Liouville Field Theory and Log-Correlated Random Energy Models

Xiangyu Cao, Alberto Rosso, and Raoul Santachiara
LPTMS, CNRS, Université Paris-Sud, Université Paris-Saclay, 91405 Orsay, France

Pierre Le Doussal
CNRS-Laboratoire de Physique Théorique de l'École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex, France

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An exact mapping is established between the $c \geq 25$ Liouville field theory (LFT) and the Gibbs measure statistics of a thermal particle in a 2D Gaussian free field plus a logarithmic confining potential. The probability distribution of the position of the minimum of the energy landscape is obtained exactly by combining the conformal bootstrap and one-step replica symmetry-breaking methods. Operator product expansions in the LFT allow us to unveil novel universal behaviors of the log-correlated random energy class. High-precision numerical tests are given.

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The problem of a thermal particle embedded in a log-correlated random potential (log-REMs) plays a key role in many physical systems ranging from 2D localization [1–4] to spin glasses [5–7], the branching process [8–12], and random matrices [13–18]. As a result of the competition between the deep minima of the log-potential and the entropic spreading of the particle, the system undergoes a second-order *freezing transition* between a high-temperature delocalized phase and a low-temperature glassy phase where the particle is frozen in few minima [5,8]. In the simplest realization of such disordered systems, the random potential is sampled from a 2D Gaussian free field (2D GFF). This allowed for exact predictions of free energy and Gibbs measure statistics [19], in cases where the particle is restricted to simple 1D curves drawn on the 2D GFF potential [20–26]. Unfortunately, no results are known in 2D, despite powerful tools of integrability and the conformal field theory, e.g., the Dotsenko-Fateev integrals [27] generalizing the Selberg integrals used for 1D curves.

One of the most studied 2D conformal field theories is the Liouville field theory (LFT) that describes 2D quantum gravity [28–30] and plays an important role in the holography correspondence with $(2+1)$-D gravity; see, e.g., [31,32], and references therein. Although the LFT is an *interacting* theory, it has strong connections to the 2D GFF. Indeed, this is a general feature of conformal field theories, as it is manifest, for instance, in the Coulomb gas approach to critical statistical models [27,33]. This viewpoint underlies also recent mathematical developments [34–37].

Ideas of relating the Gibbs measure statistics in the 2D GFF and the $c \geq 25$ LFT go back to Refs. [1,5] (see [38] for earlier work on LFT–disordered-system connections). Links were found between LFT features (scaling dimension, $c = 25$ barrier) and disordered-system phenomena (multifractal exponents and freezing, respectively). However, as pointed out in Ref. [5], these ideas were not fully exploited, because the asymptotic behavior of the LFT field is subtle to implement in the statistical model under consideration. This Letter reopens the problem using more powerful methods, based on recent progresses in the LFT and in understanding of log-REM freezing transitions. Adding a logarithmic confining potential to the 2D GFF allows us to establish an exact correspondence between the disorder-averaged Gibbs measure in 2D and the LFT four-point function. When carried through the freezing transition, this result leads to predictions for the probability distribution of the positions of the extrema in 2D and also extends to curved surfaces and higher-order Gibbs measure correlations. More generally, we use the short-distance behavior of LFT correlators to give predictions that go beyond the previous setup and apply to *all* log-REMs. This is possible thanks to the well-known dimension independence (universality) of many properties of log-REMs [5,39] and mappings between them. In particular, our results extend to an arbitrary temperature a recent work of Derrida and Mottishaw [40] on the directed polymer on a Cayley tree. The above outline is illustrated in Fig. 1.

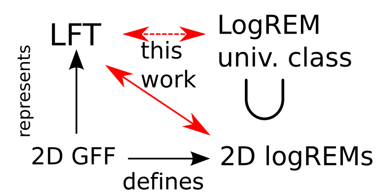

FIG. 1. The map of connections considered in this work. Building on known relations between the LFT and 2D GFF, we establish exact mappings between LFT and log-REMs defined by the 2D GFF [Eqs. (9) and (13)]. Then we exploit the universality of the log-REM class to extend Liouville OPE predictions to all log-REMs, e.g., Eq. (19).

