Chaotic Griffiths Phase with Anomalous Lyapunov Spectra in Coupled Map Networks

Kenji Shinoda* and Kunihiko Kaneko†
Department of Basic Science, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8902, Japan
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Dynamics of coupled chaotic oscillators on a network are studied using coupled maps. Within a broad range of parameter values representing the coupling strength or the degree of elements, the system repeats formation and split of coherent clusters. The distribution of the cluster size follows a power law with the exponent α, which changes with the parameter values. The number of positive Lyapunov exponents and their spectra are scaled anomalously with the power of the system size with the exponent β, which also changes with the parameters. The scaling relation $\alpha \sim 2(\beta + 1)$ is uncovered, which is universal independent of parameters and among random networks.

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After the success of extensive studies on network structures [1–3], the dynamics on networks has gathered considerable attention [4]. Apart from simple two-state dynamics as adopted in neural or gene-regulatory networks as well as in epidemic propagation, oscillatory dynamics are extensively investigated. The main focus therein lies in global synchronization among all oscillators: Depending on the network structure, synchronizability varies, and the design of easily synchronized networks has been analyzed [5,6].

When oscillators are globally synchronized, their dynamics are reduced into just that of a single oscillator. If the dynamic elements on the network are responsible for some function, global synchronization would imply the loss of the function. In the power grid network, such synchronization will lead to a global black out [7,8], while in neural networks, it may lead to the loss of cognitive function. In contrast, biological systems often avoid such global synchronization, and involve dynamics with many degrees, which are often suggested to lie at a critical state, represented by a power law in activities [9–15]. Hence, dynamics on the network, which achieves a critical state, need to be studied.

Indeed, dynamics with many degrees of elements are much richer, including spatiotemporal intermittency, split of elements into multiple coherent clusters, chaotic itinerancy that changes effective degrees of synchronization in time, and collective chaotic dynamics, as have been extensively studied in coupled map lattices (CMLs) [16–18] or globally coupled maps (GCMs) [19], where identical chaotic dynamics interact with each other. Behaviors discovered in coupled maps have been observed in fluid, optical, electronic, and chemical systems as well as in biological and neural dynamics [20], and also in direct experiments [21].

Further, behaviors of coupled maps neither on a simple lattice nor with global interactions have also been studied [22–26]. In particular, coupled maps on networks (CMNs) should be relevant for exploring the salient behaviors in high-dimensional dynamics, where conditions for chaotic synchronization [27,28] and splitting of elements into a few synchronized clusters, which also depends on network structures [29–36], have been investigated. Chaotic dynamics between synchronization and desynchronization with a critical state robust to parameter changes, however, has not been explored as yet.

In the present Letter, we study a CMN with chaotic logistic maps. After classifying the dynamics into several phases, we focus on a phase that we call the chaotic Griffiths phase, in which elements repeat synchronization and desynchronization intermittently. In this phase, the size distribution of synchronized clusters is found to follow a power law, while the Lyapunov spectra satisfy anomalous scaling. These "critical" behaviors are maintained over a broad range in the parameter values. Furthermore, the critical exponents of the clusters distribution and Lyapunov spectra change with the parameter values, while maintaining a certain scaling relationship.

To be specific, we study the following coupled map network,

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{k_i}\sum_{j=1}^{N} T_{i,j} f(x_n(j)), \quad (1)$$

with the logistic map $f(x) = 1 - ax^2$, and the coupling strength ϵ, where the adjacency matrix $T_{i,j}$ represents the Erdös-Rényi random network, and k_i is a degree of the element i, whose average is set at k. Here, we choose a sufficiently large value of a (say $1.6 < a < 2$), such that chaotic dynamics exist for the logistic map $x_{n+1} = f(x_n)$. As analyzed by linear stability analysis, the whole elements are synchronized for larger ϵ and k. The dynamics of this synchronized state are reduced to a single logistic map $x_{n+1} = f(x_n)$, and thus exhibit chaotic dynamics. With the decrease in ϵ or k, the synchronized state loses stability, and nontrivial dynamics appear. The attractor dynamics are

