

Critical Behaviors in Contagion Dynamics

L. Böttcher,^{*} J. Nagler,[†] and H. J. Herrmann

ETH Zurich, Wolfgang-Pauli-Strasse 27, CH-8093 Zurich, Switzerland

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We study the critical behavior of a general contagion model where nodes are either active (e.g., with opinion *A*, or functioning) or inactive (e.g., with opinion *B*, or damaged). The transitions between these two states are determined by (i) spontaneous transitions independent of the neighborhood, (ii) transitions induced by neighboring nodes, and (iii) spontaneous reverse transitions. The resulting dynamics is extremely rich including limit cycles and random phase switching. We derive a unifying mean-field theory. Specifically, we analytically show that the critical behavior of systems whose dynamics is governed by processes (i)–(iii) can only exhibit three distinct regimes: (a) uncorrelated spontaneous transition dynamics, (b) contact process dynamics, and (c) cusp catastrophes. This ends a long-standing debate on the universality classes of complex contagion dynamics in mean field and substantially deepens its mathematical understanding.

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In 1972 Schlögl proposed two models describing autocatalytic chemical reactions [1] that are commonly known today as Schlögl's first and Schlögl's second model (henceforth referred to as Schlögl I and Schlögl II). Schlögl I, also known as contact process [2], comprises the important case of simple contagion, i.e., the susceptible-infected-susceptible (SIS) model where healthy individuals can be infected due to the exposure to a single infectious source, eventually leading to the spread of an epidemic disease [2–5]. In contrast, Schlögl II, also known as quadratic contact process [6], requires contact with two sources. Later studies on Schlögl II sparked a debate on its critical behavior and Grassberger noticed in 1982 that a relation to the Ising universality class “would be a most remarkable extension of the universality hypothesis, from models with detailed balance to models without it” to conclude that Schlögl II “is not an example of universality between models with and without detailed balance” [7].

Closely related to this debate, but more recently, a generalized model of Schlögl II has been proposed where an arbitrary number of sources is necessary to induce a transition [8]. The study of the model's mean-field critical behavior led the authors to conjecture that such general failure-recovery dynamics belong to the Ising universality class [9]. This model is of particular interest since it not only includes simple contagions but also complex contagion phenomena such as the diffusion of innovations [10,11], political mobilization [12], and viral marketing [13] that require social reinforcement, i.e., the connection to multiple sources [14,15]. The model displays an intricate and very rich dynamics including hysteresis effects, limit cycles, and cusp catastrophes [9,16–20]. Thus, a unifying mean-field theory of the critical behavior is essential for a broad range of dynamical systems.

However, the relation to contact process dynamics and cusp catastrophes has only been shown for specific values of the model's parameters [20]. But, given the model's parameter regime, can we generally predict the dynamics type? And does the model's mean-field critical behavior belong to the Ising universality class or not? Here we answer these questions and analytically demonstrate that the mean-field critical behavior of the model is restricted to only three possible regimes: (a) uncorrelated spontaneous transition dynamics, (b) contact process dynamics, and (c) cusp catastrophes. Cusp catastrophes can display abrupt transitions and hysteresis effects—phenomena that can harm the proper functioning of real-world networked systems since small variations in the system's control parameters may cause catastrophic transitions from a seemingly well-functioning state to global malfunction or severe outages [21–30].

Model.—The general contagion dynamics is defined in a network whose constituents (i.e., nodes) are regarded as either active (e.g., not damaged) or inactive (e.g., failed). Three fundamental processes define the transitions between these two states [9,20]: (i) nodes undergo a spontaneous transition $A \rightarrow X$ from an active (*A*) to an inactive state (*X*) in a time interval dt with probability pdt ; (ii) if fewer than or equal to m nearest neighbors of a node are active, the node becomes inactive (*Y*) due to an induced transition, i.e., $A \rightarrow Y$, with probability $r dt$, and (iii) a spontaneous reverse transition with probability $q dt$ if $X \rightarrow A$ or probability $q' dt$ if $Y \rightarrow A$. The inactive states *X* and *Y* only differ in their reverse transitions and are equivalent if $q = q'$. Process (ii) describes that a node with degree k can become inactive if its number of inactive neighbors is larger or equal to $k - m$. Similar to threshold models describing complex contagion phenomena, the threshold m defines the number of contacts to inactive nodes that is necessary to induce a

