = \Gamma\left(\frac{p_1 + p_2 - s}{2}\right)\Gamma\left(\frac{p_3 + p_4 - s}{2}\right)$$
$$\times \Gamma\left(\frac{p_2 + p_3 - t}{2}\right)\Gamma\left(\frac{p_1 + p_4 - t}{2}\right)$$
$$\times \Gamma\left(\frac{p_1 + p_3 - u}{2}\right)\Gamma\left(\frac{p_2 + p_4 - u}{2}\right). \tag{13}$$

We have also defined

$$u = p_1 + p_2 + p_3 + p_4 - s - t. \tag{14}$$

Mack [14] observed that $\mathcal{M}$ behaves in some ways as an S matrix, with the dual variables s, t, u playing the role of Mandelstam invariants. Its analytic structure is very simple: for fixed t, the so-called reduced Mellin amplitude M has simple poles in s; each pole corresponds to an intermediate operator exchanged in the s-channel operator product expansion (OPE) of the four-point function. Organizing operators in conformal families, each exchanged primary of dimension Δ and spin J contributes an infinite sequence of pole at $s = \tau + 2m$, where $\tau = \Delta - J$ is the twist and $m \in \mathbb{Z}_+$. Analogous statements hold in the crossed channels.

As pointed out by Penedones [15], definition Eq. (11) is completely natural in a large N theory: dividing by $\Gamma_{p_1 p_2 p_3 p_4}$ removes the poles associated with double-trace operators, leaving in $\mathcal{M}$ only single-trace poles. Recall that $\mathcal{G}_{\text{conn}}$ is subleading at large N with respect to the disconnected part—it is $O(1/N^2)$ in $SU(N)$ SYM theory. It receives contributions from both single-trace operators and double-trace operators. For example, in the s-channel OPE $(x_{12}, x_{34} \to 0)$ there are double-trace operators of the schematic form $\mathcal{O}_{p_1} \partial^J \Box^n \mathcal{O}_{p_2}$, of twist $\tau = p_1 + p_2 + 2n + O(1/N^2)$, and $\mathcal{O}_{p_3} \partial^J \Box^n \mathcal{O}_{p_4}$, of twist $\tau = p_3 + p_4 + 2n + O(1/N^2)$. Their contribution is precisely captured by the first two Gamma functions in Eq. (13), while the other Gamma functions serve the same purpose in the t and u channels [29].

We are interested in further taking the 't Hooft coupling λ to infinity. This regime is described in the bulk by classical supergravity. The only single-trace operators that survive in this limit are one-half BPS operators and their superconformal descendants, dual to supergravity KK modes. Naively, each single-trace operator $\mathcal{O}$ appearing, e.g., in the s-channel OPE would contribute infinitely many poles to $\mathcal{M}$ at $s = \tau_{\mathcal{O}} + 2m$, $m \in \mathbb{Z}_+$, but in fact this sequence of single-trace poles *truncates* before it would start overlapping with the double-trace poles in Eq. (13) [30]. This truncation is necessary for a consistent OPE interpretation [32].

The same conclusion can be reached by a diagrammatic argument in supergravity. The $O(1/N^2)$ term of $\mathcal{G}_{\text{conn}}$ is given by a (finite) sum of tree-level Witten diagrams: s-, t-, and u-channel exchange diagrams, in correspondence with the single-trace operators exchanged in the respective channel OPE, and additional contact diagrams, arising from quartic vertices. The Mellin amplitude for an s-channel exchange Witten diagram takes the form [19]

$$\mathcal{M}_{\Delta,J}(s,t) = \sum_{m=0}^{\infty} \frac{Q_{J,m}(t)}{s-(\Delta-J)-2m} + P_{J-1}(s,t), \tag{15}$$

where Δ and J are the dimension and spin of the exchanged field, $Q_{J,m}(t)$ are polynomials in t of degree J and $P_{J-1}(s,t)$ is a polynomial in s and t of degree $J-1$. For the values of Δ and J that appear in $\mathrm{AdS}_5 \times \mathrm{S}^5$ supergravity, the sum over m truncates [34], with the same $m_{\max}$ predicted by the principle that there must be no overlap with double-trace poles; namely, $m_{\max} = \frac{\min\{\Delta_1+\Delta_2,\Delta_3+\Delta_4\}-(\Delta-J)}{2}$.

