Conformal Bootstrap in Mellin Space

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We propose a new approach towards analytically solving for the dynamical content of conformal field theories (CFTs) using the bootstrap philosophy. This combines the original bootstrap idea of Polyakov with the modern technology of the Mellin representation of CFT amplitudes. We employ exchange Witten diagrams with built-in crossing symmetry as our basic building blocks rather than the conventional conformal blocks in a particular channel. Demanding consistency with the operator product expansion (OPE) implies an infinite set of constraints on operator dimensions and OPE coefficients. We illustrate the power of this method in the ϵ expansion of the Wilson-Fisher fixed point by reproducing anomalous dimensions and, strikingly, obtaining OPE coefficients to higher orders in ϵ than currently available using other analytic techniques (including Feynman diagram calculations). Our results enable us to get a somewhat better agreement between certain observables in the 3D Ising model and the precise numerical values that have been recently obtained.

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Introduction.—The Wilsonian paradigm [1–3] for quantum field theories puts the scale invariant fixed points of the renormalization group on center stage. In the context of relativistic quantum field theories (QFTs), these critical points are believed to be conformally invariant [4]. The study of such conformal field theories (CFTs) is thus central to many areas of physics. Unfortunately, we currently have very few tools to access the dynamics of such CFTs, apart from cases where they are free or close to free. The dynamical data of a CFT are entirely in its spectrum of dimensions of primary operators as well as their three point functions [or operator product expansion (OPE) coefficients]. In principle, conformal invariance and associativity of the OPE in the four point function give powerful constraints on these data [5,6]. In practice, apart from two dimensions [7], this constraint has been difficult to effectively implement.

Recently, there has been a successful revival [8,9] of this bootstrap program, in which associativity and positivity constraints have been translated into inequalities which can be efficiently implemented numerically using linear programing [8], semidefinite programing [10], and judicious truncation [11]. This has led to rather amazing constraints on the low-lying spectrum (as well as the OPE coefficients) of various nontrivial CFTs—see Ref. [9] for references. These numerical techniques now give the best data on the low-lying operators of the 3D Ising model [10,12,13] and hint at there being special points in the domains allowed by the inequalities.

Here, we will outline a new approach to the conformal bootstrap for CFT_d which is calculationally effective, as well as being conceptually suggestive. This involves two ingredients which turn out to blend very naturally. The first involves revisiting an approach of Polyakov which has crossing symmetry from the outset but is not obviously compatible with the operator expansion. We will implement this approach in terms of conformally invariant building blocks which are exchange Witten diagrams in $(d+1)$-dimensional anti–de Sitter space (AdS_{d+1}) rather than the conventional conformal blocks. In other words, for a four point function of identical external scalars, we expand the amplitude as a function of cross ratios (u, v) in terms of the functions $W^{(s)}_{\Delta,\ell}(u, v)$, which can be written in terms of an integral of an AdS_{d+1} bulk to bulk propagator (corresponding to a CFT_d operator of dimension Δ and spin ℓ) together with bulk to boundary propagators for the four external scalars of the dimension Δ_ϕ:

$$\begin{aligned}\mathcal{A}(u,v) &= \langle \mathsf{O}(1)\mathsf{O}(2)\mathsf{O}(3)\mathsf{O}(4)\rangle \\ &= \sum_{\Delta,\ell} c_{\Delta,\ell}[W^{(s)}_{\Delta,\ell}(u,v) + W^{(t)}_{\Delta,\ell}(u,v) + W^{(u)}_{\Delta,\ell}(u,v)]. \end{aligned} \tag{1}$$

The sum here is over the entire physical spectrum of primary operators generically characterized by the dimensions (Δ) and spin (ℓ). The coefficients $c_{\Delta,\ell}$ are proportional to (the square of) the OPE coefficients. The central observation of Polyakov [6] was that there are spurious powers, in this case, u^{Δ_ϕ} (and $u^{\Delta_\phi}\ln u$) in such an expansion. Demanding cancellations of these terms (as a function of v) gives an infinite number of constraints on Δ as well as the coefficients $c_{\Delta,\ell}$.