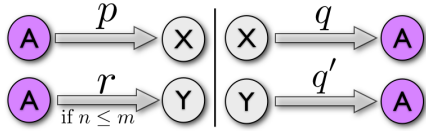


FIG. 1. Model. Spontaneous failure ($A \rightarrow X$) and spontaneous recovery ($X \rightarrow A$) occurs with rates p and q , respectively. A node may also fail (become inactive) dependent on its neighborhood, if too few active nearest neighbors $n \leq m$ sustain the node's activity ($A \rightarrow Y$ with rate r). In addition, a failed node Y recovers ($Y \rightarrow A$) with rate q' . Active nodes (A) are purple while failed ones (X and Y) are grey.

transition as defined by process (ii) [14,31–33]. A low value of m corresponds to the situation where many inactive neighbors are required to sustain spreading. In contrast, for a large value of m only a few inactive neighbors can sustain the spreading process. Processes (i)–(iii) are illustrated in Fig. 1.

Let $a(t) \in [0, 1]$ denote the total fraction of inactive nodes. Thus, $a(t) = u_{\text{spon}}(t) + u_{\text{ind}}(t)$ with $u_{\text{spon}}(t)$ and $u_{\text{ind}}(t)$ being the fractions of nodes that are inactive due to spontaneous and induced transitions, respectively. The total fraction of inactive nodes in the stationary state is referred to as a_{st} . In accordance with Ref. [20] we derive the mean-field rate equations by assuming a system with homogeneous degrees in the thermodynamic limit that exhibits perfect mixing. Here, perfect mixing either refers to a network of randomly connected nodes with a sufficiently large mean degree or dynamical rewiring [3,34]. For the fraction of nodes that spontaneously became inactive we find

$$\dot{u}_{\text{spon}} = p(1 - a) - qu_{\text{spon}}, \quad (1)$$

where the first term accounts for the fact that active nodes spontaneously become inactive with rate p [process (i)] and the second term corresponds to the spontaneous reverse transition with rate q [process (iii)]. Equation (1) is exact since the network structure is not influencing these spontaneous transitions.

Induced transitions [process (ii)] can only occur for nodes whose number of active neighbors is smaller than or equal to m . Under the assumption of a perfectly mixed population, the probability that a node of degree k is located in such a neighborhood is $E_k = \sum_{j=0}^m \binom{k}{k-j} a^{k-j} (1-a)^j$ [9,20]. The time evolution of the fraction of nodes that are inactive due to induced transitions is therefore given by

$$\dot{u}_{\text{ind}} = r \sum_k f_k E_k (1-a) - q' u_{\text{ind}}, \quad (2)$$

with f_k being the degree distribution. The first term describes the occurrence of induced transitions [process (ii)] with rate r of active nodes in a neighborhood where the number of active neighbors is smaller than or equal to m , whereas the second term accounts for the spontaneous reverse transition to an active state with rate q' [process

(iii)]. In order to study the influence of different threshold values m on the mean-field critical behavior of Eqs. (1) and (2), we consider a regular network with degree k , i.e., the degree distribution $f_k = \delta_{kk'}$. We demonstrate below that the model defined by processes (i)–(iii) can only exhibit three different regimes depending on the choice of m . It is important to notice that for more general degree distributions the mean-field critical behavior still falls into these classes; see Supplemental Material [35].

The coupled equations (1) and (2) admit oscillatory behavior for $q' > q$ [20] as a dynamical feature that does not belong to the critical behavior [18]. The equations describing the critical behavior, i.e., $\dot{u}_{\text{spon}} = 0$ and $\dot{u}_{\text{ind}} = 0$, can be decoupled by multiplying one of them with an appropriate constant excluding limit cycles [18]—tantamount to setting $q = q' = 1$. This yields

$$\dot{a} = f(a, r, p) = rS(a) + p(1 - a) - a, \quad (3)$$

with $S(a) = \sum_{j=0}^m \binom{k}{k-j} a^{k-j} (1-a)^{j+1} = (1-a)E_k$. We use $S(a)$ as shorthand notation for the probability that an active node is located in a neighborhood that is able to induce a transition. Thus, differences in the inactive states X and Y , i.e., different q and q' , do not influence the critical behavior of Eq. (3) but only rescale r and p . In the following we analyze the stationary states of Eq. (3) that are reached in the long-time limit.

