Quantum Steering Inequality with Tolerance for Measurement-Setting Errors: Experimentally Feasible Signature of Unbounded Violation

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Quantum steering is a relatively simple test for proving that the values of quantum-mechanical measurement outcomes come into being only in the act of measurement. By exploiting quantum correlations, Alice can influence—steer—Bob's physical system in a way that is impossible in classical mechanics, as shown by the violation of steering inequalities. Demonstrating this and similar quantum effects for systems of increasing size, approaching even the classical limit, is a long-standing challenging problem. Here, we prove an experimentally feasible unbounded violation of a steering inequality. We derive its universal form where tolerance for measurement-setting errors is explicitly built in by means of the Deutsch–Maassen–Uffink entropic uncertainty relation. Then, generalizing the mutual unbiasedness, we apply the inequality to the multisinglet and multiparticle bipartite Bell state. However, the method is general and opens the possibility of employing multiparticle bipartite steering for randomness certification and development of quantum technologies, e.g., random access codes.

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Introduction.—Bell nonlocality is demonstrated when the correlations between the outcomes of spacelike separated measurements violate a Bell inequality. To violate existing Bell inequalities, two spacelike separated parties, Alice and Bob, with d-dimensional local systems need to perform an exponential (in d) number of measurements [1–4]. According to the monogamy relation [5], Alice and Bob could violate a Bell inequality using $O(d)$ measurements; however, existing inequalities are far from this regime and, so, are experimentally unfeasible [6].

Quantum steering inequalities have recently been considered as weaker, and correspondingly more feasible, tests of the quantumness of correlations compared to Bell inequalities [7–9]. They can expect to see significant applications [10,11] and such inequalities for large systems enable tests of quantum mechanics outside of typical regimes. Quantum steering inequalities quantify the amount of quantum steering [12,13] that can be achieved, that is, the discrepancy between the full quantum-mechanical treatment and the local hidden state (LHS) model, where one of the observers, e.g., Alice, performs classical measurements. The assumption of the objective existence of local states that specifies the outcome of local measurements bounds the amount Alice can steer Bob's state. Violation of a steering inequality by a factor of $O(\sqrt{d})$ requires $d+1$ observables in the form of mutually unbiased bases (MUBs) [14]. This scenario necessitates the complementarity relation among the bases to be fulfilled exactly, which is experimentally impossible to attain. Steering is intimately linked to Bell nonlocality. This relation is rather complex [15] but, for the Clauser-Horne-Shimony-Holt Bell inequality, steering is limited by the strength of some fine-grained uncertainty relations [16].

Here, we provide the first general formulation of a quantum steering inequality with tolerance to errors in measurement settings. This is a significant step towards observing an unbounded violation of a steering inequality for a multiparticle bipartite Bell-singlet state. The method provides robustness with respect to the degree of unbiasedness of bases: it employs generalized MUBs where unbiasedness varies from one observable (or its eigenvector) to another, and reveals a link to the Deutsch-Maassen-Uffink entropic relations. Our formula fits various physical systems, but since mesoscopic quantum effects are subtle, technological requirements for an apparatus are high. Here, we focus on a quantum-optical implementation based on polarization entangled squeezed vacuum states generated by parametric down-conversion, polarization rotations, and photon-counting detection.

Violation of steering inequalities has been demonstrated for a single photon [17,18], two-photon singlet [19], and Werner states [20]. Since quantum steering is interpreted as a quantum-information task where classical measurements simulate an untrusted device, it was extended to a multipartite scenario useful for semi-device-independent certification of entanglement in quantum networks [21]. Our approach employing generalized MUBs will boost the young field of quantum random access codes by enabling more near-optimal and robust scenarios [22].