Setup.—Our central object is the normalized Gibbs measure of a particle on the plane:

$$p_\beta(z) \overset{\text{def}}{=} \frac{1}{Z} e^{-\beta[\phi(z)+U(z)]}, \quad z \in \mathbb{C}, \tag{1}$$

$$Z \overset{\text{def}}{=} \int_{\mathbb{C}} e^{-\beta[\phi(z)+U(z)]} d^2 z. \tag{2}$$

Here, Z is the canonical partition function at temperature $1/\beta$, $U(z)$ is a confining potential defined as the sum of two logarithms:

$$U(z) \overset{\text{def}}{=} 4a_1 \ln|z| + 4a_2 \ln|z-1|, \quad a_1, a_2 > 0, \tag{3}$$

and $\phi(z)$ is the 2D GFF. The latter is well defined only in a finite geometry of size R with a lattice spacing ϵ. In the regime $\epsilon \ll |z-w| \ll R$, the covariance is

$$\overline{\phi(z)\phi(w)} = 4\ln(R/|z-w|), \tag{4}$$

supplemented by $\overline{\phi(z)^2} = 4\ln(R/\epsilon)$ and $\overline{\phi(z)} = 0$. Figure 2 shows a simulation of $\phi + U$. To prepare for the field theory connection below, we now discuss the $\epsilon \to 0$, $R \to \infty$, *thermodynamic* limit of the model. For later convenience, the zero mode, immaterial for the Gibbs measure, is adjusted to vanish, i.e., $\int \phi(z) d^2 z = 0$ for each realization.

If one sets $U(z) = 0$, the model belongs to the class of standard log-REMs, for which the mean free energy is universal [modulo an $O(1)$ correction] and displays a freezing transition at $\beta = 1$ [5,8,19]:

$$\overline{F} = -Q\ln M + \eta \ln\ln M + O(1), \qquad M = (R/\epsilon)^2, \tag{5}$$

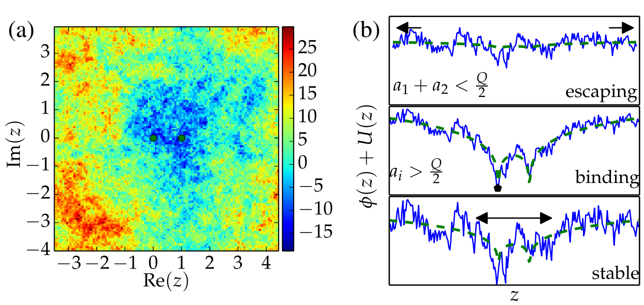

FIG. 2. (a) Color plot of a 2D GFF (4) sample plus the log confining potential $U(z)$ (3) with $a_{1,2} = 0.8,\ 0.6$. The two singularities $z = 0,\ 1$ are indicated by dots. The domain has lattice spacing $\epsilon = 2^{-5}$ and size $R = 8$, with a periodic boundary condition. (b) Top: When Eq. (8) is violated ($a_{1,2} = 0.1$, $Q = 2$), the particle is not confined, and the $R \to \infty$ limit is ill defined. Middle: When Eq. (7) is violated ($a_{1,2} = 2$, $Q = 2$), the particle is trapped, and the Gibbs measure becomes a δ peak as $\epsilon \to 0$. Bottom: When both Eqs. (7) and (8) are met ($a_{1,2} = 0.8,\ 0.6$, $Q = 2$, the extent of the central region is stable as $R \to \infty$, $\epsilon \to 0$.

$$Q = b + b^{-1}, \qquad b = \min(1, \beta). \tag{6}$$

Here, $\eta \ln\ln M$ is the universal subleading correction [5,8,41]. In the $\beta < 1$ phase, it is absent ($\eta = 0$). At the critical point $\beta = 1$, the subleading term appears with $\eta = \frac{1}{2}$. In the glassy phase $\beta > 1$, the leading term $-2\ln M$ displays freezing, and the correction coefficient becomes $\eta = \frac{3}{2}$. Note that the leading behaviors are shared by the *uncorrelated* REM [42], the first signature of the log-correlated universality being the subleading term $\frac{3}{2}\ln\ln M$ [43,44].