We see from Eq. (15) that exchange diagrams grow at most linearly at large s and t, because $J \le 2$ in supergravity. In Mellin space, a contact diagram is a polynomial in s and t [15], of degree equal to half the number of spacetime derivatives in the quartic vertex. The AdS_5 effective action [8] contains quartic vertices with up to four spacetime derivatives, which would naively give a quadratic asymptotic growth for large s and t, but in fact the final answer is expected to grow at most linearly [35]. Indeed, a larger asymptotic growth would be inconsistent with the flat-space space limit [15].

Bootstrap problem.—We are ready to enumerate several properties of $\mathcal{M}$. First, there are structural algebraic properties, valid for any N and λ.

1. Bose symmetry.—$\mathcal{M}$ is invariant under permutation of the Mandelstam variables, if the quantum numbers of the external operators are permuted accordingly. For example, for equal weights $p_i = p$, this gives the usual crossing relations

$$\sigma^p \mathcal{M}(u,t;1/\sigma,\tau/\sigma) = \mathcal{M}(s,t,;\sigma,\tau),$$
$$\tau^p \mathcal{M}(t,s;\sigma/\tau,1/\tau) = \mathcal{M}(s,t;\sigma,\tau), \tag{16}$$

where u was defined in Eq. (14).

2. Superconformal Ward identity.—We need to translate Eq. (9) into Mellin space. In parallel with Eq. (12), we take the integral transform of the dynamical $\mathcal{H}$ function,

$$\widetilde{M}(s,t;\sigma,\tau) = \int_0^\infty dV V^{-t/2+(\min\{p_1+p_4,p_2+p_3\}/2)-1} \times \int_0^\infty dU U^{-s/2+(p_3+p_4)/2-L-1} \mathcal{H}(U,V;\sigma,\tau), \tag{17}$$

and then define

$$\widetilde{\mathcal{M}}(s,t;\sigma,\tau) = \frac{\widetilde{M}(s,t;\sigma,\tau)}{\tilde{\Gamma}_{p_1p_2p_3p_4}}, \tag{18}$$

where $\tilde{\Gamma}_{p_1p_2p_3p_4}$ is obtained by replacing $u \to \tilde{u} = u-4$ in Eq. (13). This shift in u is useful to make the crossing symmetry properties of $\widetilde{\mathcal{M}}$ more transparent. For example, for equal weights,

$$\sigma^{p-2}\widetilde{\mathcal{M}}(\tilde{u},t;1/\sigma,\tau/\sigma) = \widetilde{\mathcal{M}}(s,t,;\sigma,\tau),$$
$$\tau^{p-2}\widetilde{\mathcal{M}}(t,s;\sigma/\tau,1/\tau) = \widetilde{\mathcal{M}}(s,t;\sigma,\tau). \tag{19}$$

With this definition of $\widetilde{\mathcal{M}}$, Eq. (9) is equivalent to

$$\mathcal{M}(s,t;\sigma,\tau) = \hat{R}\circ\widetilde{\mathcal{M}}(s,t;\sigma,\tau), \tag{20}$$

where $\hat{R}$ is given by Eq. (10) with each U^mV^n replaced by a difference operator $\widehat{U^mV^n}$ acting as

$$\begin{aligned}\widehat{U^mV^n}\circ\widetilde{\mathcal{M}}(s,t;\sigma,\tau) &= \widetilde{\mathcal{M}}(s-2m,t-2n;\sigma,\tau) \\ &\times \left(\frac{p_1+p_2-s}{2}\right)_m \left(\frac{p_1+p_3-u}{2}\right)_{2-m-n} \\ &\times \left(\frac{p_1+p_4-t}{2}\right)_n \left(\frac{p_2+p_3-t}{2}\right)_n \\ &\times \left(\frac{p_2+p_4-u}{2}\right)_{2-m-n} \left(\frac{p_3+p_4-s}{2}\right)_m,\end{aligned}$$

with $(h)_n = \Gamma[h+n]/\Gamma[h]$ denoting the Pochhammer symbol. Contrasting Eqs. (9) and (20), it may appear that we have forgotten the term $\mathcal{G}_{\text{free}}$. In fact, the Mellin transform of the free part is "zero" (a sum of delta functions) and can be consistently ignored. While the direct Mellin transform Eq. (12) is unambiguous, the inverse Mellin transform from $\mathcal{M}$ back to $\mathcal{G}_{\text{conn}}$ requires us to prescribe an integration contour—one must integrate inside the "fundamental strips" for s and t where the integrals in Eq. (12) converge. The correct choice of contour reproduces $\mathcal{G}_{\text{free}}$ automatically. Details will appear in Ref. [33].

Second, we have argued that at leading $O(1/N^2)$ order and for $\lambda \to \infty$, $\mathcal{M}$ becomes a very constrained rational function.

