Symmetry Protection of Critical Phases and a Global Anomaly in $1+1$ Dimensions

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We derive a selection rule among the $(1+1)$-dimensional SU(2) Wess-Zumino-Witten theories, based on the global anomaly of the discrete $\mathbb{Z}_2$ symmetry found by Gepner and Witten. In the presence of both the SU(2) and $\mathbb{Z}_2$ symmetries, a renormalization-group flow is possible between level-k and level-k' Wess-Zumino-Witten theories only if $k \equiv k' \bmod 2$. This classifies the Lorentz-invariant, SU(2)-symmetric critical behavior into two "symmetry-protected" categories corresponding to even and odd levels, restricting possible gapless critical behavior of translation-invariant quantum spin chains.

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Introduction.—The classification of quantum phases is a central problem in condensed-matter and statistical physics. They are first classified into gapped and gapless phases. There has been significant progress in further classification of gapped phases, which are relatively easy to be handled theoretically. In particular, symmetry protected topological (SPT) phases [1,2] have become an important concept in the classification. That is, even when two states have no long-range entanglement and are indistinguishable in terms of any local observables, they could still belong to distinct phases separated by a quantum phase transition, in the presence of a certain symmetry. In contrast, classification of gapless quantum phases remains very much open. Symmetries are naturally expected to play an important role also in the classification of gapless quantum phases.

The issue of classification of quantum phases is also deeply related to that of quantum field theories. Since a quantum field theory can be regarded as an effective description of universal low-energy behaviors of quantum many-body systems, we may expect that they are essentially the same problem. An intriguing feature of quantum field theory is the *anomaly* [3–6]. A particularly interesting anomaly is the global anomaly, which emerges after promotion of a global symmetry to gauge symmetry [6]. As it is the case with many connections between quantum field theory and condensed-matter physics, investigation of the consequences of the anomaly in quantum field theory in a condensed-matter physics context has been quite fruitful, including the discovery of the "Chern insulator" [7]. Nevertheless, the exact correspondence in many concrete cases is not yet understood.

An anomaly in a field theory may imply that the field theory cannot be realized in a condensed-matter system or a lattice model in the same dimensions. A renowned case is the impossibility of realization of a chiral fermion with (noninteracting) fermions on a lattice, known as the Nielsen-Ninomiya theorem [8,9], which is deeply related to the chiral anomaly, a representative example of the quantum anomaly. Nevertheless, such a field theory could be realized at the boundary (edge or surface) of a condensed-matter system in higher dimensions. For example, the chiral fermion in $1+1$ dimensions indeed appears as the edge state of a two-dimensional quantum Hall system. Ryu *et al.* [10,11] generalized this observation to classification of gapped SPT phases: the edge or surface state of a SPT phase exhibits an anomaly with respect to the relevant symmetry, which implies the "ingappability" of the edge state in the presence of the symmetry [12]. Conversely, such an anomaly can be identified with an edge or surface state, and thus with an SPT phase in higher dimensions.

These developments motivate us to question if there is a mechanism of symmetry protection of the universality class of *bulk* gapless critical phases. In this Letter, we argue that there is a protection of bulk gapless critical phases by discrete symmetry. This symmetry protection is analogous to that of the well-known (gapped) SPT phases; here we show that the concept can be generalized to bulk *gapless critical* phases. We demonstrate this for the SU(2)-symmetric quantum antiferromagnetic chains and their effective field theory, SU(2) Wess-Zumino-Witten (WZW) theory as an example. The SU(2) WZW theory is characterized by a non-negative integer k, which is called level. Hereafter we denote the level-k SU(2) WZW theory as WZW_k. WZW_k's with $k = 1, 2, \ldots$ are a complete classification of the universality classes of critical points in $1+1$ dimensions with the Lorentz and SU(2) symmetry only. We can also identify WZW_0 with a gapped phase with a unique ground state.

Our main claim in the present Letter is as follows: in the presence of the SU(2) and a certain discrete $\mathbb{Z}_2$ symmetry of the WZW theory, which corresponds to the translation symmetry of the spin chain, a renormalization-group (RG) flow is possible between WZW_k and $\mathrm{WZW}_{k'}$ only if $k \equiv k'$ mod 2. That is, the gapless critical phases in $1+1$ dimension with the SU(2), the $\mathbb{Z}_2$, and the Lorentz symmetries are classified into two "symmetry-protected" categories: one corresponds to even levels and the other to odd levels. In terms of spin chains, as long as the SU(2) spin rotation and the lattice translation symmetries are unbroken (either explicitly or spontaneously), a spin chain with $S \in \mathbb{Z}$ can only realize WZW_k with an even k, while one with $S \in \mathbb{Z} + 1/2$ can only realize WZW_k with an odd k.