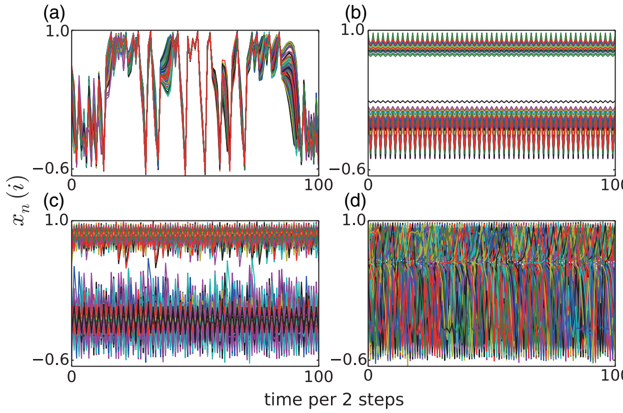

FIG. 1. Examples of typical time series in the CMN (1): $x_{2n}(i)$ as a function of time step n are overlaid for all elements i, after discarding initial transients of 10^6 steps. Instead of plotting every time step, the variables are plotted every two time steps to make them clearly discernible, since the period-two oscillation is inherent in the logistic map. $a = 1.7$ and $N = 200$. (a) $\epsilon = 0.5$, $k = 20$ [phase (ii)]. (b) $\epsilon = 0.35$, $k = 15$ [phase (iii)]. (c) $\epsilon = 0.2$, $k = 10$ [phase (iv)]. (d) $\epsilon = 0.05$, $k = 10$ [phase (v)].

roughly classified into the following phases (Figs. 1 and 2), as are also quantitatively characterized by the Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ (see Supplemental Material [37], Fig. S1). (i) Chaotic synchronization: All elements are synchronized, and thus, the dynamics are reduced to a single logistic map. $\lambda_1 > 0$, and $\lambda_j < 0$ for $j > 1$. (ii) Chaotic Griffiths phase (CGP): $\lambda_j > 0$ up to a certain number N_p with $1 \ll N_p \ll N$. As shown in Fig. 1(a), $x_n(i)$'s are almost synchronized for some time and are desynchronized later. Criticality with power-law statistics is preserved within the phase, as will be discussed in detail below. (iii) Ordered phase: After transients, chaos disappears and is replaced by a periodic or quasiperiodic attractor [see Fig. 1(b)], such that $\lambda_1 \leq 0$. The phase corresponds to the ordered phase in GCM [19] or pattern selection in CML [17]. Near the boundary to phase (ii), the chaotic transient before reaching the attractor is quite long. (iv) Frozen chaos phase with macroscopic order: Dynamics of each element are desynchronized and chaotic, while maintaining the period-2 band motion as $x > x^* \leftrightarrow x < x^*$, where x^* is the unstable fixed point of the map $x_{n+1} = f(x_n)$ [see Fig. 1(c)]. Here, many but not all of the Lyapunov exponents are positive. The number of positive Lyapunov exponents N_p increases linearly with N; i.e., $N_p = O(N) < N$. The phase corresponds to the frozen random pattern in CML. (v) Fully chaotic phase: All the Lyapunov exponents are positive, i.e., $N_p = N$, and dynamics are fully chaotic [see Fig. 1(d)]. The phase corresponds to the turbulent state in CML and GCM.

Hereafter, we focus on phase (ii) (CGP), as it is inherent to a network system, including both random connection and global coupling, and no correspondent phase exists in CML and GCM [with the increase in k, the phase shrinks and is replaced by the chaotic synchronization phase (see Supplemental Material [37], Fig. S2)]. First, we analyze the repetition of the synchronization-desynchronization process by defining a cluster with a given precision: By introducing bins with the precision δ, which is sufficiently small, we define a cluster as elements $x_n(i)$ that fall within the same bin. In Fig. 3, the time series of maximal cluster size is plotted. As shown, many of the elements are synchronized from time to time to form a large cluster, and then they are desynchronized. This represents the repetition of the synchronization-desynchronization process. The behavior is analogous with chaotic itinerancy in GCM [19,38], but remarkably, this phase exists not only at a critical point but also in a broad parameter region (see Fig. 2). To confirm this criticality, we computed the