Main result.—Consider the quantum steering scenario shown in Fig. 1. Alice and Bob have local access to

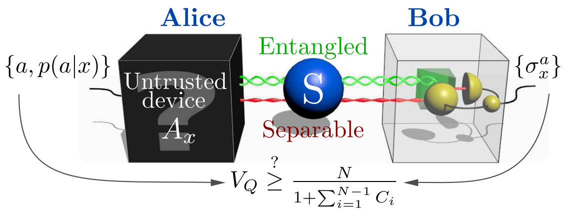

FIG. 1. Quantum steering scenario: Alice and Bob share a bipartite state ρ, either entangled (green) or separable (red), and Bob would like to verify which one this is. Alice performs a measurement using an untrusted device of one of her N observables A_x, and communicates to Bob the outcome a which she receives with probability $p(a|x)$. Bob applies his measurement σ_x^a and checks violation of the steering inequality (8).

subsystems of a bipartite quantum state ρ. Alice chooses one of her settings $x \in \mathbb{N}_N$, $\mathbb{N}_j \equiv \{1, \ldots, j\}$, measures a nondegenerate observable A_x with eigenvectors $\{\varphi_x^a\}$, and receives a result $a \in \mathbb{N}_d$ with probability $p(a|x) = \mathrm{Tr}\{(|\varphi_x^a\rangle\langle\varphi_x^a| \otimes I)\rho\}$. When Alice obtains the result a, the conditional state

$$\sigma_x^a = \mathrm{Tr}_A\{(|\varphi_x^a\rangle\langle\varphi_x^a| \otimes I)\rho\}, \tag{1}$$

is "created at a distance" at Bob's location. For a maximally entangled state and ideal measurements in MUBs, $\{\sigma_x^a\}$ (renormalized to have unit trace) is an orthonormal basis for each value of a. In nonideal circumstances, the $\{\sigma_x^a\}$ will be close to some orthonormal bases $\{\phi_x^a\}$. This can be quantified by the steering functional

$$S_\mathrm{Q} = \sum_{x=1}^{N}\sum_{a=1}^{d} \mathrm{Tr}\{|\phi_x^a\rangle\langle\phi_x^a|\sigma_x^a\}. \tag{2}$$

The maximal quantum value of (2) equals the number of settings, $S_\mathrm{Q} = N$, and reveals the perfect match. Please note that the quantity (2) is a function of a set of measurements $M = \{|\varphi_x^a\rangle\langle\varphi_x^a|\}$, $F = \{|\phi_x^a\rangle\langle\phi_x^a|\}$ and the state ρ on which it is performed, so more formally $S_\mathrm{Q} = S_\mathrm{Q}(M, F, \rho)$.

If the shared state ρ is separable, by measuring its subsystems, Alice can only generate states (at Bob's side) that possess the LHS model [14]. In this case, we denote ρ by ρ_LHS. Within this model, Alice's measurements are untrusted; i.e., they are treated as a black box which receives inputs (x's) and reports outcomes (a's) with probability $p_\lambda(a|x)$, where λ labels the hidden (physical) state. We denote these classical measurements by M_C. The Bob's conditional state is an average over the ensemble of local hidden states σ_λ [9]

$$\bar{\sigma}_x^a = \sum_{\lambda\in\Lambda} q_\lambda p_\lambda(a|x)\sigma_\lambda, \tag{3}$$

where Λ is a finite set of indices λ, and non-negative coefficients q_λ fulfill $\sum_{\lambda\in\Lambda} q_\lambda = 1$. Then, the steering functional equals

$$S_\mathrm{LHS} = \sum_{x=1}^{N}\sum_{a=1}^{d} \mathrm{Tr}\{|\phi_x^a\rangle\langle\phi_x^a|\bar{\sigma}_x^a\}. \tag{4}$$

Its maximal value depends on the choice of Bob's bases $\{\phi_x^a\}$ and will be estimated below. Equation (4) can be considered as a function of classical measurements M_C, quantum measurements F, and the state ρ_LHS on which they are performed, $S_\mathrm{LHS} = S_\mathrm{LHS}(M_C, F, \rho_\mathrm{LHS})$. For the given measurement F, the general form of a steering inequality is $S_Q(M, F, \rho) \leq \sup_{M_C, \rho_\mathrm{LHS}} S_\mathrm{LHS}(M_C, F, \rho_\mathrm{LHS})$. The maximal degree of violation of this inequality by quantum states is

$$V_Q = \frac{\sup_{M,\rho} S_Q}{\sup_{M_C,\rho_\mathrm{LHS}} S_\mathrm{LHS}} > 1. \tag{5}$$

An unbounded quantum violation is observed if V_Q is an increasing function of some experimental parameters, e.g., of the amount of entanglement in ρ or number of settings. An unbounded violation of (5) has been shown for a sequence of maximally entangled states with increasing local dimension d (e.g., larger and larger spin) [14]. However, it requires Bob to perform complex measurements, namely, in MUBs with a linearly diverging number of settings $N = d + 1$. For a large dimension, their existence is only known for special cases, and precise experimental verification of their defining feature $|\langle\phi_x^a|\phi_y^b\rangle|^2 = 1/d$ for $x \neq y$ is unfeasible.