Our argument is based on the global anomaly of a discrete symmetry in the WZW theory originally found by Gepner and Witten in 1986 [13], providing a new link between the anomaly in quantum field theory and condensed-matter physics. As we will discuss later, the present result includes, as special cases, the earlier semi-classical ($k \to \infty$) analysis [14], and the Lieb-Schultz-Mattis theorem [15–17] applied to SU(2)-symmetric one-dimensional systems. However, the present result is much stronger than the Lieb-Schultz-Mattis theorem, in restricting the possible universality class of the gapless critical phase. Furthermore, we shall discuss an experimental consequence of the present result, in terms of Raman spectroscopy as an example.

Model.—The standard Heisenberg antiferromagnetic (HAFM) chain is defined by the Hamiltonian $\mathcal{H}_{\mathrm{HAFM}} = J_1 \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$ with $J_1 > 0$, which possesses the SU(2) symmetry of the global spin rotation, the lattice translation symmetry, and the lattice inversion symmetry. We can also consider various generalizations of this model by including next-nearest-neighbor interaction [18], biquadratic interaction [19], and so on [20,21].

In order to explore the possible quantum critical behavior of quantum spin chains, non-Abelian bosonization is useful. In the non-Abelian bosonization of quantum antiferromagnetic chains, first the spin-S spin chain is represented in terms of fermions with $2S$ "colors" [14]. The resulting effective field theory is WZW_k defined by the action

$$\mathcal{S}_k = -\frac{1}{\lambda}\int_{S^2} dx_0 dx_1 \mathrm{Tr}[(g^{-1}\partial_\mu g)(g^{-1}\partial_\mu g)] + k\Gamma_{\mathrm{wz}}, \qquad (1)$$

with the SU(2) matrix field $g(x_0, x_1)$ with the spin indices, the coupling constant $\lambda > 0$, and the Wess-Zumino term $k\Gamma_{\mathrm{WZ}}$. We consider the space-(imaginary) time compactified as the two-dimensional sphere S^2. The Wess-Zumino term Γ_{WZ} is defined on the three-dimensional ball B^3, extended from the spacetime manifold S^2. It thus appears to depend on the extension of g to B^3, which is arbitrary. However, it is a topological term unaffected by any infinitesimal variation of the extension. Nevertheless, Γ_{WZ} can take values different by integral multiples of 2π, corresponding to topologically inequivalent extensions. To define the theory consistently, the partition function should be independent of the arbitrary extension to B^3, and thus the level k is quantized to be $k \in \mathbb{Z}$ [22]. WZW_k is a conformal field theory (CFT) with the SU(2) symmetry, for each level k. Exact features of WZW_k are known, thanks to its infinite dimensional symmetry governed by Kac-Moody algebra [13,22]. Each value of k represents the different critical behaviors.

In the non-Abelian bosonization treatment of spin-S chains [14], the level is naturally given as $k = 2S$. Indeed, an integrable model called the Takhtajan-Babujian model is known for each S, and its exact solution shows that its low-energy limit is described by WZW_{2S}. However, generically the effective theory of the spin-S chain contains various perturbations to WZW_{2S}. The general principle is that, all the possible perturbations allowed by the symmetries should be present, unless parameters in the Hamiltonian are fine-tuned. In fact, the Takhtajan-Babujian model corresponds to the special multicritical point where the parameters are fine-tuned so that all the relevant perturbations vanish. More generic models usually have relevant perturbations which drive the system away from the original WZW_{2S} fixed point, under RG.

To discuss the RG flow, we need to identify symmetries of the system and their representation in the field theory. In this Letter, we limit ourselves to models with the global SU(2) symmetry of spin rotation. Furthermore, we consider the models which are invariant under the translation by one site, $T_1 : \boldsymbol{S}_j \to \boldsymbol{S}_{j+1}$. The lattice translation symmetry T_1 is represented by the $\mathbb{Z}_2$ symmetry under $g \to -g$, in the WZW theory [14].