The second ingredient exploits the Mellin representation [14–18] of CFT amplitudes, which is a close counterpart of momentum space in the usual QFTs. For the above amplitude, this representation is essentially a Mellin transform with respect to the cross ratios:

$$\mathcal{A}(u,v)=\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{dt}{2\pi i}u^s v^t \rho_{\Delta_\phi}(s,t)\mathcal{M}(s,t). \quad (2)$$

Here, $\rho_{\Delta_\phi}(s,t)=\Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi-s)$ is a convenient kinematic factor, while $\mathcal{M}(s,t)$ contains the dynamics. The integral is evaluated by closing the contour appropriately, picking up the poles of the integrand. The expansion in Eq. (1) can now be translated into Mellin space. Each of the Witten exchange functions $W^{(s)}_{\Delta,\ell}(u,v)\to M^{(s)}_{\Delta,\ell}(s,t)$ will be discussed below. We use these functions as our basis for an expansion of the (reduced) Mellin amplitude [19]:

$$\mathcal{M}(s,t)=\sum_{\Delta,\ell}c_{\Delta,\ell}[M^{(s)}_{\Delta,\ell}(s,t)+M^{(t)}_{\Delta,\ell}(s,t)+M^{(u)}_{\Delta,\ell}(s,t)]. \quad (3)$$

The spurious powers u^{Δ_ϕ} (and $u^{\Delta_\phi}\ln u$) in $\mathcal{A}(u,v)$ translate into spurious single and double poles in the full Mellin amplitude

$$\Gamma^2(\Delta_\phi-s)\mathcal{M}(s,t)=\frac{q^{(2)}_{\rm tot}(t)}{(s-\Delta_\phi)^2}+\frac{q^{(1)}_{\rm tot}(t)}{(s-\Delta_\phi)}+\cdots. \quad (4)$$

The ellipsis refers to physical contributions as well as spurious descendant poles. Compatibility with the operator expansion demands that we set both residues,

$$q^{(a)}_{\rm tot}(t)=\sum_{\Delta,\ell}c_{\Delta,\ell}[q^{(a,s)}_{\Delta,\ell}(t)+q^{(a,t)}_{\Delta,\ell}(t)+q^{(a,u)}_{\Delta,\ell}(t)]=0, \quad (5)$$

for $(a=1,2)$. The terms on the rhs come from the obvious expansion of the individual terms in Eq. (3) in terms of the poles, as in Eq. (4). This is our central constraint equation.

We will see that this scheme is calculationally effective by revisiting the ϵ expansion in $(d=4-\epsilon)$ dimensions for a single real scalar at the Wilson-Fisher fixed point. We will find that we can reproduce the answers [2,3] for the dimensions of ϕ and ϕ^2 to $O(\epsilon^3)$ and $O(\epsilon^2)$, respectively. For the higher spin currents $J^{(\ell)}$ of the schematic form $\phi\partial^\ell\phi$, we reproduce the known anomalous dimensions to $O(\epsilon^2)$ [3] as well as to the $O(\epsilon^3)$ piece [20]. More nontrivially, we also determine OPE coefficients, which are usually difficult to compute using Feynman diagram techniques. Thus, we find, for the first time, the three point function C_ℓ of two ϕ's with $J^{(\ell)}$ to $O(\epsilon^3)$. Specifically, this enables one to compute the central charge c_T, which is related to the stress tensor coefficient $C_{\ell=2}$ to this order. Similarly, we will also indicate how to reproduce and go beyond some of the existing results for large spin operators, which were obtained with the (double) light cone expansion [21–23].

Implementing the Mellin bootstrap.—Witten exchange functions $W^{(s)}_{\Delta,\ell}(u,v)$ are computed from a tree level four point function with the exchange of a field in AdS_{d+1} of spin ℓ (and corresponding to a conformal dimension Δ on the boundary) in the s channel [24]. By construction, they preserve all the isometries of AdS_{d+1} and are conformally covariant. Their expressions are, unfortunately, quite complicated in position space [25]. As has been stressed in the literature, there is dramatic simplification in Mellin space. Thus, for a scalar exchange, $M^{(s)}_{\Delta,\ell=0}(s,t)$ can be written in terms of a ${}_3F_2$ hypergeometric function (evaluated at unit argument). See, for instance, Refs. [15,16]. It is a meromorphic function (only of s, in this case) which has simple (physical) poles at $2s=\Delta+2m$, where $m=0,1,2\ldots$.