Class (a): Uncorrelated spontaneous transitions.—We start with the case $m = k$ where the number of active nodes necessary to sustain spreading has to be smaller or equal to the node's degree k according to the definition of process (ii). This describes the regime where spreading occurs independently of the neighborhood's state such as in exogenously driven adoption dynamics [36,37],

$$\dot{a} = (r + p)(1 - a) - a \quad (4)$$

since $E_k = 1$ and $S(a) = 1 - a$. Equation (4) has only one stationary state, i.e., $a_{st}(r, p) = (r + p)/(1 + r + p)$; see Fig. 2 (left).

Class (b): Contact dynamics.—By definition $m = k - 1$ implies that $k - 1$ or fewer neighbors of a node have to be active to induce a transition. This case describes a contact process where one inactive neighbor is sufficient to sustain spreading [3]. As demonstrated in Supplemental Material [35], we find for $p = 0$ that there exists a critical $r_c = k^{-1}$ separating an absorbing and an active phase, i.e., $a_{st}^{(1)}(r, 0) = 0$ as $r \leq r_c$ and $a_{st}^{(2)}(r, 0) > 0$ as $r > r_c$. In the limit of $r \rightarrow r_c$ Eq. (3) takes the form

$$\dot{a} = rka(1 - a) - a. \quad (5)$$

Equation (5) describes the mean-field contact process, SIS dynamics, or Schlögl I [3,7]. In the limit of $r \rightarrow r_c$ the order parameter scales as $a_{st}(r, 0) \sim |r - r_c|^\beta$ with $\beta = 1$ and adding the fieldlike contribution $p(1 - a)$ to Eq. (5) yields