Modular invariance.—To see the consistency of a CFT, it is convenient to consider the system on a torus. The torus can be defined in terms of complex coordinates z and $\bar{z}$ with the identifications $z \sim z + 2\pi$ and $z \sim z + 2\pi\tau$ with the modulus $\tau \in \mathbb{C}$. The conformal invariance enhanced by the SU(2) symmetry (Kac-Moody algebra) dictates that the partition function on the torus is generally given as

$$Z(\tau, \bar{\tau}) = \sum_{j,j'=0}^{k/2} \chi_j(\tau) X_{j,j'} \bar{\chi}_{j'}(\bar{\tau}), \qquad (2)$$

where $\chi_j(\tau)$ and $\bar{\chi}_{j'}(\bar{\tau})$ are Kac-Moody characters as functions of the modulus τ, corresponding to holomorphic and anti-holomorphic parts [23]. The characters are labeled by the "spin" $j = 0, 1/2, \ldots, k/2$. The coefficients $X_{j,j'}$ count the number of the primary field with the spin (j, j'). Thus $X_{j,j'}$ must be non-negative integers in a consistent CFT.

The same torus can be represented by different modular parameters, which are related by modular transformations generated by $\mathcal{T} : \tau \to \tau + 1$ and $\mathcal{S} : \tau \to -1/\tau$. Here $\mathcal{T}$ twists the spatial boundary condition and $\mathcal{S}$ exchanges the space and imaginary-time directions [24]. Since these keep the underlying torus unchanged, a physically sensible partition function should be invariant under them. This requirement, which is called modular invariance, in fact leads to quite a powerful constraint on a possible consistent CFT. In the present context, $X_{j,j'}$ are strongly constrained by the modular invariance so as to be a non-negative integer [23,24].

The standard partition function of WZW_k on the torus with the periodic boundary conditions is given by

$$Z_{\mathrm{SU}(2)}(\tau,\bar{\tau}) = \mathrm{Tr}(e^{-2\pi \mathrm{Im}\tau \mathcal{H}} e^{i2\pi \mathrm{Re}\tau \mathcal{P}}), \tag{3}$$

where $\mathcal{H}$ and $\mathcal{P}$ are the total energy and momentum, respectively. This indeed turns out to be modular invariant partition function with $X_{j,j'} = \delta_{j,j'}$.

However, this is not the only possible modular invariant partition function. Since WZW_k also possesses the discrete $\mathbb{Z}_2$ symmetry $g \to -g$, we also consider projecting the Hilbert space onto the subspace which is symmetric under $g \to -g$. The resulting partition function reads

$$Z_{+}^{\mathrm{proj}}(\tau,\bar{\tau}) = \mathrm{Tr}(P_{+} e^{-2\pi \mathrm{Im}\tau \mathcal{H}} e^{i2\pi \mathrm{Re}\tau \mathcal{P}}), \tag{4}$$

where P_{+} is the projection operator onto the subspace which is even under $g \to -g$. In the path-integral formalism, the insertion of P_{+} is equivalent to averaging over the periodic and the antiperiodic boundary conditions on g in the imaginary time direction. As a consequence, Z_{+}^{proj} is *not* modular invariant. We can construct a modular invariant partition function based on Z_{+}^{proj}, by "gauging" the $\mathbb{Z}_2$ symmetry also in the spatial direction. This procedure is known as orbifold construction, and the resulting partition function of the $\mathbb{Z}_2$ orbifold of WZW_k reads

$$Z_{+}(\tau,\bar{\tau}) = (1 + \mathcal{S} + \mathcal{TS}) Z_{+}^{\mathrm{proj}}(\tau,\bar{\tau}) - Z_{\mathrm{SU}(2)}(\tau,\bar{\tau}). \tag{5}$$

Despite the construction to make it modular invariant, in fact, $Z_{+}(\tau,\bar{\tau})$ is modular invariant only if k is even; it is modular noninvariant if k is odd [13,24]. This is an example of a global anomaly in quantum field theory. While the presence of this anomaly can be interpreted by a construction of SO(3) WZW theory [13] as SU(2) modulo $g \sim -g$ is SO(3), the physical implications of the anomaly has not been elucidated.