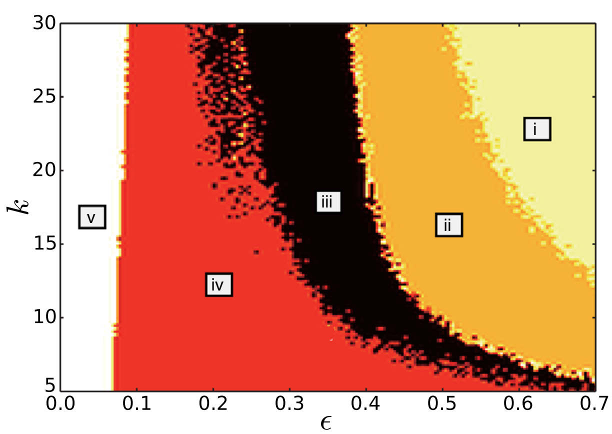

FIG. 2. Phase diagram of the CMN (1) with $a = 1.7$ and $N = 200$. Each phase (i)–(v) (see text) is determined by the Lyapunov exponents as described in the text (see Supplemental Material [37] for the diagrams on λ_1, λ_2, N_p, and an index for macroscopic order defined therein). The configuration of the phase diagram is independent of a, while the phase boundary is shifted.

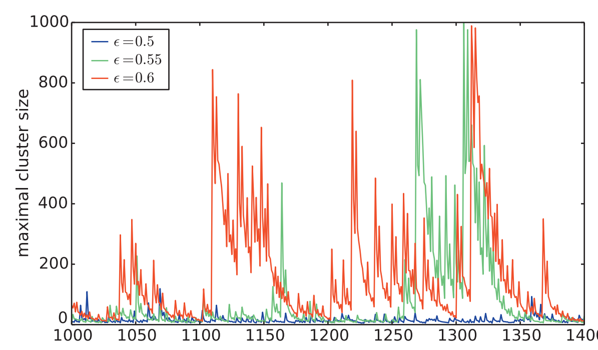

FIG. 3. Temporal evolution of the maximal cluster size. $a = 1.7$, $k = 20$, and $N = 1000$. The cluster is computed by using the threshold $\delta = 10^{-3}$, while this intermittent behavior does not vary as long as it is sufficiently small. $\epsilon = 0.5$ (blue line), $\epsilon = 0.55$ (green line), $\epsilon = 0.6$ (red line), in the chaotic Griffiths phase.

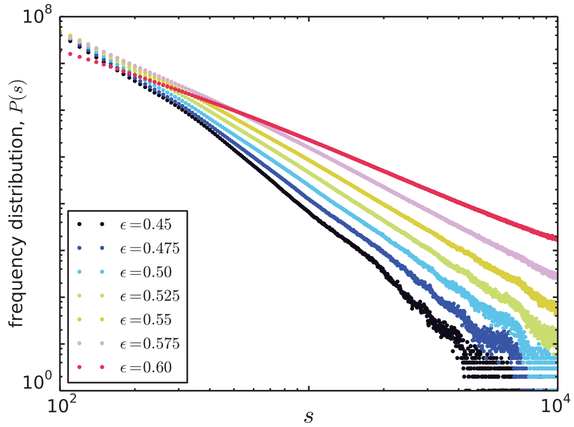

FIG. 4. The distribution $P(s)$ of cluster size s. Log-log plot. $a = 1.7$, $k = 20$, and $N = 16384$. The results from $\epsilon = 0.45$, 0.475, 0.5, 0.525, 0.55, 0.575, and 0.6 are plotted with different colors. The distribution is obtained by sampling over 10^3 steps, with 100 initial conditions, over 100 networks, by using the threshold $\delta = 10^{-3}$, while the exponents do not vary as long as this threshold is sufficiently small, and also the network sample dependence is negligible.

distribution $P(s)$ of cluster sizes s by long-term sampling of them. As shown in Fig. 4, the distribution obeys a power law $P(s) \sim s^{-\alpha}$ at this phase, where the exponent α changes with the parameters ϵ, k, and a. For given a, as ϵ or k is increased to approach the chaotic synchronization phase (i), the exponent α approaches 2, while it increases monotonically to ~4 as the parameters decrease towards the boundary value to phase (iii). This suggests that the criticality is maintained throughout phase (ii), while the exponent α changes monotonically with the parameters. (As for the change against k, see Supplemental Material [37], Fig. S3).

Existence of power-law behavior over a finite range of parameters is not typically observed in a regular lattice or a mean-field coupling system. Indeed, in the spatiotemporal chaos in CML, the correlation decays exponentially in space [39], while in the turbulent phase in GCM, the distribution of synchronized cluster decays exponentially such that a large cluster is not generated [19]. In these cases, the number of positive Lyapunov exponents N_p increases linearly with the number of elements N, i.e., they are extensive variables [40–43]. In contrast, in phase (ii) with a power-law behavior, the number of positive Lyapunov exponents N_p increases with N anomalously, as N^β [see Fig. 6(a)]. The exponent is ~1 at the boundary with phase (iii), and decreases monotonically as ϵ or k is increased, until it approaches 0 at the boundary with phase (i). Furthermore, the Lyapunov exponents $\lambda(z)$ plotted as a function of the scaled variable $z = i/N^\beta$ follow a single curve independent of N for positive exponents, except for the first few Lyapunov exponents [see Fig. 6(b) and Supplemental Material [37], Fig. S3]. This anomalous scaling is in stark contrast with the Lyapunov exponents in CML and GCM, where only the normal scaling $\beta = 1$ is observed (except at the critical point) [44].

