Modification of the Porter-Thomas Distribution by Rank-One Interaction

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The Porter-Thomas (PT) distribution of resonance widths is one of the oldest and simplest applications of statistical ideas in nuclear physics. Previous experimental data confirmed it quite well, but recent and more careful investigations show clear deviations from this distribution. To explain these discrepancies, Volya, Weidenmüller, and Zelevinsky [Phys. Rev. Lett. **115**, 052501 (2015)] argued that to get a realistic model of nuclear resonances is not enough to consider one of the standard random matrix ensembles which leads immediately to the PT distribution, but it is necessary to add a rank-one interaction which couples resonances to decay channels. The purpose of this Letter is to solve this model analytically and to find explicitly the modifications of the PT distribution due to such an interaction. Resulting formulas are simple, in good agreement with numerics, and could explain experimental results.

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Random matrix (RM) theory has undeniable success in describing nuclear physics data, in particular statistical properties of nuclear resonances and the distribution of their widths (see, e.g., reviews [1–3] and references therein). One of the simplest and widely used RM predictions is the statement that resonance widths are distributed as modulus square of RM eigenfunctions. For large dimensional invariant RM ensembles the latter are described by the Gaussian distribution which leads to the famous Porter-Thomas (PT) law [4]:

$$P_\beta(x) = \frac{1}{(2\pi x)^{1-\beta/2} l^{\beta/2}} \exp\left(-\frac{\beta x}{2l}\right). \tag{1}$$

In nuclear physics x is reduced resonance width, and in RM theory $x = N|\Psi|^2$, where Ψ is any eigenfunction component and N is the matrix dimension. Index $\beta = 1$ or 2 for, respectively, time-invariant or time-noninvariant systems. Constant l equals the mean value of x. The standard choice is $\langle x\rangle = 1$ and $l = 1$.

In RM theory the PT law is a theorem for invariant ensembles in the limit $N \to \infty$ (see, e.g., Ref. [5]). For physical problems like nuclear resonances its applicability is not guaranteed and requires experimental verification. Older experiments (cf. Refs. [1,2]) were in reasonable good agreement with this law. Nevertheless, recent experimental results and more careful treatment of old results demonstrate a clear disagreement with the PT distribution [6–8]. As RM theory is one of the cornerstones of quantum chaos in nuclear physics, it is important to understand the origin of the discrepancy. Different scenarios have been proposed so far (see Refs. [9–12] among others).

After a careful analysis, the authors of Ref. [12] came to the conclusion that a realistic model of nuclear s-wave resonances should include in addition to the RM term a rank-one interaction which couples resonances to decay channels, and they argued that the effective Hamiltonian (see Refs. [2,13]) can be chosen in the form

$$M_{ij} = G_{ij} + Z\delta_{i1}\delta_{j1}. \tag{2}$$

Here $G_{ij} \equiv G_{ij}^{(\beta)}$ is either a $N \times N$ real symmetric random matrix ($\beta = 1$) or a complex Hermitian one ($\beta = 2$) with the Gaussian distribution

$$P(G_{ij}) \sim \exp\left(-\frac{\beta}{4\sigma^2}\mathrm{Tr}(GG^\dagger)\right). \tag{3}$$

The mean eigenvalue density of such matrices when $N \to \infty$ is given by the Wigner semicircle law (see, e.g. Ref. [5]):

$$\rho_W(E) = \frac{1}{2\pi\sigma^2}\sqrt{4N\sigma^2 - E^2}. \tag{4}$$

The case of Hermitian matrices Eq. (2) (i.e., with real Z) is investigated here. According to Ref. [12], the imaginary part of Z is related with conjectural nonstatistical gamma decays, and the calculation of its influence is beyond the scope of this Letter. Notice also that the exact solution for pure imaginary Z has been known for a long time (see Ref. [3] and references therein).

To get a nontrivial limit, it is assumed that

$$\kappa = \frac{Z}{\sigma\sqrt{N}} \tag{5}$$

remains constant when $N \to \infty$.

In Ref. [12] it was noted that for $\beta = 1$ and real κ the distribution of $x = N|\Psi_1|^2$, where Ψ_1 is the first component of eigenvectors of matrix Eq. (2), does deviate from the PT law. But this conclusion was based only on numerical calculations and no clear physical picture had emerged.