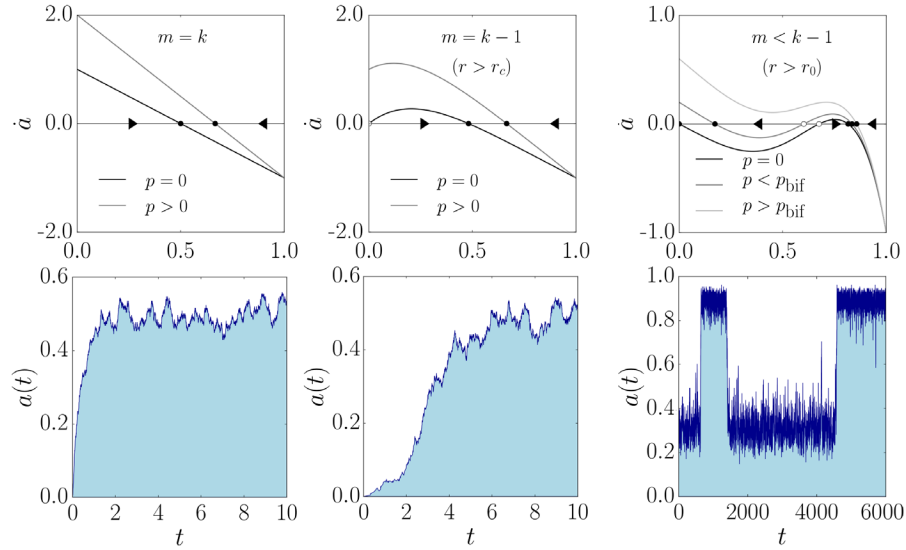


FIG. 2. Phase portraits of the general contagion model's regimes. (Top panel) The phase portrait is shown for $k = 4$ and different values of m with arrows indicating the sign of \dot{a} (right arrow, $\dot{a} > 0$; left arrow, $\dot{a} < 0$). Black circles correspond to stable fixed points and white circles to unstable ones. (Left) For $m = k$ there exists only one stable fixed point, whose position depends on p . We set $r = 1$ and for the grey solid line $p = 1$. (Center) For $m = k - 1$ the dynamics resembles the phase space of a contact process with a stable nonzero fixed point for $r > r_c$, where r_c defines the critical spreading rate below which a approaches 0 if $p = 0$. For $p > 0$ and $r > r_c$ the second-order phase transition gets smeared out. We set $r = 1$ and for the grey solid line $p = 1$. (Right) If $m < k - 1$, r exceeding r_0 and $0 < p < p_{\text{bif}}$ (inside the hysteresis region enclosed by the bifurcation lines) it is possible to find two stable fixed points. For $p > p_{\text{bif}}$ the dynamics exhibits only one stable fixed point. We set $r = 10$ and $m = 0$. For the grey solid line ($p < p_{\text{bif}}$) we set $p = 0.2$ and for the silver solid line ($p > p_{\text{bif}}$) we used $p = 0.6$. (Bottom panel) The time evolution for different parameters in a regular random graph with $N = 256$ nodes and $k = 4$. (Left) For $m = k = 4$, $p = 0.01$, $r = 1.0$ the dynamics grows until a stationary state is reached. (Center) For $m = k - 1 = 3$, $p = 0.01$, $r = 1.0$ we find the typical logistic growth pattern. (Right) For $m < k - 1 = 1$, $p = 0.24$, $r = 10$ we encounter phase switching.

$a_{st}(r_c, p) \sim p^{1/\delta_h}$ as $p \rightarrow 0$ with the field exponent $\delta_h = 2$. We illustrate in Fig. 2 (center) the occurrence of only one stable fixed point for $p > 0$. This also results in a smeared-out transition for $p > 0$ instead of a second-order phase transition for $p = 0$ as shown in Fig. 3 (top panel). For $p = 0$ and $r > r_c$ one clearly sees that $a_{st}^{(1)}(r, 0)$ is unstable but $a_{st}^{(2)}(r, 0)$ is a stable fixed point.

Class (c): Cusp catastrophes.—For some values of m , we find a metastable region as illustrated in Fig. 3 (right). Inside this hysteresis region two stable fixed points coexist. Phase switching is observed when fluctuations in systems of finite size push the dynamics close to the unstable fixed point, cf., Fig. 2 (right). Between the switching events the dynamics remains in one of the two phases for some time. The waiting times thus depend on the fluctuation strength and the distance from one phase to the unstable state in the phase portrait, cf., Fig. 2 (right). For $m < k - 1$ where either two or more inactive neighbors are necessary to induce a transition, we show that the corresponding metastable regions always exist due to the relation to cusp catastrophes [17]. For a detailed analytical treatment we refer the reader to Supplemental Material [35]. The cusp point where the two bifurcation lines intersect [cf., Fig. 3 (right)] is given by $a_0(k, m) = (k - 1 - m)/(k + 1)$ together with the corresponding control parameters,

$$r_0(k, m) = \frac{1}{S(a_0) + S'(a_0)(1 - a_0)}, \quad (6)$$

and

$$p_0(k, m) = \frac{S'(a_0)a_0 - S(a_0)}{S(a_0) + S'(a_0)(1 - a_0)}. \quad (7)$$

We illustrate the influence of different values of m on (r_0, p_0) and on the extent of the hysteresis area in Supplemental Material [35]. Studying Eq. (3) in the vicinity of (a_0, r_0, p_0) , i.e., setting $a = a_0 + \tilde{a}$, $r = r_0 + \tilde{r}$, $p = p_0 + \tilde{p}$, yields for the Taylor expansion (omitted tilde)

$$\dot{a} = rS(a_0) + p(1 - a_0) + a[rS'(a_0) - p] + \frac{r_0}{6}S'''(a_0)a^3. \quad (8)$$

We thus find by setting p or r to 0 respectively and solving for the fixed point of Eq. (8),

$$a(r) = \left(-\frac{6S(a_0)}{r_0S'''(a_0)} \right)^{1/3} r^{1/3} + \mathcal{O}(r^{2/3}), \quad (9)$$

$$a(p) = \left(-\frac{6(1 - a_0)}{r_0S'''(a_0)} \right)^{1/3} p^{1/3} + \mathcal{O}(p^{2/3}). \quad (10)$$