It is more generally true that $M^{(s)}_{\Delta,\ell}(s,t)$ is the sum of a meromorphic function with poles at $2s=\Delta-\ell+2m$ plus an additional polynomial in (s,t) of a degree of at most $\ell-1$. Thus, our building blocks are polynomially bounded in Mellin space, unlike the conformal blocks, which have an exponential behavior [26,27]. This is what makes them a better choice for a basis to expand in terms of. Moreover, they exhibit the right factorization on the physical poles in having the same residues as the conformal blocks. The way the Witten exchange functions differ from the conformal blocks is that, unlike the latter, they additionally contain the contribution of so-called double trace operators. These are operators of dimension for, e.g., $(\Delta_1+\Delta_2)$—i.e., $2\Delta_\phi$ in our case of identical scalars. In a large N CFT, there are indeed physical operators with this dimension ("two particle states") with $(1/N)$ corrections. However, in a generic CFT this is not the case, and the term "double trace" operators is a misnomer for these contributions [28]. They are really spurious contributions which need to cancel out in the full amplitude—in both position space and Mellin space, as discussed above.

Many of these properties of Witten diagrams are transparent in a spectral (or "split") representation of these diagrams [15,16,29]. In position space, this can be used to write the Witten functions as

$$W^{(s)}_{\Delta,\ell}(u,v)=\int_{-i\infty}^{i\infty}d\nu\mu_{\Delta,\ell}(\nu)F^{(s)}_{\nu,\ell}(u,v), \quad (6)$$

where the conformal partial waves $F^{(s)}_{\nu,\ell}(u,v)$ are purely kinematic in nature—their exact form can be found, e.g., in Ref. [30] and will not be important in the following. The spectral function for identical external scalars given by $2\pi i\mu_{\Delta,\ell}(\nu)=\xi_{\Delta,\ell}(\nu)\xi_{\Delta,\ell}(-\nu)$, with

$$\xi_{\Delta,\ell}(\nu)=\frac{\Gamma^2(\frac{2\Delta_\phi-h+\ell+\nu}{2})}{[(\Delta-h)+\nu]\Gamma(\nu)(h+\nu-1)_\ell}, \quad (7)$$

contains the information about the exchanged operators. The poles (in ν) are at the physical value (together with its shadow) $h\pm\nu=\Delta$ (where $h=(d/2)$). However, there are additional poles corresponding to the double trace operator $2\Delta_\phi$ [31].

In Mellin space we can write the corresponding spectral representation as

$$M^{(s)}_{\Delta,\ell}(s,t) = \int_{-i\infty}^{i\infty} d\nu \mu_{\Delta,\ell}(\nu)\Omega^{(s)}_{\nu,\ell}(s)P^{(s)}_{\nu,\ell}(s,t). \quad (8)$$

The conformal partial waves go over to a set of so-called Mack polynomials $P^{(s)}_{\nu,\ell}(s,t)$ of degree ℓ in (s,t) [14,30,32]. We also have an additional factor

$$\Omega^{(s)}_{\nu,\ell}(s) = \frac{\Gamma(\frac{h+\nu-\ell}{2}-s)\Gamma(\frac{h-\nu-\ell}{2}-s)}{\Gamma^2(\Delta_\phi - s)}. \quad (9)$$

We now pick out the spurious poles as in Eq. (4). First, Eq. (9) has a denominator piece which cancels against the $\Gamma^2(\Delta_\phi - s)$ in $\rho_{\Delta_\phi}(s,t)$. Second, note that the poles at $2\Delta_\phi - h + \ell - \nu = 0$ in the numerator of Eq. (7) give rise, upon doing the ν integration, to the required single and double spurious poles at $s = \Delta_\phi$. Finally, we observe that the Mack polynomial defines, through

$$Q^{\Delta}_{\ell,0}(t) = \frac{4^\ell}{(\Delta-1)_\ell(2h-\Delta-1)_\ell} P^{(s)}_{\Delta-h,\ell}\left(s = \frac{\Delta-\ell}{2}, t\right), \quad (10)$$

a single variable orthogonal polynomial (labeled by ℓ and explicitly expressible in terms of hypergeometric functions) known as a continuous Hahn polynomial—see Refs. [32,33] for details. They are the analogue of Legendre polynomials in a partial wave expansion. This is a particularly nice feature of the Mellin expansion since it gives us a way to decompose the residues in Eq. (4) in a natural basis and impose the condition of vanishing on the coefficients term by term. Moreover, what we have just seen is that, in the s channel, a field of a given spin ℓ only contributes to the $Q^{\Delta}_{\ell,0}(t)$ with the same ℓ. Thus, we can write this contribution to Eq. (5) as