The purpose of this Letter is to demonstrate that the eigenfunction distribution of matrix Eq. (2) can be found analytically for large N and real Z. The main result of this Letter is that this distribution has the same functional form as the PT distribution Eq. (1) but with an important difference that l (and, consequently, the mean value of x) in these expressions is not a universal constant but a certain function of state energy E and a dimensionless coupling constant. As a consequence, such a simple Gaussian character of the resulting distribution is valid only for eigenfunctions belonging to eigenvalues in a narrow energy interval. The distribution of eigenfunctions in a large window is not Gaussian but equals a weighted integral over Gaussian functions.

The model Eq. (2) is an old one (see, e.g., Ref. [14]), but it seems that its eigenfunction distribution has not been calculated so far. The possibility of explicit calculations is based on the rank-one structure of interaction which permits us to express the eigenvector modulus through eigenvalues of unperturbed and perturbed matrices. We denote eigenvalues and eigenfunctions of matrices G and M as follows (eigenstates are indexed by greek letters):

$$\sum_{j=1}^{N} G_{ij}\Phi_j(\alpha) = e_\alpha \Phi_i(\alpha), \qquad \sum_{j=1}^{N} M_{ij}\Psi_j(\alpha) = E_\alpha \Psi_i(\alpha). \tag{6}$$

All eigenfunctions are assumed to be orthogonal. Expanding eigenfunctions of the new matrix M into a series of eigenfunctions of matrix G, $\Psi_j(\alpha) = \sum_{\gamma=1}^{N} C_{\alpha\gamma}\Phi_j(\gamma)$, and substituting this expansion into Eq. (6), one gets the relation

$$C_{\alpha\gamma} = Z\frac{\Psi_1(\alpha)\Phi_1^*(\gamma)}{E_\alpha - e_\gamma} \tag{7}$$

and the consistency (quantization) conditions

$$Z\sum_\gamma \frac{|\Phi_1(\gamma)|^2}{E_\alpha - e_\gamma} = 1, \qquad Z\sum_\gamma \frac{|\Psi_1(\gamma)|^2}{E_\gamma - e_\alpha} = 1. \tag{8}$$

It is plain that eigenvalues E_α and e_α are interlacing. More precisely, if $\{e_\alpha\}$ are ordered (i.e., $e_1 \le e_2 \le \cdots \le e_N$), then $e_i \le E_i \le e_{i+1}$ $(i = 1, \ldots, N-1)$, $e_N \le E_N$ for all $Z > 0$, and $E_1 \le e_1$, $e_{i-1} \le E_i \le e_i$ $(i = 2, \ldots, N)$ for all $Z < 0$.

Equation (8) can be solved for the numerators. Using the Cauchy determinant formula one concludes that

$$Z|\Phi_1(\alpha)|^2 = \frac{\prod_\gamma (E_\gamma - e_\alpha)}{\prod_{\gamma\neq\alpha}(e_\gamma - e_\alpha)}, \tag{9}$$

$$Z|\Psi_1(\alpha)|^2 = \frac{\prod_\gamma (E_\alpha - e_\gamma)}{\prod_{\gamma\neq\alpha}(E_\alpha - E_\gamma)}. \tag{10}$$

With such values matrix Eq. (7) is automatically unitary, $CC^\dagger = 1$. Many other relations can be derived for the above coefficients. In particular,

$$Z\sum_\alpha |\Phi_1(\alpha)|^2 = Z\sum_\alpha |\Psi_1(\alpha)|^2 = \sum_\alpha (E_\alpha - e_\alpha). \tag{11}$$

By construction eigenvalues e_α and eigenfunctions $\Phi_1(\alpha)$ of matrix G^β are distributed as in standard random matrix ensembles [5]:

$$P(\{e_\alpha\},\{r_\alpha\}) \sim \prod_{\alpha<\gamma}|e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1}\delta\left(\sum_\alpha r_\alpha - 1\right)\exp\left(-\frac{\beta}{4\sigma^2}\sum_\alpha e_\alpha^2\right), \qquad r_\alpha = |\Phi_1(\alpha)|^2. \tag{12}$$