Rather than trying to cure these problems, our method shall overcome them. We will relax the MUB condition by allowing the unbiasedness to vary from one observable or its eigenvector to another and will seek for the maximal overlap between the bases. In this way, our result is naturally linked to the Deutsch-Maassen-Uffink uncertainty relations [23,24] saying that the Shannon entropies of measurement outcomes of two nondegenerate observables obtained in two bases $\{\phi_x^a\}$ and $\{\phi_y^b\}$ for a quantum state ρ satisfy

$$H(\{\phi_x^a\}|\rho) + H(\{\phi_y^b\}|\rho) \geq -2\log C_{xy}, \tag{6}$$

where the maximal overlap $C_{xy} = \max_{a,b}|\langle\phi_x^a|\phi_y^b\rangle|$ quantifies the complementarity of the bases.

Our main result is the following.

Theorem 1.—Given a quantum steering scenario involving $x \in \mathbb{N}_N$ settings, $a \in \mathbb{N}_d$ outcomes, and a set of N orthonormal eingenbases $F = \{\phi_x^a\}$ defining the receiver's (Bob's) measurements, the LHS steering functional is bounded from above by

$$\sup_{M_C,\rho_\mathrm{LHS}} S_\mathrm{LHS} \leq 1 + \sum_{i=1}^{N-1} C_i, \tag{7}$$

where $C_i = \max_x C_{x(N+x-i \bmod N)}$ and $C_{xy} = \max_{a,b}|\langle\phi_x^a|\phi_y^b\rangle|$ for $x, y \in \mathbb{N}_N$ is defined as in the Deutsch-Maassen-Uffink uncertainty relations. This implies

$$V_Q \geq \frac{N}{1 + \sum_{i=1}^{N-1} C_i}. \tag{8}$$

In particular, a weaker bound holds

$$V_Q \geq \frac{N}{1 + (N-1)C} \tag{9}$$

with $C = \max_i C_i = \max_{x\neq y} C_{xy}$.

The proof is included in the Supplemental Material (SM) [25], Section 1.

Consequently, any ρ for which there exists M, F such that $S_Q(M, F, \rho) > 1 + \sum_{i=1}^{N-1} C_i$ violates the steering inequality. The dependence solely on the maximal overlap C frees us from the necessity of precisely controlling all MUB conditions one by one, as required in previous approaches [14]. Unbounded violation of the steering inequality is obtained when the ratio $S_Q(M^{(d)}, F^{(d)}, \rho^{(d)})/(1 + \sum_{i=1}^{N-1} C_i^{(d)})$, where d is a local dimension of ρ, goes to infinity with d.

It is unclear whether the bound in the Theorem is tight. For the scenario where $N = d + 1$ measurements in MUBs are performed, $V_Q \geq (d+1)/(1+\sqrt{d}) \approx \sqrt{d}$, as in Ref. [14]. We can obtain the upper bound $V_Q \leq N$ by noting that, for the measurements $M = \{\varphi_x^a\}$, $F = M^*$, $S_Q = N$ is achieved by the maximally entangled state and $S_{\mathrm{LHS}} \geq 1$ for the measurements M on a pure product state. Determining whether this naive upper bound or our lower bound is loose is an open problem.

However, the key application of our bound is the following unbounded violation with respect to the local dimension that is robust to experimental imperfections.

We take the number of measurements to be a function increasing sublinearly with the dimension and set Bob's measurements to be bases such that $C = \sqrt{d^{\epsilon-1}}$, $0 \leq \epsilon < 1$. Such bases we define as the ϵ-generalized MUBs. Then $V_Q \geq \sqrt{d^{1-\epsilon}}$ is unbounded using the number of measurements increasing slower than linearly with the dimension. For $\epsilon = 0$, the ϵ-generalized MUBs reproduce the original MUBs and V_Q the scaling of [14].