$$q^{(a,s)}_{\Delta,\ell}(t) = q^{(a,s)}_{\Delta,\ell} Q^{2\Delta_\phi+\ell}_{\ell,0}(t), \quad (11)$$

with $q^{(2,s)}_{\Delta,\ell}, q^{(1,s)}_{\Delta,\ell}$ being the coefficients of the constant and the $(s-\Delta_\phi)$ term from

$$q^{(s)}_{\Delta,\ell}(s) = -\frac{4^{1-\ell}\Gamma(\Delta_\phi+s+\ell-h)^2}{(\ell+2s-\Delta)(\ell+2s+\Delta-2h)\Gamma(2s+\ell-h)}, \quad (12)$$

under a Taylor expansion around $s = \Delta_\phi$.

This was for the s channel, but we can add in the t and u channels easily by an appropriate exchange of u, v variables, which translates into exchanging the Mellin variables with some shifts:

$$t\,\text{channel}:\ s \to t + \Delta_\phi, \qquad t \to s - \Delta_\phi,$$
$$u\,\text{channel}:\ s \to \Delta_\phi - s - t, \qquad t \to t. \quad (13)$$

We need to extract the corresponding contributions $q^{(a,t)}_{\Delta,\ell'}(t)$ to the residues and decompose them in an expansion in the same orthogonal basis of $Q^{2\Delta_\phi+\ell}_{\ell,0}(t)$. Now, however, a spin ℓ' exchange in these channels will give a contribution in all partial waves. With the change of variables explained above, we have

$$M^{(t)}_{\Delta,\ell'}(s,t) = M^{(s)}_{\Delta,\ell'}(t+\Delta_\phi, s-\Delta_\phi). \quad (14)$$

Now $\rho_{\Delta_\phi}(s,t)$ gives rise to the spurious poles, and thus we just need to evaluate $M^{(t)}_{\Delta,\ell'}(s,t)$ and its first order expansion around $s = \Delta_\phi$ to obtain $q^{(a,t)}_{\Delta,\ell}(t)$. Furthermore, the individual contributions to the $Q^{2\Delta_\phi+\ell}_{\ell,0}(t)$ expansion can be picked out using their orthogonality properties. The end results for $q^{(2,t)}_{\Delta,\ell}, q^{(1,t)}_{\Delta,\ell}$ are obtained, as before, by Taylor expanding

$$c_{\Delta,\ell} q^{(t)}_{\Delta,\ell}(s) = \kappa_\ell(s)^{-1} \sum_{\ell'} c_{\Delta,\ell'} \int dt d\nu \Gamma^2(s+t)\Gamma^2(-t) \times \mu_{\Delta,\ell'}(\nu)\Omega^{(t)}_{\nu,\ell'}(t)P^{(t)}_{\nu,\ell'}(s-\Delta_\phi, t+\Delta_\phi)Q^{2s+\ell}_{\ell,0}(t) \quad (15)$$

around $s = \Delta_\phi$. Here, $\kappa_\ell(s)$ is a normalization factor [33] and $P^{(t)}_{\nu,\ell'}(s,t) = P^{(s)}_{\nu,\ell'}(t,s)$. It can be shown straightforwardly, using the properties of the continuous Hahn polynomials, that the u channel gives an identical contribution, i.e., $q^{(a,u)}_{\Delta,\ell} = q^{(a,t)}_{\Delta,\ell}$.

The sum over the physical spectrum also includes the identity operator ($\Delta = \ell = 0$). It will be convenient to separate out this piece. It gives a position space contribution to $\mathcal{A}(u,v)$, which is $[1 + (u/v)^{\Delta_\phi} + u^{\Delta_\phi}]$. We will take the corresponding Mellin amplitude to be given by the poles that reproduce this power law behavior. Thus,

$$M_{\Delta=0,\ell=0}(s,t) = \rho_{\Delta_\phi}(s,t)^{-1}\left(\frac{1}{st} + \text{crossed}\right), \quad (16)$$

where the crossed channels are obtained from the s channel using Eq. (13). In this case, only the t and u channels contribute to a spurious single pole at $s = \Delta_\phi$. The contribution to $Q^{2\Delta_\phi+\ell}_{\ell,0}(t)$ can be evaluated by using the above amplitude and orthogonality. The answer is

$$q^{(1,t)}_{\Delta=0,\ell} = q^{(1,u)}_{\Delta=0,\ell} = -\kappa_\ell(\Delta_\phi)^{-1} Q^{2\Delta_\phi+\ell}_{\ell,0}(0). \quad (17)$$