3. Analytic structure.—It follows from the truncation of the sum in Eq. (15) and from the spectrum of $\mathrm{AdS}_5 \times \mathrm{S}^5$ supergravity that $\mathcal{M}$ has a finite number of simple poles in s, t, u, at the locations

$$s_0 = s_M - 2a, \qquad s_0 \ge 2,$$
$$t_0 = t_M - 2b, \qquad t_0 \ge 2,$$
$$u_0 = u_M - 2c, \qquad u_0 \ge 2,$$

where

$$s_M = \min\{p_1+p_2, p_3+p_4\} - 2, \tag{21}$$
$$t_M = \min\{p_1+p_4, p_2+p_3\} - 2, \tag{22}$$
$$u_M = \min\{p_1+p_3, p_2+p_4\} - 2, \tag{23}$$

and a, b, c are non-negative integers. Furthermore, the residue at each pole is a polynomial in the other Mandelstam variable.

4. Asymptotics.—$\mathcal{M}$ grows linearly at large values of the Mandelstam variables:

$$\mathcal{M}(\beta s, \beta t; \sigma, \tau) \sim O(\beta) \quad \text{for } \beta \to \infty. \tag{24}$$

Taken together, these conditions define a very constrained bootstrap problem.

Our solution.—Some experimentation at low KK levels leads us to the ansatz

$$\widetilde{\mathcal{M}}(s,t;\sigma,\tau) = \sum_{\substack{i+j+k=L-2 \\ 0\le i,j,k\le L-2}} \frac{a_{ijk}\sigma^i\tau^j}{(s-s_M+2k)(t-t_M+2j)(\tilde{u}-u_M+2i)}. \tag{25}$$

This is the most symmetric expression compatible with Bose symmetry, the scaling Eq. (24), and the expected pole structure. Imposing that $\mathcal{M} = \hat{R}\circ\widetilde{\mathcal{M}}$ has poles with polynomial residues fixes the coefficients a_{ijk} uniquely, up to overall normalization:

$$a_{ijk} = \left(1+\frac{|p_1-p_2+p_3-p_4|}{2}\right)^{-1}_i \times \left(1+\frac{|p_1+p_4-p_2-p_3|}{2}\right)^{-1}_j \times \left(1+\frac{|p_1+p_2-p_3-p_4|}{2}\right)^{-1}_k \binom{L-2}{i\ j\ k} C_{p_1p_2p_3p_4}, \tag{26}$$

where $\binom{L-2}{i\ j\ k}$ is the trinomial coefficient. The normalization constant $C_{p_1p_2p_3p_4} = f(p_1,p_2,p_3,p_4)/N^2$ cannot be determined from our homogeneous consistency conditions [37]. We have checked that our proposal reproduces all the available supergravity calculations: the equal weights cases $p_i = 2$ [10], $p_i = 3$ [21], and $p_i = 4$ [22], as well as the general expression [23,24,36] for next-to-next extremal correlators (i.e., the cases $p_1 = n+k$, $p_2 = n-k$, $p_3 = p_4 = k+2$) [38]. We have not yet been able to prove, but find it very plausible, that Eq. (25) is the most general ansatz compatible with the bootstrap conditions.

Position space method.—The power of maximal supersymmetry can also be appreciated by an independent method in position space, which will be fully illustrated in Ref. [33]. This method mimics the conventional holographic calculation of correlation functions, writing the answer as a sum of exchange and contact Witten diagrams,

$$\mathcal{A}_{\text{sugra}} = \mathcal{A}_{\text{exchange}} + \mathcal{A}_{\text{contact}}, \tag{27}$$

but it eschews knowledge of the precise cubic and quartic couplings, left as undetermined coefficients. Using the results of Ref. [39], the exchange diagrams are expressed as finite sums of contact diagrams ($\bar{D}$ functions). All in all, one is led to an ansatz in terms a finite sum of $\bar{D}$ functions, depending linearly on a set of coefficients, to be fixed by imposing the superconformal Ward identity. The task of obtaining the correct vertices from the effective action and working out tedious combinatorics is replaced by an easier linear algebra problem. In practice, one uses the fact that $\bar{D}$ functions can be uniquely written as

$$\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} = R_\Phi \Phi(U,V) + R_V \log V + R_U \log U + R_0, \tag{28}$$