Using Eq. (9) the joint distribution of the old eigenvalues e_α and the new ones E_α without the confinement term was calculated in Ref. [15]:

$$\prod_{\alpha<\gamma}|e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1}\delta\left(\sum_\alpha r_\alpha - 1\right) \sim \frac{\prod_{\gamma>\alpha}(e_\gamma - e_\alpha)(E_\alpha - E_\gamma)}{\prod_{\gamma,\alpha}|e_\gamma - E_\alpha|^{1-\beta/2}}\delta\left(\sum_\alpha (E_\alpha - e_\alpha) - Z\right). \tag{13}$$

Our next step consists of changing N variables e_α to N variables $x_\alpha = |\Psi_1(\alpha)|^2$. It is plain that $\partial x_\alpha/\partial e_\beta = -x_\alpha/(E_\alpha - e_\beta)$. Using the Cauchy determinant it is possible to calculate explicitly $\det(\partial x_\alpha/\partial e_\beta)$ and to find that after a rank-one perturbation the joint distribution of new eigenvalues and eigenfunctions has the same form as the distribution of initial quantities [cf. Eq. (13)]

$$\prod_{\alpha<\gamma}|e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1}\delta\left(\sum_\alpha r_\alpha - 1\right)\prod_\alpha de_\alpha dr_\alpha = \prod_{\alpha<\gamma}|E_\gamma - E_\alpha|^\beta \prod_\alpha x_\alpha^{\beta/2-1}\delta\left(\sum_\alpha x_\alpha - 1\right)\prod_\alpha dE_\alpha dx_\alpha.$$

It seems that this key identity has been overlooked in previous studies. The result could be anticipated without calculations when one notices that Eq. (2) can be written in the symmetric form $G_{ij} = M_{ij} - v_i^* v_j$, which interchanges variables, $e_\alpha \leftrightarrow -E_\alpha$, but Eq. (13) is symmetric under this transformation.

Such symmetry is valid only without the confinement term. Using the representation $M_{ij} = \sum_\alpha E_\alpha \Psi_i(\alpha)\Psi_j^*(\alpha)$ and calculating $\mathrm{Tr}G^2 = \mathrm{Tr}(M - Z\delta_{i1}\delta_{j1})^2$ (or from the direct calculations as in the Appendix of Ref. [16]), one concludes that

$$\sum_\alpha e_\alpha^2 = \sum_\alpha E_\alpha^2 - 2Z\sum_\alpha E_\alpha x_\alpha + Z^2. \tag{14}$$

Consequently, the total joint distribution of new eigenvalues E_α and new eigenvectors, $x_\alpha \equiv |\Psi_1(\alpha)|^2$, is the following:

$$P(\{E_\alpha\},\{x_\alpha\}) \sim \prod_{\alpha<\gamma}|E_\gamma - E_\alpha|^\beta \prod_\alpha x_\alpha^{\beta/2-1}\delta\left(\sum_\alpha x_\alpha - 1\right)\exp\left[-\frac{\beta}{4\sigma^2}\left(\sum_\alpha E_\alpha^2 - 2Z\sum_\alpha E_\alpha x_\alpha\right)\right]. \quad (15)$$

For initial distribution Eq. (12) eigenvalues and eigenvectors were independent, but after a rank-one perturbation the distribution of eigenvectors depends on eigenvalues due to the term $\sum_\alpha E_\alpha x_\alpha$ in the exponent.

Expression (15) is exact. Below, only the most interesting case of large N is considered, though for $\beta = 2$ certain analytical calculations are possible for finite N [17].

Let us assume that when $N \gg 1$ all components $x_\alpha = |\Psi_1(\alpha)|^2$ are of order N^{-1} (below it is demonstrated that it is valid only for $\kappa^2 < 1$). Then the condition $\sum_\alpha x_\alpha = 1$ can be taken into account as usual by the introduction of the Lagrange multiplier $\delta(\sum_\alpha x_\alpha - 1) \to \exp[-\mu(\sum_\alpha x_\alpha - 1)]$. Now the probability distribution Eq. (15) is factorized and different x_α become independent, each $x_\alpha \equiv x(E_\alpha)$ being distributed as in Eq. (1) but with $l = (2\mu/\beta - ZE/\sigma^2)^{-1}$.