When $U(z)$ is turned on, Eq. (5) may not persist. Indeed, when a log-singularity of $U(z)$ (say, at $z = 0$) is too deep, there can be a *binding transition* [5,20] dominating the free energy [see Fig. 2(b), middle]. This happens when the energy at its bottom is $4a_1 \ln\epsilon \ll \overline{F}$ as $\epsilon \to 0$, i.e., when $a_1 > Q/2$. This work excludes such bound phases, in which the Gibbs measure is a trivial δ, by requiring

$$a_1, a_2 < Q/2. \tag{7}$$

Moreover, the potential must also confine the particle at $z \sim O(1)$ in the $R \to +\infty$ limit; otherwise, p_β would be non-normalizable in that limit [see Fig. 2(b), top]. Thus, we require $\overline{F} + U(R) \to +\infty$ as $R \to +\infty$, or

$$a_1 + a_2 > Q/2. \tag{8}$$

When (7) and (8) are satisfied, $p_\beta(z)$ has a well-defined nontrivial limit (in law) as $\epsilon \to 0$, $R \to \infty$ [see Fig. 2(b), bottom]. Thus, adding a confining potential is sufficient to make the position problem well posed. By contrast, the free energy distribution is dominated by long-wavelength fluctuations of ϕ and suffers from an $R \to \infty$ divergence, whose proper subtraction is an open question (see, however, discussions in 1D [20,21,23]).

Connection to LFT in $\beta < 1$ phase.—Let us first introduce some notations. Let $\langle \prod_{i=1}^n \mathcal{V}_{a_i}(z_i) \rangle_b$ be the Euclidean n-point correlation function of the LFT defined on the complex plane plus a point at ∞, $\mathbb{C}\cup\{\infty\}$, and with central charge $c = 1 + 6Q^2$, $Q = b + b^{-1}$. The field $\mathcal{V}_a(z)$ is a primary field with scaling dimension $\Delta_a = a(Q-a)$ [29,45]. We first demonstrate the connection between the Gibbs measure statistics and the LFT on the simplest example. We claim

$$\overline{p_\beta(z)} \overset{\beta<1}{\propto} \langle \mathcal{V}_{a_1}(0)\mathcal{V}_{a_2}(1)\mathcal{V}_b(z)\mathcal{V}_{a_3}(\infty)\rangle_b, \tag{9}$$

where $a_3 = Q - a_1 - a_2$ [51].

In order to show the above identity, we will use the LFT functional integral representation. This is defined, on any closed surface Σ, from the action $\mathcal{S}_b$:

$$\mathcal{S}_b = \int_\Sigma \left(\frac{1}{16\pi}(\nabla\varphi)^2 - \frac{1}{8\pi}Q\hat{R}\varphi + \mu e^{-b\varphi}\right)dA, \quad (10)$$

where μ is the coupling constant, $\hat{R}$ is the Ricci curvature, and dA is the surface element. Note that, in our case, the surface $\Sigma = \mathbb{C}\cup\{\infty\}$ has the topology of a sphere with the curvature concentrated at ∞ and vanishing elsewhere: $\hat{R}(z) = 8\pi\delta^2(z-\infty)$, $dA = d^2z$. In this representation, the primary fields are exponential fields, $\mathcal{V}_a(w) \rightsquigarrow e^{-a\varphi(w)}$, also called *vertex operators*. The four-point correlation function in (9) can be written as

$$K_4 \stackrel{\text{def}}{=} \int \mathcal{D}\varphi e^{-\mathcal{S}_b - b\varphi(z) - a_1\varphi(0) - a_2\varphi(1) - a_3\varphi(\infty)}, \quad (11)$$

where we noted $K_4 \stackrel{\text{def}}{=} \langle \mathcal{V}_{a_1}(0)\mathcal{V}_{a_2}(1)\mathcal{V}_b(z)\mathcal{V}_{a_3}(\infty)\rangle_b$ for better readability. To derive (9), we recall that the Liouville field is decomposed into a zero mode and a fluctuating part, $\varphi(z) = \varphi_0 + \tilde{\varphi}(z)$, where φ_0 is the zero mode [52] (see [36] for recent rigorous work in a related setting). Accordingly, the functional integral is written as $\int \mathcal{D}\varphi = \int_\mathbb{R} d\varphi_0 \int \mathcal{D}\tilde{\varphi}$. Once we performed the integration over φ_0, the one over $\tilde{\varphi}$ can be written as an expectation over the 2D GFF *without* a zero mode, i.e., over ϕ defined in Eq. (4); that is, for any observable $\mathcal{O}$, we have $\int \mathcal{D}\tilde{\varphi} e^{-\int (1/16\pi)(\nabla\tilde{\varphi})^2 d^2 z}\mathcal{O}[\tilde{\varphi}] = \overline{\mathcal{O}[\phi]}$. With these considerations, one can obtain

$$\mu b K_4 = \overline{e^{-a_1\phi(0) - a_2\phi(1) + (a_1+a_2)\phi(\infty) - b\phi(z)}/Z_0}, \quad (12)$$

where $Z_0 = \int_\mathbb{C} e^{-b\phi(z)} d^2 z$. The choice of a_3 in (9) is crucial for the apparition of Z_0^{-1}. Then, a complete-the-square trick allows us to identify the average in (12) to $\overline{p_\beta(z)}$, leading to (9) (see [45] for details).