Consequences of the global anomaly.—Now we shall argue that there are indeed very profound consequences. First, we consider a RG flow from WZW_k to $\mathrm{WZW}_{k'}$ induced by perturbations allowed within the SU(2) and the $\mathbb{Z}_2$ $(g \to -g)$ symmetries, for an even k. While both fixed points have the SU(2) Kac-Moody and $\mathbb{Z}_2$ symmetries for any k and k', the level k' of the infrared fixed point is not arbitrary. Since k is assumed to be even, for the ultraviolet fixed point WZW_k, the $\mathbb{Z}_2$ orbifold is a consistent conformal field theory with a modular invariant partition function. Because the orbifold construction is given by summation over partition functions with various boundary conditions, it should not affect the RG flow in the bulk. In other words, once the projection onto the symmetric subspace is done consistently, the RG flow can be followed under the projection. Thus the RG flow between WZW theories under the $\mathbb{Z}_2$ symmetry implies a corresponding RG flow between their $\mathbb{Z}_2$ orbifolds. This means that, the infrared fixed point $\mathrm{WZW}_{k'}$ should have a consistent $\mathbb{Z}_2$ orbifold and thus k' must be even.

Next we consider the RG flow from WZW_k to $\mathrm{WZW}_{k'}$, when k is odd. Now the ultraviolet fixed point has the global anomaly and the $\mathbb{Z}_2$ orbifold is ill defined. There is no *a priori* reason that the infrared fixed point has the same anomaly, since symmetries can, generally speaking, emerge in the infrared limit. Nevertheless, below we shall argue that the global anomaly in the ultraviolet fixed point is "inherited" by the infrared limit.

Naively, since the theory has the discrete $\mathbb{Z}_2$ symmetry $g \to -g$, we expect that we can consider projection of the entire Hilbert space onto the subspace which is symmetric under $g \to -g$. However, the anomaly for odd k precisely means that there is no consistent CFT defined within the symmetric (or antisymmetric) subspace. The odd-k WZW_k is inconsistent unless both symmetric and antisymmetric sectors are included. This observation can be related to the "gappability" of the theory. In general, a CFT has relevant operators in its spectrum. Once the theory is perturbed by a relevant operator, generically the theory would become massive; the excited states would be separated from the ground state by a nonvanishing mass gap. Usually the ground state in such a system is unique. If this is the case, because of the $\mathbb{Z}_2$ symmetry of the theory, the unique ground state is either symmetric or antisymmetric with respect to the $\mathbb{Z}_2$ symmetry. Then, in order to describe the low-energy physics, we can consider a projection onto symmetric or antisymmetric sector of the Hilbert space. However, for odd k, the global anomaly of WZW_k means that such a projection does not yield a consistent quantum field theory, Therefore, in order to open a mass gap, the global anomaly for odd k requires that the ground states below the gap exist in both the symmetric and antisymmetric sectors and are doubly degenerate. This signals the spontaneous breaking of the $g \to -g$ symmetry [25]. The above statement corresponds to a field-theory version of the Lieb-Schultz-Mattis theorem [15–17], as it will become clearer in the correspondence with spin chains discussed below.

Now suppose there is a RG flow from WZW_k with an odd k to $\mathrm{WZW}_{k'}$ with an even k', we can further perturb the infrared fixed point to obtain a massive (gapped) field theory with a unique ground state (corresponding to WZW_0). This contradicts with the ingappability of WZW_k discussed above. Thus k' must also be odd. Combining the results for even and odd k, we obtain the following statement.

When there is a RG flow from WZW_k to $\mathrm{WZW}_{k'}$, if the SU(2) symmetry and the $\mathbb{Z}_2$ symmetry $g \to -g$ are respected, $k' < k$ and $k' \equiv k \bmod 2$.

In addition, it has been known that $k' < k$ thanks to Zamolodchikov's c-theorem [27], which dictates that the central charge of the infrared fixed point $\mathrm{WZW}_{k'}$ should be smaller than that of the ultraviolet one WZW_k.

The implications of the above field-theory constraint on spin chains is as follows. As discussed earlier, non-Abelian bosonization of a spin-S HAFM chain yields WZW_k with $k = 2S$, with perturbations allowed by the symmetries. Thus we find the following:

The critical behavior of a general spin-S HAFM chain is described by WZW_k with $k \equiv 2S$ mod 2, as long as the Hamiltonian possesses the SU(2) spin rotation symmetry and the lattice translation symmetry.

In other words, critical phenomena in $1 + 1$ dimensions with SU(2) and $\mathbb{Z}_2$ symmetry of $g \to -g$ are grouped into two symmetry-protected classes: one consists of WZW_k with even k and the other with odd k. In the presence of the SU(2) spin rotation symmetry and the lattice translation symmetry, general HAFM chains with integer spin S can only realize the former, while those with half-odd-integer spin S can only realize the latter. The present argument can also be generalized to the case of the site-centered inversion symmetry [24,28,29].