The value of μ has to be calculated from the requirement that $\sum_\alpha x_\alpha = 1$. It leads to

$$\sum_\alpha \frac{1}{\mathcal{E} - E_\alpha} = \frac{Z}{\sigma^2}, \qquad \mathcal{E} = \frac{2\mu\sigma^2}{\beta Z}. \quad (16)$$

The sum in this equation for $N \gg 1$ can be approximated by the mean unperturbed Green function $G_0(E)$ [18]:

$$G_0(E) = \int \frac{\rho_W(e)}{E-e}de = \frac{E - \sqrt{E^2 - 4\sigma^2 N}}{2\sigma^2}. \quad (17)$$

After simple algebra, one finds that when $\kappa^2 < 1$,

$$\mu = \beta N\frac{\kappa^2+1}{2}. \quad (18)$$

Consequently, the eigenfunction distribution of the perturbed problem has the same functional form as the PT distribution Eq. (1), but l [and the mean value of $x = N|\Psi_1(E)|^2$] depends on the energy and the coupling constant

$$l(E) \equiv N\langle|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}. \quad (19)$$

In the limit $\kappa \to 0$, $l(E) \to 1$, as it should be to recover the usual PT distribution. But when $\kappa^2 > 1$, the condition $\sum_\alpha x_\alpha = 1$ breaks down [due to the necessity of changing the branch of the square root in Eq. (17)] and $\sum_\alpha x_\alpha = \kappa^{-2}$. The reason for such behavior is well known. When $\kappa^2 > 1$, one collective state (called an outlier) becomes separated from the other levels and is situated at $E_c = \sigma\sqrt{N}(\kappa + \kappa^{-1})$. All states except the collective one have values of x_α of order N^{-1} and their probability distribution is given by the local PT distribution Eqs. (1) and (19). The only difference is that the square modulus of the outlier eigenvector component $x_c \equiv |\Psi_1(E_c)|^2$ is independent of N, $x_c = 1 - \kappa^{-2}$, so the total normalization is conserved. The above parameters of the collective state can be derived by various methods. For completeness, in Ref. [19] it is demonstrated how they can be calculated directly from Eq. (15).

The obtained simple Gaussian-like formulas Eqs. (1) with (19) correspond to the distribution of eigenfunctions in small energy intervals $|\delta E| \ll \sigma\sqrt{N}$. For practical reasons, it is important to know the distribution of eigenfunctions $x_\alpha = N|\Psi_1(\alpha)|^2$ whose energies E_α are in a finite interval $E_1 < E_\alpha < E_2$. The above results specify that the moments of the resulting distribution have to be calculated from the expression

$$\langle x^q\rangle = \frac{c_\beta(q)}{\delta N}\int_{E_1}^{E_2}\frac{\rho_W(E)}{l^q(E)}dE, \qquad \delta N = \int_{E_1}^{E_2}\rho_W(E)dE, \quad (20)$$

where $l(E)$ is determined by Eq. (19), $\rho_W(E)$ is the Wigner spectral density Eq. (4), and $c_\beta(q)$ are the Gaussian moments, $c_1(q) = 2^q\Gamma(q+1/2)/\sqrt{\pi}$, $c_2(q) = \Gamma(q+1)$.

Similarly, the full distribution in an interval $[E_1, E_2]$ is

$$\mathcal{P}_\beta(x) = \frac{1}{\delta N}\int_{E_1}^{E_2}\frac{\rho_W(E)e^{-\beta x/2l(E)}}{(2\pi x)^{1-\beta/2}l^{\beta/2}(E)}dE. \quad (21)$$

When all states (except the collective one, if any) are taken into account, $E_1 = -2\sigma\sqrt{N}$ and $E_2 = 2\sigma\sqrt{N}$. Straightforward calculations show that in this case

$$\mathcal{P}_1(x) = \sqrt{\frac{2}{\pi^3 x}}\int_0^\pi d\phi \sin^2\phi\sqrt{\kappa^2+1-2\kappa\cos\phi}\,e^{-(\kappa^2-2\kappa\cos\phi+1)x/2}, \qquad \mathcal{P}_2(x) = \frac{I_1(2\kappa x)}{\kappa x}e^{-(\kappa^2+1)x}. \quad (22)$$

Here, $I_1(x)$ is the modified Bessel function. The obtained formulas agree well with numerical calculations [19].