The exponent $\beta < 1$ in the CMN implies that the degree of chaos does not properly increase with the system size, thus allowing room for generation of a large coherent cluster intermittently. Then, it is expected that with the increase from $\beta = 0$ to 1, the fraction of a larger cluster decreases, such that the exponent α is increased. The relationship between the two exponents is plotted in Fig. 5, where the data are roughly fitted with $\alpha \sim 2(\beta + 1)$.

This critical behavior within a finite parameter range in a system with quenched randomness is reminiscent of Griffiths phase, first predicted in the diluted Ising model [45]. The existence of Griffiths phase in network dynamics was recently reported [46–48]: Muñoz *et al.* studied a quenched contact process to find a power-law relaxation process within a finite range of remaining rate of edges. Indeed, in a contact process on a regular lattice, there exists a percolation threshold beyond which active states persist, and below which active states disappear, to be replaced by the absorbing state that is reached within a finite time. For a quenched disordered system, in contrast, the transition point is blurred, and the critical behavior is stretched in

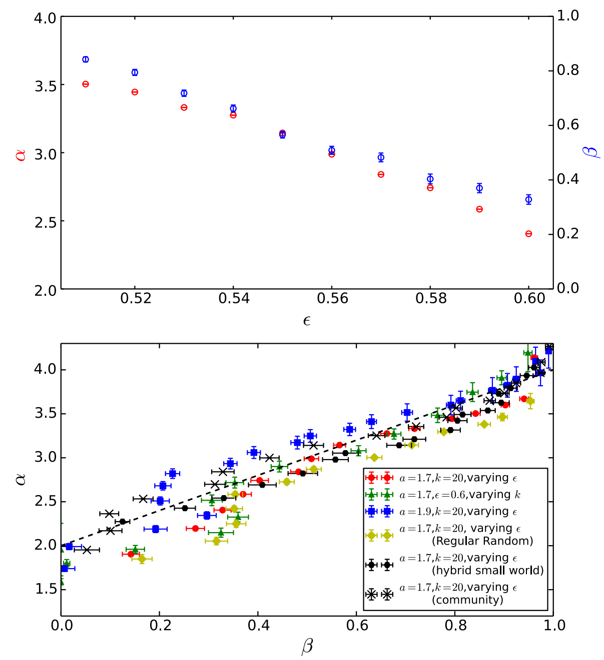

FIG. 5. (a) Dependence of the exponent α (red) and β (blue) upon ϵ, from the data of Figs. 4 and 6. (b) Relationship between the exponents α (abscissa) and β (ordinate). Apart from the data of Fig. 4 (○,red), the data with varying k by fixing ϵ at 0.6 (△,green), the data $a = 1.9$ (□,blue), and $a = 1.7$ on a regular random network (◇, yellow), as well as that with increasing the small-world structure (⋆, purple) and that with increasing community structure (×, black), with fixed k and varying ϵ are plotted. (see also Supplemental Material [37], Fig. S5). The error bars are computed from the least-square fit of the power law.