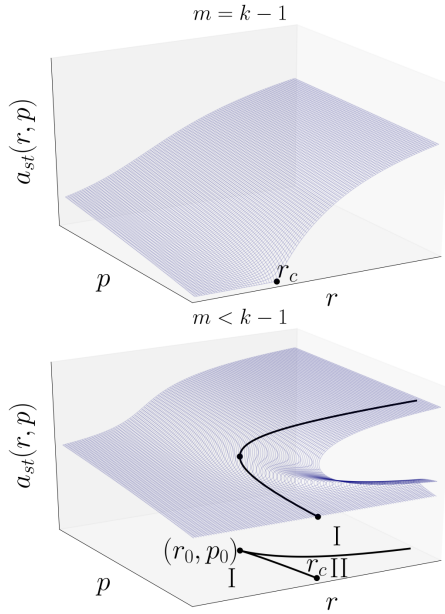


FIG. 3. Phase diagrams of contact dynamics and cusp catastrophes. (Top panel) In the case of $m = k - 1$, the phase space corresponds to the one of Schlögl I or contact dynamics. For vanishing p , a second-order phase transition from an absorbing to an active phase occurs at r_c . For a positive value of p , the transition gets smeared out [3,38]. (Bottom panel) For all $m < k - 1$, cusp catastrophes define the phase space. Two in the thermodynamic limit stable states coexist inside the hysteresis region (II) that is surrounded by bifurcation lines (black solid lines) merging at the cusp point (r_0, p_0) . Outside the hysteresis region (I) only one stable state exists. The critical point r_c is defined as the transition point for vanishing p .

In previous work, the critical behavior at the cusp point of a regular random network ($k = 10$ and $m = 4$) has been conjectured to belong to the Ising universality class although by definition the dynamics corresponds to a general contact process [20].

Final remarks.—We find that the critical behavior of the general contagion model as formulated in Eq. (3) does not belong to the Ising universality class but to exactly three regimes. The first regime, $m = k$, corresponds to purely

spontaneous failure and recovery dynamics. For $m = k - 1$ the model recovers the critical behavior of the contact process. A cusp catastrophe is found for all $m < k - 1$ with the typical critical behavior at the cusp point [Eqs. (9) and (10)]. This sheds analytical light onto a broad range of spreading processes that are determined by the network's connectivity k and the threshold parameter m , cf., examples in Table I.

We have demonstrated that the phase diagram corresponds to a cusp catastrophe, when two or more inactive nodes are needed to trigger induced node transitions. This scenario typically implies dramatic and uncontrollable global transitions in the network for many systems involving complex contagion dynamics. One could naively expect that it could be beneficial for failure control to design systems such that a component only fails if many of its neighbors already failed, i.e., delaying the failure dynamics. Our results suggest, however, that this delaying procedure might facilitate uncontrollable transitions, hence achieving exactly the opposite as initially intended. This result agrees well with previous findings on delaying procedures that have been applied to a SIS model [44,45]. For low spatial dimensions or highly structured networks, the assumptions of perfect mixing or independent node-to-node interactions are not guaranteed. Still mean-field approximations qualitatively describe a given dynamics [3,4,46]; see examples given in Supplemental Material [35].

Future work should establish the behavior of transients as a function of threshold parameter m and the topology of the network. It has been demonstrated that opinions as well as coinfections may spread faster in clustered networks compared to random ones [42,47]. This links our result to the multiple exposure condition in complex contagion phenomena.

In the study of collective behaviors, such as the adoption of innovations, the distinction between exogenous and endogenous factors is of great interest but often solely based on a contact processlike adoption model [36,37]. Our results suggest studying these processes within our more general framework that incorporates contact processlike adoption as one special case and can account for spreading that relies on multiple contacts.

TABLE I. Examples of models and processes that are related to the classes (a)–(c).

(a) $m = k$	(b) $m = k - 1$	(c) $m < k - 1$
Exogenous factors influencing adoption of innovations [37] ^a Social response to exogenous factors [36] ^b	Schlögl I [1,7] Contact process [2,3,38] ^a SIS model [4,5] ^a Reggeon field theory [39] ^a Directed percolation [40] ^a Bass model [37,41] ^a	Schlögl II [1,7] ^a Quadratic contact process [6] ^a General contact process [8] ^a Behavioral adoption [42] ^b Threshold models of complex contagions [11,13–15,31–33] ^{ab} or coordination games [43] ^b

^aExact mean-field correspondence.

^bPhenomenological correspondence.

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*lucasb@ethz.ch

†jnagler@ethz.ch

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