Affleck and Haldane [14] argued, based on the large-k semiclassical analysis, WZW_k with the leading perturbation allowed under the $g \to -g$ symmetry, $(\mathrm{tr}g)^2$, can be mapped to the O(3) nonlinear sigma model at the topological angle $\theta = \pi k$. Given that it is equivalent to WZW_0 or WZW_1 when k is even or odd, respectively, their result is a special case of the present one. On the other hand, the present result generalizes significantly that of Ref. [14] in restricting RG flows among $k > 1$ fixed points induced by generic perturbations.

Examples.—Let us now discuss several concrete cases, in light of the present result.

Ziman and Schulz studied translation invariant $S = 3/2$ antiferromagnetic chains numerically and confirmed that, while the Takhtajan-Babujian model is described by WZW_3, away from the special Takhtajan-Babujian point the system is described by WZW_1 [30]. Namely, there is a RG flow from WZW_3 to WZW_1, in accordance with the present result. The critical behavior corresponding to WZW_2 is not found in the model, although it is allowed by the c-theorem.

Another interesting example is the translation invariant spin-S model [20]

$$\mathcal{H}_{J_1\text{-}J_3} = \sum_j [J_1 \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + J_3\{(\boldsymbol{S}_{j-1} \cdot \boldsymbol{S}_j)(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}) + \mathrm{H.c.}\}]. \tag{6}$$

This model has the WZW_{2S} quantum critical point $J_{3c} > 0$ [31], even though the model is not at the integrable Takhtajan-Babujian point. Again this is consistent with our selection rule.

An $S = 1$ spin chain of Ref. [18] is a highly nontrivial example of our theory. It possesses a WZW_4 multicritical point whose level is higher than $2S$ and still consistent with our selection rule.

We stress that our selection rule is protected by the one-site translation symmetry. In fact, it can be removed by breaking the translation symmetry explicitly. For example, an extended model [21]

$$\mathcal{H}_{J_1\text{-}J_3\text{-}\delta} = \mathcal{H}_{J_1\text{-}J_3} - J_1\delta \sum_j (-1)^j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \tag{7}$$

breaks the one-site translation symmetry explicitly, when $\delta \neq 0$. When $S = 1$, the model Eq. (7) exhibits a critical line of $c = 1$ connected to the multicritical point with $c = 3/2$. This means that WZW_2 can flow to WZW_1 in the absence of the translation symmetry. A similar RG flow from the level 2 to the level 1 is found in the $S = 1$ bilinear-biquadratic chain with the bond alternation [19]. The bond alternation breaks the lattice translation and the site-centered inversion symmetries, but keeps the time reversal and the bond-centered inversion symmetries. This is consistent with our analysis that either the lattice translation or the site-centered inversion symmetry protects the two categories, while the time reversal nor the site-centered inversion symmetry does not [24].

Observable consequences.—The Raman spectroscopy is important in studying one-dimensional spin systems [32], probing the dynamical correlation of the Raman operator $R \sim \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$ in spin chains [33,34]. Dimensional analysis leads to the Raman spectrum $I(\omega) \propto (\omega/J)^{2x-2}$ in the low energy limit, $\hbar\omega \ll k_B T \ll J$, where ω is the Raman frequency shift, J is the (typical) spin-spin interaction energy scale, and x is the scaling dimension of R. Near the WZW_k quantum critical point of a translation-invariant spin chain, we can derive $R \propto \int dx (\mathrm{tr}g)^2$, which implies $x = 2/(2 + k)$. If the one-site translation symmetry is broken, $R \propto \int dx \mathrm{tr}g$ and $x = 3/[4(2 + k)]$. Thus the Raman spectrum $I(\omega)$ provides a rather direct probe of the level k of the effective WZW_k field theory of the spin chain, and would also be useful in studying crossover among WZW theories with different levels as discussed in this Letter.

Conclusions.—In this Letter, we proposed the concept of symmetry protection of critical phases. As an example, we demonstrated that the SU(2) WZW theories are classified into two categories of even and odd levels, in the presence of the discrete $\mathbb{Z}_2$ symmetry of WZW theory which corresponds to the one-site translation symmetry of spin chains. The present result provides a new direction for the classification of quantum phases, as well as a novel link between the anomalies in field theory and condensed-matter physics. It would be interesting to find more examples with different symmetries or in higher dimensions.

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