Implementation.—We now turn our abstract mathematical result into a form which could be tested in a laboratory. First, we consider independent multiple copies of a singlet state $|\Psi\rangle = |\psi_-\rangle^{\otimes k}$ with a single-pair fidelity $F < 1$, and the measurements in ϵ-generalized MUBs. The measurements are taken for each pair individually but for all pairs at the same time. The conclusive version of the experiment requires introducing a nonunit efficiency $\eta < 1$ for each detector at Alice's side [19]. The local dimension of the state $d = 2^k$ allows the existence of $d + 1$ original MUBs with $C = \sqrt{d^{-1}}$, see, e.g., [29]. We take the number of settings growing slower than the dimension, $N = d^{1-\sigma} = 2^{k(1-\sigma)}$ for $0 \leq \sigma < 1$ (N is assumed to be integer), and the overlap $C = \sqrt{d^{\epsilon-1}} = \sqrt{2^{k(\epsilon-1)}}$. This leads to $V_Q^\eta = [2^{k(1-\sigma)}(\eta F)^k]/[1 + (2^{k(1-\sigma)} - 1)\sqrt{2^{k(\epsilon-1)}}]$ (see SM [25], Section 2). Exponential unbounded violation of order of $O((\sqrt{2^{1-\epsilon}}\eta F)^k)$ is observed if $\epsilon < 2\log_2(\eta F) + 1$ (similar analysis can be performed for $N = \lfloor d^{1-\sigma} \rfloor$ leading to complex formulas). Remarkably, for any fidelity and efficiency satisfying $\eta F > 1/\sqrt{2}$, there exist ϵ such that the violation grows exponentially with the number of pairs k.

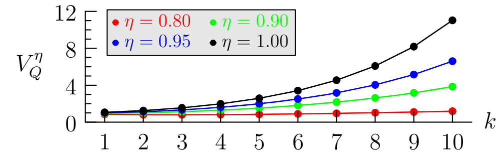

FIG. 2. Quantum violation V_Q^η of steering inequality (9) for the ϵ-generalized MUBs with $\epsilon = 0.1$, singlet state fidelity $F = 0.98$, and $\sigma = 0.5$ as a function of detection efficiency η.

However, $\eta F < 1$ limits the set of ϵ MUBs which lead to the violation [$\log_2(\eta F) < 0$]. Ultimately, for ηF approaching $1/\sqrt{2}$, only the original MUBs ($\epsilon = 0$) can be used.

It is clear that the best violation of (9) is obtained for the original MUBs, $\epsilon = 0$. Increasing ϵ results in monotonic decrease of the violation, but its exponential character is preserved. Figure 2 depicts dependence of the violation on the efficiency η for $F = 0.98$, $\epsilon = 0.1$, and $\sigma = 0.5$. For these parameters, the minimal required efficiency for observing the violation equals $\eta = 0.75$. The figure reveals an interesting property of quantum steering: exponential decay of the global fidelity F^k and efficiency η^k, observed for multipair sources of entangled qubits, is suppressed by an exponential number of measurements. The latter is possible for handling a small number of qubits (as in quantum tomography). In fact, this proposal has recently been demonstrated for the case of $k = 4$ copies and $N = 2$ settings corresponding to two MUB elements [30]. The main challenge in this type of experiments is extending the number of settings and creating more MUBs. This seems to require a significant nonlinearity, a resource which is rare. Hence, below, we shall consider a system where an unbounded violation is possible using solely linear optics.

We will now employ a quantum-optical scheme based on a parametric-down-conversion source generating a polarization entangled squeezed vacuum [31]. Its quantum correlations posses the same rotational invariance as the two-photon polarization singlet and can be seen as two copies of approximate original EPR correlations. Because of this property, these states have recently been successfully used to reveal a Bell nonlocality which does not vanish in the limit of large populations and numbers of settings [32]. Using the same key feature and the ϵ-generalized MUBs implemented by merely polarization rotations, we will show that entangled squeezed vacuum states lead to unbounded violation of our steering inequality.