The above steps generalize easily to the multipoint correlations of powers of the Gibbs measure $p_\beta^{q_i}(z_i) = [p_\beta(z_i)]^{q_i}$, $q_i \geq 0$, with $U(z) = \sum_{j=1}^k 4a_j \ln|z - w_j|$ such that $\forall\ a_j < Q/2$ and $a_{k+1} \stackrel{\text{def}}{=} Q - \sum_{j=1}^k a_j < Q/2$ [compare to (7) and (8)]. The result is stated as

$$\overline{\prod_{i=1}^n p_\beta^{q_i}(z_i)} \stackrel{\beta<1}{\propto} \left\langle \prod_{j=1}^{k+1} \mathcal{V}_{a_j}(w_j) \prod_{i=1}^n \mathcal{V}_{\beta q_i}(z_i) \right\rangle_b, \quad (13)$$

where $w_{k+1} = \infty$ and $\forall\ q_i < Q/(2\beta)$ [45]. Moreover, (13) holds, in general closed surfaces [45]. While mapping Gibbs measure correlations onto LFT correlations on a sphere requires a potential with ≥ 3 singular points [e.g., 0, 1, and ∞ for (3)], on a torus the potential is unnecessary. In general, the sum of the charges must be equal to $Q\chi/2$, where χ is the Euler characteristics of the surface ($\chi = 2$ for the sphere and 0 the torus) [45].

We now use known properties of the LFT to obtain new results for our log-REM model and beyond.

$\beta > 1$ phase.—The four-point function in (9) is invariant under the transform $b \to 1/b$ [53]. Hence, from the *freezing-duality conjecture* [20,22], we expect that $\overline{p_{\beta>1}} = \overline{p_1}$ *freezes*, so the prediction (9) *still holds* thanks to the notation $b = \min(1,\beta)$; this can be also shown by replica symmetry breaking (RSB) [21,54]. Taking the $\beta \to \infty$ limit gives the position distribution of the minimum of $\phi(z) + U(z)$. Note that the freezing of $\overline{p_\beta}$ does *not* imply that of p_β, as revealed by its multipoint correlations. Indeed, $p_{\beta>1}$ develops δ peaks, which are absent when $\beta < 1$ and which give rise to a δ contact singularity in two-point correlations of $p_{\beta>1}$. For example, an RSB calculation as in Refs. [21,54] gives

$$\overline{p_\beta(z_1)p_\beta(z_2)} = (1-T)\delta_{1,2}\overline{p_1(z_1)} + T\overline{p_1(z_1)p_1(z_2)}, \quad (14)$$

where $\delta_{1,2} = \delta(z_1 - z_2)$ and $T = 1/\beta < 1$. We will further apply and discuss this result below; see Eq. (19).

At $\beta \to \infty$, the positions of the deepest minima of the 2D GFF can be also studied by RSB [54] (see also some rigorous results [55]). That allows us to show, for instance, that the joint probability distribution of the first and second minima positions ($\xi_{1,2}$) is (see [45])

$$P(\xi_1, \xi_2) = c_0\delta(\xi_1 - \xi_2)\overline{p_1(\xi_1)} + (1 - c_0)\overline{p_1(\xi_1)p_1(\xi_2)} \quad (15)$$

and thus also relates to the LFT using (13). Here $c_0 = 1 - \bar{g}$ is the probability that the two lowest minima belong to the same "cluster," and g is the energy gap between them, which depends on model-specific details at the $\sim\epsilon$ scale.

Numerical test.—The LFT four-point functions are exactly calculated by the conformal bootstrap [45,53], implemented by the code base [56], and extended to take into account the *discrete terms* (they have important consequences; see below). The lhs of (9) is measured on extensive simulations of the discrete 2D GFF, as shown in Fig. 2. The results validate unambiguously the predictions; see Fig. 3.