Thus, the simplest set of bootstrap equations [37] in Mellin space read

$$\sum_{\Delta\neq 0,\ell} c_{\Delta,\ell}(q^{(2,s)}_{\Delta,\ell} + 2q^{(2,t)}_{\Delta,\ell}) = 0 = 2q^{(1,t)}_{\Delta=0,\ell} + \sum_{\Delta\neq 0,\ell} c_{\Delta,\ell}(q^{(1,s)}_{\Delta,\ell} + 2q^{(1,t)}_{\Delta,\ell}). \quad (18)$$

We have an infinite number of equations, one for each ℓ. The first term corresponds to the vanishing of the log term and the second to the spurious power law piece in position space. Typically, the latter constraint determines the anomalous dimensions, and the former the OPE coefficients.

Results.—The scalar ϕ^4 theory in $d < 4$ has an interacting fixed point in the IR known as the Wilson-Fisher fixed point. This fixed point is accessible perturbatively in an ϵ expansion where $d = (4 - \epsilon)$. The anomalous dimension of ϕ and ϕ^2 are known up to ϵ^5 order (see, e.g., Refs. [38,39]) while, for the higher spin operators, $J^{(\ell)}$, the result is known to ϵ^4 order [20,40]. However, Feynman diagram computations for OPE coefficients for the stress tensor exchange have only been carried out to a couple of low orders in ϵ. Here, we will apply the above bootstrap procedure to the four point function of ϕ. We will also assume the existence of a unique stress tensor with ($\Delta = d, \ell = 2$) as the lowest member of a tower of twist two primaries $J^{(\ell)}$ of even spin ℓ. By demanding the cancellation of the spurious terms [33], we find $\Delta_\phi = 1 - (\epsilon/2) + \frac{1}{108}\epsilon^2 + \frac{109}{11664}\epsilon^3 + O(\epsilon^4)$, $\Delta_{\phi^2} = 2 - \frac{2}{3}\epsilon + \frac{19}{162}\epsilon^2 + O(\epsilon^3)$, which reproduce three-loop Feynman diagram results. What is nontrivial to obtain diagrammatically is the OPE coefficient [41], with ϕ^2 exchange, where we have a prediction at $O(\epsilon^2)$.

$$\frac{C_0}{C_0^{\text{free}}} = 1 - \frac{1}{3}\epsilon - \frac{17}{81}\epsilon^2 + O(\epsilon^3). \tag{19}$$

Here, we have normalized the result with the free theory OPE coefficient and have written $C^2_{\phi\phi\phi^2} = C_0$. The $O(\epsilon^2)$ result yields $C_0/C_0^{\text{free}} \approx 0.457$ on setting $\epsilon = 1$, as compared to 0.553 from numerics [10]. One can go onto studying the sector with higher spin currents $J^{(\ell)}$ in an analogous fashion. We simply state the results (details to appear in Ref. [27])

$$\Delta_\ell = d - 2 + \ell + \left(1 - \frac{6}{\ell(\ell+1)}\right)\frac{\epsilon^2}{54} + \delta_\ell^{(3)}\epsilon^3 + O(\epsilon^4). \tag{20}$$

We recover the known $O(\epsilon^2)$ [3] and $O(\epsilon^3)$ results [20]

$$\delta_\ell^{(3)} = \frac{373\ell^2 - 384\ell - 324 + 109\ell^3(\ell+2) - 432\ell(\ell+1)H_\ell}{5832\ell^2(\ell+1)^2}, \tag{21}$$

where H_n denotes the harmonic number. We also have

$$\frac{C_\ell}{C_\ell^{\text{free}}} = 1 + \frac{\epsilon^2}{54\ell(\ell+1)}[6(\ell+1)^{-1} + 2(\ell^2+\ell-3)H_\ell - (\ell-2)(\ell+3)H_{2\ell}] + C_\ell^{(3)}\epsilon^3, \tag{22}$$

where $C_{\Delta_\ell,\ell} = C_\ell$. This is a completely new result. The $O(\epsilon^3)$ term can also be calculated case by case for any given spin [27]. Specifically, this implies that the central charge $c_T = [d^2\Delta_\phi^2/(d-1)^2 C_2]$ is given by

$$\frac{c_T}{c_{\text{free}}} = 1 - \frac{5\epsilon^2}{324} - \frac{233\epsilon^3}{8748} + O(\epsilon^4). \tag{23}$$