An entangled squeezed vacuum can be expressed as a superposition of $2n$-photon polarization singlet states $|\psi_n\rangle = 1/\sqrt{n+1}(a_H^\dagger b_V^\dagger - a_V^\dagger b_H^\dagger)^n|0\rangle$ with a probability amplitude λ_n, $|\Psi\rangle = \sum_{n=0}^{\infty} \lambda_n|\psi_n\rangle$, where $a^\dagger$ ($b^\dagger$) is creation operator for a spatial mode a (b) and H (V) denotes horizontal (vertical) polarization [33,34], (see SM [25], Section 3A). Perfect correlations present in each multiparticle singlet are manifested by equal photon numbers in orthogonal polarizations in the spatial modes

$$|\psi_n\rangle = \frac{1}{\sqrt{n+1}} \sum_{m=0}^{n} (-1)^m |m_H, (n-m)_V\rangle_a |(n-m)_H, m_V\rangle_b. \tag{10}$$

They are preserved with respect to the global rotations of polarization. Each spatial mode in $|\psi_n\rangle$ contains a fixed number of particles equal to n. Projections on Fock states constitute a natural framework for Bob's measurements. They reveal the correlations but, also, distinguish $|\psi_n\rangle$ from $|\psi_{n'}\rangle$, which leads to post-selective creation of $|\psi_n\rangle$ from the squeezed vacuum state $|\Psi\rangle$. Thus, in our considerations, we will now focus on a particular $|\psi_n\rangle$.

To show unbounded violation (9) for (10) we will adopt a strategy similar to the one used in [32]: we will examine the correlations after applying incremental rotations on the spatial modes (the setup is shown in the SM [25], Section 3B), further denoted by θ_x. Each mode represents a $d = (n+1)$ dimensional Hilbert space spanned by one of the bases enumerated by $x \in \mathbb{N}_N$, generated by an appropriate polarization rotation

$$|\phi^m(\theta_x)\rangle \coloneqq |(n-m)_{H+\theta_x}, m_{V+\theta_x}\rangle. \tag{11}$$

They correspond to Bob's $\{\phi_x^m\}$ bases discussed in the Theorem and represent the ϵ-generalized MUBs (see Fig. S1 in the Supplemental Material [25]). Alice's measurements are given by $|\phi^{n-m}(\theta_x)\rangle$. We then have the following.

Proposition.—Given a set of N Bob's measurement bases $\{|\phi_x^m\rangle\} \coloneqq \{|\phi^m(\theta_x)\rangle\}$ with $m = \{0\} \cup \mathbb{N}_n$ and $x \in \mathbb{N}_N$, defined by some set of angles $0 \leq \theta_x < \pi/2$, $C = \max_{x,y,a,b} |\langle \phi_x^a | \phi_y^b \rangle|$ equals the maximal overlap between $\{|\phi^m(0)\rangle\}$ and $\{|\phi^m(\theta)\rangle\}$ with $\theta = \min_{x,y} |\theta_x - \theta_y|$

$$C^{(n)}(\theta) = \max_{m,l} |\langle \phi^l(0) | \phi^m(\theta)\rangle| = \sqrt{\binom{n}{q_{\theta,n}}} (\cos\theta)^n (\tan\theta)^{q_{\theta,n}}, \tag{12}$$

where $q_{\theta,n} \coloneqq \lfloor n \sin^2\theta - \cos^2\theta \rfloor + 1$ and $\lfloor \ldots \rfloor$ denotes the floor function. $C^{(n)}(\theta)$ goes to zero as fast as $1/\sqrt{4}n$ ($\epsilon = 1/2$).

Including experimental imperfections in their simplest form, we assume efficiency η for each of the two detectors at Alice's side. For the states (10), this modifies the quantum value of the steering functional to $\eta^n S_Q$ and condition (9) to $V_Q^\eta \geq [\eta^n N^{(n)}]/[1 + (N^{(n)} - 1)C^{(n)}(\theta)]$. Figure 3 depicts the violation V_Q (V_Q^η for $\eta = 0$) as a function of the local number of photons n, for the optimal angle $\theta = \pi/2N$ and number of settings N_{opt} minimizing $C^{(n)}(\theta)$. The dependence $N_{\text{opt}}^{(n)}$ is rather complex (see SM [25], Section 3C) and is displayed in the upper inset of Fig. 3. The gray area indicates the values of V_Q achieved for $N < N_{\text{opt}}^{(n)}$. Figure 4 presents V_Q^η as a function of n and detection efficiency η. As expected, the violation gets stronger for the increasing population of the system (larger n-s) but, also, for higher efficiencies. We notice a discrepancy between the value of critical efficiency for $n = 1$ ($d = 2$, the singlet case) shown in this figure $\eta = 0.86$ and the one reported in literature $\eta = 0.62$ [19]. It probably reveals that our estimation of the classical bound is not tight.