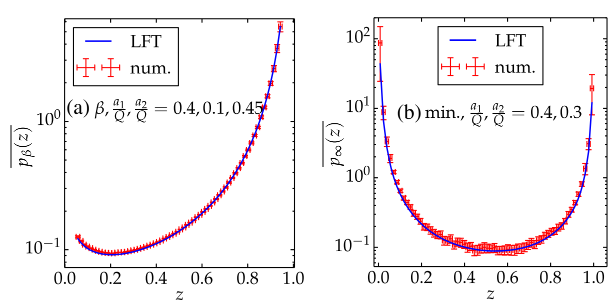

FIG. 3. Test of (9) on the segment $z \in [0,1]$. (a) High-T regime ($\beta = 0.4$). (b) Minimum position distribution versus the LFT with $b = 1$. Numerical parameters: $R = 8$, $\epsilon = 2^{-9}$, and 5×10^6 independent samples.

Now that the advocated relation has been confirmed in a particular setting, the next goal is to extract more universal physical consequences from the LFT.

Liouville OPE.—As can be seen in Fig. 3, $\overline{p_\beta(z)}$ diverges as z comes near a log singularity of the potential $U(z)$, say, as $z \to 0$ where $U(z) \approx 4a_1 \ln|z|$. This asymptotic behavior depends only on β and a_1 and can be obtained from an operator product expansion (OPE) $\mathcal{V}_\alpha(0)\mathcal{V}_{\alpha'}(z)$ [57]. Such OPEs have been obtained by conformal bootstrap [45] and read as follows:

$$\langle \mathcal{V}_a(0)\mathcal{V}_{a'}(z)\ldots\rangle_b \overset{z\to 0}{\sim} \begin{cases} |z|^{-2\delta_0}, & a'' \overset{\mathrm{def}}{=} a+a' < \frac{Q}{2}, \\ |z|^{-2\delta_0}\ln^{-(1/2)}|1/z|, & a'' = \frac{Q}{2}, \\ |z|^{-2\delta_1}\ln^{-(3/2)}|1/z|, & a'' > \frac{Q}{2}, \end{cases}$$
$$\delta_0 = 2aa', \qquad \delta_1 = \Delta_a + \Delta_{a'} - \Delta_{Q/2}, \qquad \Delta_a = a(Q-a). \tag{16}$$

These asymptotic behaviors hold for generic LFT correlations, as long as the distance $|z|$ is much smaller than that to the other operators (as well as R). Note, moreover, that field theory predictions break down when $|z| \sim \epsilon$. To obtain the divergence of $\overline{p_\beta(z \to 0)}$ shown in Fig. 3 from (9), we must set $a = a_1$ and $a' = b$ in (16).

The abrupt behavior change as the parameters cross the line $a + a' = Q/2$ comes from a peculiar feature of the LFT and corresponds to the presence or absence of the discrete terms [30,58–60] (see also [57], Ex. 3.3, and [45]). To discuss the physical consequences of this feature, we consider two independent thermal particles in one realization and the disorder-averaged joint position distribution $\overline{p_\beta(w)p_\beta(z+w)}$. If w is fixed far from singularities, and $z \to 0$, the asymptotic dependence on z is given by Eq. (16) (with $a = a' = b$), independently of the other details. In particular, combining with (15) gives the following asymptotics of the first-second minima position distribution:

$$P(\xi_1, \xi_2) \sim |\xi_1 - \xi_2|^{-2}\ln^{-(3/2)}|1/(\xi_1 - \xi_2)|, \tag{17}$$

which holds for $1 \gg |\xi_1 - \xi_2| \gg \epsilon$ [while the δ in (15) takes over as $|z| \sim \epsilon$].

Beyond 2D.—The robustness of the above results suggests their generalization beyond 2D GFF models to general log-REMs, such as the directed polymer on disordered Cayley tree model [8]. This is the best studied log-REM, due to its relevance in classical (e.g., Kardar-Parisi-Zhang class [61]) and quantum (e.g., Anderson transition [62]) disordered systems. It is defined on a Cayley tree (see Fig. 3 of [45]) of depth t and branching number κ ($\kappa = e$ for branching Brownian motion). Each edge has an independent Gaussian random energy of zero mean and variance $2\ln\kappa$ (so that freezing occurs at $\beta = 1$). A directed polymer (DP) is a simple path from the root to some leaf, and its energy is the sum of the edge energies. Then, the energy of all the DPs $\phi_1, \ldots, \phi_M$, $M = \kappa^t$ are centered Gaussian with covariance

$$\overline{\phi_i\phi_j} = 2\hat{\mathsf{q}}_{ij}\ln\kappa, \tag{18}$$

where $\hat{\mathsf{q}}_{ij} \in [0, t]$ is the common length of i and j. An interesting question is the distribution $P(\hat{\mathsf{q}})$ of the common length of two independent thermal DPs drawn from a single Gibbs measure $p_i \propto e^{-\beta\phi_i}$ for $t \to \infty$. This quantity is *different* from the more studied distribution of the *overlap* $\mathsf{q} = \hat{\mathsf{q}}/t$ for $t \to \infty$ [63]. Our results correspond to the leading finite-t correction of the latter near $\mathsf{q} = 0$.