While $O(\epsilon^2)$ is known (see, e.g., Ref. [42]), the $O(\epsilon^3)$ order is new. If we put $\epsilon = 1$ and compare it with the 3D Ising model numerical result, $c_T/c_{\text{free}} = 0.946534(11)$, from bootstrap [12], we get, with $\epsilon = 1$,

$$c_T/c_{\text{free}} \approx 0.957933, \tag{24}$$

which is a better estimate than what one gets from only the $O(\epsilon^2)$ part (~0.98). Our $O(\epsilon^3)$ explicit results for OPE coefficients gives $C_4/C_4^{\text{free}} = 1.07872$ for $\epsilon = 1$. Numerical bootstrap results for this coefficient are scarce and, as yet, with undetermined errors [43]. Using the results in Ref. [43], numerics yield 1.11345 [44]. In fact, the $O(\epsilon^2)$ results (22) as well as the $O(\epsilon^3)$ results [27] show that, as a function of ℓ, $C_\ell/C_\ell^{\text{free}}$ exhibits a minimum at $\ell = 4$. It will be interesting to see whether a numerical bootstrap has a similar feature.

Denoting the anomalous dimension of the higher spin $J^{(\ell)}$'s by γ_ℓ and that of ϕ by γ_ϕ, using our methods, it is also possible to derive the following universal form for γ_ℓ in the limit $\ell \gg 1$ for weakly coupled theories (with a small twist gap and coupling $g \ll 1$) in d dimensions:

$$\gamma_\ell - 2\gamma_\phi = \frac{\sum_{p=0}^\infty \alpha_p(g)(\log\ell)^p}{\ell^{d-2}}, \tag{25}$$

whose form agrees with Ref. [23], but our method gives explicitly, for $d = 4 - \epsilon$ (where $g = \epsilon$),

$$\alpha_p(\epsilon) = -\frac{\epsilon^{2+p}}{9p!}\left(\frac{2}{3}\right)^p + O(\epsilon^{3+p}), \tag{26}$$

which can be cross-checked for $p = 0, 1$ using Eq. (21). The general p formula is a prediction. Notice that plugging the leading order α_p into Eq. (25) resums into $-\epsilon^2/(9\ell^{2-2\epsilon/3}) \approx -\epsilon^2/(9\ell^{\Delta_{\phi^2}}) = -\epsilon^2/(9\ell^{\tau_{\phi^2}})$, where τ_O is the twist of the operator O. This spin dependence and the coefficient are in agreement with what would follow from Ref. [21]. A similar analysis can be done for any weakly coupled theory [27].

Our approach can also be used to get the leading anomalous dimensions for the ϕ^3 theory in six dimensions and the ϕ^6 theory in three dimensions [27], as well as the results for $O(N)$ [45], at both a fixed d and a large N, as well as in the ϵ expansion. It will also be interesting to extend our techniques to the theories being investigated in Ref. [46].

Outlook.—The new approach to bootstrap that we have outlined worked remarkably well for the Wilson-Fisher fixed point, reproducing analytically known results and producing new results for OPE coefficients. In contrast to the complexity of higher loop Feynman diagram computations, with all of their divergences and regularizations, our method yields finite, scheme independent physical results with relative ease of calculation. The main reason for this efficacy is that, in all of the results that we have discussed, the crossed channels involved, at most, the identity operator and ϕ^2, with other operators contributing only at higher orders. This simplification does not occur in the conventional approach to bootstrap [8]—there, one typically needs to sum over an infinite set of operators [21], even to produce results at leading order in ϵ at large spin [47]. At higher orders this phenomenon is unlikely to persist, and we will have to perhaps make use of the additional spurious poles [37]. It is likely, however, that, in the presence of a small parameter $[\epsilon, (1/\ell), (1/N), \ldots]$, we can always obtain the leading results analytically.

The conceptual suggestiveness of the present approach lies in the AdS_{d+1} Feynman diagramlike expansion. When combined with the Mellin representation, this holds out the tantalizing possibility of deciphering a dual string theory interpretation for CFTs, at least where a large N limit exists.

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