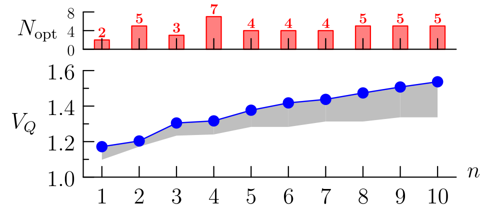

FIG. 3. Quantum violation V_Q of steering inequality (9) for multiparticle Bell-singlet states $|\psi_n\rangle$ (10) and rotation angle $\theta = \pi/2N$. The upper inset shows the optimal number of settings N_{opt} for a given n used in this computation. Gray area indicates a range of values of V_Q for $N < N_{\text{opt}}$.

Discussion.—We have derived the first fault-tolerant steering inequality and provided an experimentally feasible method of obtaining a quantum violation for large quantum systems. The fault-tolerance is obtained using the Deutsch-Maassen-Uffink uncertainty relations [4]. The proof of concept involves the ϵ-generalized MUBs and can be demonstrated for several two-photon singlets, leading to a violation of order of $O(\sqrt{n})$. The second implementation, remarkably, deploys linear optics to show that slowly increasing the number of settings with the local dimension of a maximally entangled state leads to an unlimited discrepancy of order of $O(\sqrt[4]{n})$ between the classical and quantum description of the experiment from Fig. 1. It involves entangled squeezed vacuum, polarization rotations ($\epsilon = 1/2$-generalized MUBs) and photon-number-resolving detection. Our analysis includes losses.

Squeezed vacuum states produced in parametric down-conversion, with a mean photon-number of order of ten, are available in laboratories [31,34,35]. Each $|\psi_n\rangle$ is created by post-selection: measurements at Alice's (or Bob's) side reveal photon numbers in all modes. Realizations could utilize techniques presented in [36,37] or integrated optics equipped with superconducting transition-edge sensors possessing near-perfect efficiencies and well-resolved

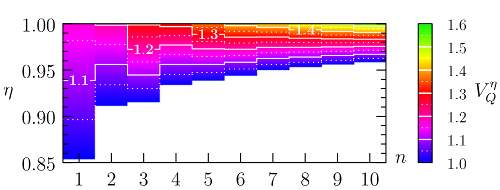

FIG. 4. Quantum violation V_Q^η of steering inequality (9) computed for multiparticle Bell-singlet states $|\psi_n\rangle$ (10), detection efficiency η, and the best N found for given n (see Fig. 3). White area indicates no violation.

photon-number peaks [38,39]. Angular momentum of light provides a good alternative [40–42].

The discussed unbounded violations emphasize disparity between steering and Bell-nonlocality correlations. Violation of Bell inequalities with a small number of settings vanishes for singlet states built from macroscopic qubits [43,44] and multiparticle Bell-singlet states [32]. Clarifying the possibility of loophole-free steering and the role of the fair sampling assumption similar to [45] necessitates further research.

Our approach may be exploited in several quantum-information tasks of great technological interest. It addresses the idea of partial characterization of devices developed in terms of semi-device-independent scenarios [46]. Multiparticle steering certifies randomness, and it may foster optimal strategies in quantum random access codes.

Our findings pose intriguing open questions. How would the number of settings scale in the best steering monogamy relations [47]? Furthermore, could an asymmetric steering reveal quantumness more adequately than the original one (cf. [48])? An affirmative answer may result in applications for information processing. Within the quantum-information resources theory, can a rigidity theorem [48] be established for multiparticle bipartite steering? It would also be interesting to examine our result within general probabilistic theories of the receiver's system (cf. [49]).

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Note added.—Recently, it has been brought to our attention that the result equivalent to a weaker part of the present theorem has been provided and analyzed in [50].

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