Equation (19) can also be derived by another method based on the fact that the averaged Green function in the limit $N \to \infty$ tends to a deterministic expression (see

Refs. [18,20,21] and references therein). A simple method to find this expression consists of application of the exact Schur complement formula [$\hat{G}(E) \equiv (E-M)^{-1}$]:

$$\hat{G}_{11}(E) = \left(E - Z - \sum\nolimits_{j,k\neq 1} M_{1j} \tilde{G}_{jk} M_{k1} \right)^{-1}, \quad (23)$$

where $\tilde{G}_{jk}$ is the Green function of the $(N-1)\times(N-1)$ matrix obtained from matrix M by removing line 1 and row 1. For matrices distributed as in Eq. (3) one can show that $M_{1j}M_{k1} \underset{N\to\infty}{\to} \sigma^2\delta_{jk}$, and this relation takes the form

$$G_{11}(E) \underset{N\to\infty}{\to} [E - Z - \sigma^2 G_0(E)]^{-1}, \quad (24)$$

where $G_0(E)$ can be approximated by Eq. (17). From eigenfunction expansion of the Green function, it follows that

$$\langle |\Psi_1(E)|^2 \rangle \approx \frac{\mathrm{Im} G_{11}(E - i0)}{\pi \rho_W(E)}, \quad (25)$$

which leads to Eq. (19). The generalization of the last method to finite-rank interactions is feasible, but due to possible outlier interaction [21] requires additional investigation and will be discussed elsewhere.

To sum up, when an ensemble of standard random matrices with Gaussian distribution Eq. (3) is perturbed by a rank-one interaction $Z\delta_{1i}\delta_{1j}$ with real Z, the distribution of $x = N|\Psi_1(E)|^2$ has the same functional form as the PT distribution, but parameter l entered Eq. (1) is not a universal constant but depends on energy E and the coupling constant $\kappa = Z/(\sigma\sqrt{N})$ as in Eq. (19). A similar result has been obtained in Ref. [9] where energy dependent $l(E)$ was associated with the existence of a virtual or weakly bound state near threshold.

Expression (19) is valid for $\kappa^2 < 1$ as well as for $\kappa^2 > 1$. In the latter case there exists one collective state whose mean energy is $E_c = \sigma\sqrt{N}(\kappa + \kappa^{-1})$. The modulus square of the corresponding eigenvector component is not of the order of N^{-1} as all the others, but it is independent of N. More precisely, $\langle |\Psi_{1c}|^2 \rangle = 1 - \kappa^{-2}$. The Gaussian character of the obtained distributions is valid only for eigenfunctions in small energy intervals. When all eigenfunctions from a large energy interval are considered their distribution is not Gaussian but is given by an integral over Gaussian functions Eq. (21). In the limit $N \to \infty$ all other components of the eigenfunctions [except $\Psi_1(E)$] remain distributed according to the usual PT distribution.

The important difference between the calculated distribution and the PT one is that the latter is universal, but the former is not. The interaction couples eigenfunctions with eigenenergies and forces the distribution to depend on the coupling constant, state energy, and the form of confinement potential. For different resonances (e.g., for different nuclei) these quantities may and will be different. A simple way to check these ideas experimentally is to fit a width distribution for a particular resonance in a small energy window to the PT formula (1) and find the corresponding $l(E)$ from the fit. The absence of *a priori* restrictions on the dependence $l(E)$ on energy makes this approach quite flexible to describe various experimental data. Taking into account together different resonances with different energies as is often done to increase statistics is not a sensitive way to investigate this phenomenon.

An interesting feature of the considered model Eq. (2) is that the introduction of rank-one interaction does not change local spectral statistics [17,22], as can be seen from the joint eigenvalue distribution Eq. (15). Experimentally it was observed that the Δ_3 statistics of nuclear resonances at small distances does agree well with RM prediction but starts to deviate from it at distances of the order of 40–70 mean level spacings [7]. Large distance deviation from RM formulas are typical for dynamical systems [23] and, in general, is not an argument against applicability of RM theory.

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