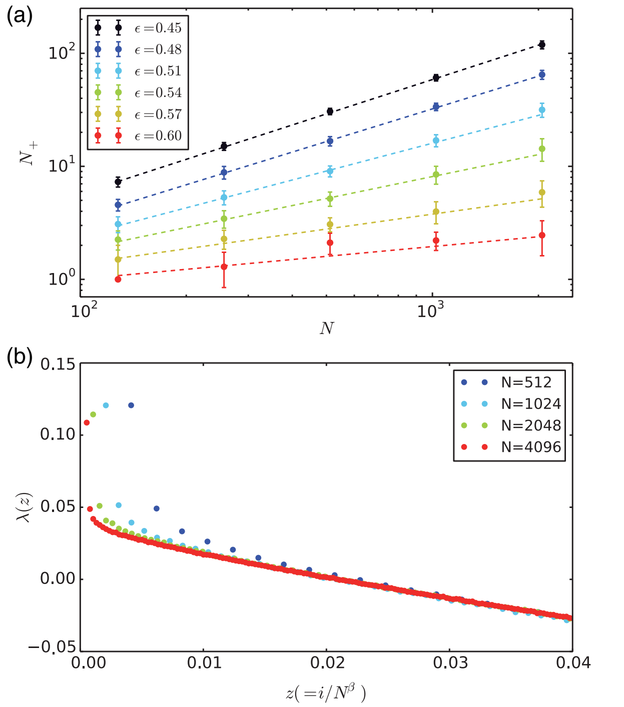

FIG. 6. (a) The number of positive Lyapunov exponents N_p plotted as a function of the system size N, for $\epsilon = 0.45$, 0.48, 0,51, 0.54, 0.57, and 0.6 with different colors. $a = 1.7$, $k = 20$. (b) The scaled Lyapunov spectra $\lambda(z)$ with $z = i/N^{\beta}$. The Lyapunov exponents are computed from 2000 time steps over 12 networks samples and 4 initial conditions.

the parameter space, leading to power-law relaxation in time to the absorbing state.

In GCM, the transition to chaotic synchronization occurs at a certain ϵ value, while in the present CMN, the transition is blurred, leading to perpetual repeat of formation and collapse of synchronized clusters, and the power-law distribution. In this sense, phase (ii) corresponds to the Griffiths phase. Here, however, the global synchronization is not an absorbing state, and the power-law behavior exists as an attractor, not in the relaxation, in contrast to the network Griffiths phases studied so far. Hence, we call phase (ii) the "chaotic Griffiths phase."

The scaling relationship $\alpha \sim 2(\beta + 1)$ seems to be valid, independent of ϵ and k, as well as of a [49]. A rough theoretical argument is as follows. In the limit to the boundary to chaotic synchronization, $\beta \to 0$ and $\alpha \to 2$. For simple approximation, let us represent the change in cluster size s by a random walk of s. In this case, as the size s increases the probability of a move at each step increases linearly with s, as each element in the cluster can synchronize or desynchronize with others. Then the stationary distribution approaches $P(s) \sim s^{-2}$, as is derived from the Fokker-Planck equation (FPE) $\partial P(s,t)/\partial t = \partial^2 s^2 P(s,t)/\partial s^2$, corresponding to the stochastic differentiation $ds = s dB_t$ of Ito calculus. Next, when $\beta > 0$, the probability of a move is expected to increase with the cluster size s as $s^{1+\beta}$, since the degrees for chaos in s elements increase with the power β, leading to an additional increase of move probability with s^{β}. Thus, the above equation is replaced by $ds = s^{1+\beta} dB_t$. The stationary distribution of the corresponding FPE is given by $\sim s^{-2(1+\beta)}$, leading to $\alpha \sim 2(1 + \beta)$. This argument is rather rough, and needs to be elaborated by complete theoretical explanation in the future [50].

Apart from the present Erdös-Rényi random network as well as regular random networks of coupled maps, we have confirmed that even for CMNs with some structure, the relationship between exponents α and β is again valid as long as the chaotic Griffiths phase exists for a finite range of parameter values. They cover networks with increased community structure or hybrid networks between random and small-world [see Fig. 5(b), Supplemental Material [37], Figs. S5 and S6] [54].

To sum up, we have reported the chaotic Griffiths phase in CMN: elements repeat formation of synchronization and desynchronization, where the size of synchronized clusters follows the power-law distribution with the power α. The exponent α changes monotonically in the phase, where the Lyapunov exponents follow the anomalous scaling with another exponent β, while roughly maintaining a monotonic relationship with $\alpha \sim 2(\beta + 1)$. This relationship is valid for a universal class of networks in which random, nonlocal paths are dominant over structured paths, where CGP exists in a finite range of parameters. Although we gave a rough theoretical explanation for the relationship, renormalization group theory needs to be developed to confirm its universality.

The prevalence of critical states is extensively reported in biological networks, especially in brain dynamics as a correlation of neural activities [13,14,55,56]. The module structure in neural connectivity is often explored as the origin of criticality. The present chaotic Griffiths phase may provide an alternative view on this, considering that chaotic neural dynamics are often reported.

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*shinoda@complex.c.u-tokyo.ac.jp
†kaneko@complex.c.u-tokyo.ac.jp

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