To calculate $P(\hat{\mathsf{q}})$, we compare positions (distance) in 2D to DPs (common length) on the tree, by matching the respective covariances (4) and (18). This gives $|z| = r = \kappa^{-\hat{\mathsf{q}}/2} \in [\kappa^{-(t/2)} = \epsilon, 1 = R]$, leading to the transformation $P(\hat{\mathsf{q}})d\hat{\mathsf{q}} = \overline{p_\beta(w)p_\beta(w+r)}2\pi r dr$. Then applying (16) leads to [see [45], Eq. (35)]

$$P(\hat{\mathsf{q}}) \sim \begin{cases} \kappa^{(2\beta^2-1)\hat{\mathsf{q}}}, & \beta < 3^{-(1/2)}, \\ \kappa^{-\hat{\mathsf{q}}/3}\hat{\mathsf{q}}^{-(1/2)}, & \beta = 3^{-(1/2)}, \\ \kappa^{-(\beta-\beta^{-1})^2\hat{\mathsf{q}}/4}\hat{\mathsf{q}}^{-(3/2)}, & \beta \in (3^{-(1/2)}, 1) \\ \hat{\mathsf{q}}^{-(3/2)}\beta^{-1}, & \beta \geq 1, \hat{\mathsf{q}} \ll t. \end{cases} \tag{19}$$

For the $\beta \geq 1$ case we used also the RSB result (14), which can be interpreted as follows: With probability $T = 1/\beta$, the common length $\hat{\mathsf{q}}$ remains finite, and the Liouville OPE applies, while with probability $1 - T$, $\hat{\mathsf{q}} \sim O(t)$. In the 2D context, the latter case corresponds to two particles *frozen* at a distance $\sim\epsilon$. The field theory results are valid only in the continuum regime $r \gg \epsilon$, which corresponds to $\hat{\mathsf{q}} \ll t$ for the DP model. For this reason, our $\beta \geq 1$ result matches the exact solution of Ref. [40] when $\mathsf{q} = \hat{\mathsf{q}}/t \ll 1$ and loses validity at $\mathsf{q} \to 1$ ($\hat{\mathsf{q}} \to t$).

The results for the $\beta < 1$ phase are new and display, in the high-T phase, log-corrections typical of the freezing transition. As will be reported in upcoming work, they are universal signatures of the termination point transition (called "*prefreezing*" in [64]) in log-REMs. This transition manifests itself in the *annealed* average of inverse partition ratio $P_q \overset{\mathrm{def}}{=} \sum_{i=1}^M (e^{-\beta\phi_i}/Z)^q$, $q > 0$. Indeed, one can show

$$-\ln\overline{P_q} \overset{\beta<1}{\sim} \begin{cases} \tau(q)\ln M & q\beta < \frac{Q}{2}, \\ \tau(q)\ln M + \frac{1}{2}\ln\ln M & q\beta = \frac{Q}{2}, \\ \tau(q)\ln M + \frac{3}{2}\ln\ln M & q\beta > \frac{Q}{2}, \end{cases} \tag{20}$$

where $\tau(q) = \Delta_{\min(q\beta, Q/2)} - 1$ [see (16)]. Note that, for the uncorrelated REM [64], we would have the same $\tau(q)$ but $\frac{1}{2}\ln\ln M$ correction when $q\beta > Q/2$ [65]. In the LFT, the latter phase is where p_β^q can no longer be represented by $\mathcal{V}_{q\beta}$ as it would violate a Seiberg bound [(13) and [45]].

Conclusion.—We related the $c \geq 25$ LFT to the Gibbs measure of the 2D GFF plus a log potential and found indications that the LFT may describe universal features of general log-REMs. We mention two exciting perspectives. The first is extending the mapping to log-REMs with imaginary temperature ($b \to \mathbf{i}b$), where relations to $c \leq 1$ conformal field theories, β-random matrix ensembles, and 2D log-gases are natural to expect. The other concerns the glassy phase $\beta > 1$, in which the LFT must be supplemented by an RSB or freezing-duality conjecture in order to make correct predictions. However, the termination point transition predicted by the LFT alone resembles strikingly the freezing transition. This points to the intriguing question: Does the glassy phase have a field theory description?

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