where $\Phi(U,V) = \bar{D}_{1111}$ is the scalar box diagram and $R_{\Phi,U,V,0}$ are rational functions of the cross ratios U and V. The ansatz for $\mathcal{A}_{\text{sugra}}$ can be decomposed similarly, with rational coefficient functions $R^{\text{sugra}}(z,\bar{z};\alpha,\bar{\alpha})$ that also depend on the R-symmetry cross ratios. The superconformal Ward identity then becomes a set of conditions on the rational coefficient functions,

$$R^{\text{sugra}}_\Phi(z,\bar{z};\alpha,1/\bar{z}) = 0,$$
$$R^{\text{sugra}}_V(z,\bar{z};\alpha,1/\bar{z}) = 0,$$
$$R^{\text{sugra}}_U(z,\bar{z};\alpha,1/\bar{z}) = 0, \tag{29}$$

giving a set of linear equations for the undetermined coefficients. The uniqueness of the maximally supersymmetric action guarantees the existence of a unique solution up to overall rescaling. Finally, the overall normalization is determined by matching the protected part of the correlator with free field theory:

$$R^{\text{sugra}}_0(z,\bar{z};\alpha,1/\bar{z}) = \mathcal{G}_{\text{free}}(z,\bar{z};\alpha,1/\bar{z}). \tag{30}$$

This method is fully rigorous, relying entirely on the structure of the supergravity calculation with no additional assumption. Despite being much simpler than the conventional approach, even this method quickly becomes unwieldy as the KK level is increased. We have so far obtained results for the equal weights correlators with $p = 2, 3, 4, 5$. The result for $p = 5$ is new. It agrees both with our Mellin formula Eq. (25) and with a previous conjecture by Dolan, Nirschl, and Osborn [40], who proposed a general answer for arbitrary equal weights, as a sum of $\bar{D}$ functions. Unfortunately the complexity of their expression grows very rapidly with p, making a check against Eq. (25) very cumbersome for $p > 5$.

Discussion.—The remarkable simplicity of the general formula Eq. (25) is a welcome surprise. Like the Parke-Taylor formula [41] for tree-level maximally helicity violating (MHV) gluon scattering amplitudes, it encodes in a succinct expression the sum of an intimidating number of diagrams. It appears that holographic correlators are much simpler than previously understood. We believe that they should be studied following the blueprint of the modern on-shell approach to perturbative gauge theory amplitudes. While we have obtained Eq. (25) as the solution of a set of bootstrap conditions, a more constructive approach based on on-shell recursion relations (*à la* Britto-Cachazo-Feng-Witten [42]?) may also exist, and lend itself more easily to the generalization to higher n-point correlators [43].

An important direction to pursue is the generalization of our results to include the 't Hooft coupling dependence. For large λ, one could study α' corrections perturbatively, by relaxing the asymptotic behavior Eq. (24). It would be interesting to make contact with the results of Ref. [46]. In the opposite limit of small λ, it would be worthwhile to explore whether a pattern similar to Eq. (25) can be recognized in the Mellin transformation of perturbative correlators [47]. On a more practical note, Eq. (25) implicitly contains a large amount of CFT data, such as the order $O(1/N^2)$ anomalous dimensions of arbitrary double-trace operators in the strong coupling limit, which are proportional to the residues of the double poles in M. These are useful data for comparison with the superconformal bootstrap [49,50], and it will be nice to extract them explicitly.

Finally, a direct generalization of the approach pursued here gives structurally similar results for holographic correlators in $\mathrm{AdS}_7 \times \mathrm{S}^4$, as we shall report elsewhere.

Our work is supported in part by NSF Grant No. PHY-1316617. We are grateful to Gleb Arutyunov, Sergey Frolov, Carlo Meneghelli, João Penenones, and Volker Schomerus for useful conversations.

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[34] This truncation can be seen from the explicit expressions in Ref. [19] and is equivalent to the observation in Ref. DHoker:1999aa that the exchange Witten diagrams relevant for $\mathrm{AdS}_5 \times \mathrm{S}^5$ supergravity can always be written as finite sums of contact diagrams.
[35] In all explicit supergravity calculations performed so far [10,21–24,36] the contribution of vertices with four spacetime derivatives can be effectively rewritten in terms of vertices with zero or two derivatives. It would be nice to prove that this reduction of order occurs in the general case, as we have argued indirectly from compatibility with the flat space limit. We are indebted to G. Arutyunov and S. Frolov for a very useful discussion on this point.
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[38] The supergravity result in Ref. [24] gives an $\mathcal{H} \propto UV^k \bar{D}_{k+2,k+2,n+k+2,n-k}$ from which one immediately finds $\tilde{\mathcal{M}} \propto 1/(s-2k-2)(t-n)(s+t-n-2k)$, in agreement with our Eq. (25).
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