Eigenvalue Outliers of Non-Hermitian Random Matrices with a Local Tree Structure

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Spectra of sparse non-Hermitian random matrices determine the dynamics of complex processes on graphs. Eigenvalue outliers in the spectrum are of particular interest, since they determine the stationary state and the stability of dynamical processes. We present a general and exact theory for the eigenvalue outliers of random matrices with a local tree structure. For adjacency and Laplacian matrices of oriented random graphs, we derive analytical expressions for the eigenvalue outliers, the first moments of the distribution of eigenvector elements associated with an outlier, the support of the spectral density, and the spectral gap. We show that these spectral observables obey universal expressions, which hold for a broad class of oriented random matrices.

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Introduction.—Directed graphs represent graphically the causal relations between a discrete number of degrees of freedom of a dynamical system. Neural networks, transportation networks, and the Internet are examples of systems modeled by directed graphs. The dynamics of processes governed through directed graphs can be modeled with sparse non-Hermitian matrices, for example, Markov matrices define the dynamics of stochastic processes [1,2], and Jacobian matrices determine the stability of dynamical systems [3].

The dynamics of complex systems can be studied from the spectra of sparse non-Hermitian *random* matrices, even when the interactions between the relevant degrees of freedom are not known. Sparse non-Hermitian random matrices generalize random-matrix ensembles with independent and identically distributed matrix elements [4–12]. A general theory has been developed for the spectral density of sparse and non-Hermitian random matrices [13–20], but other spectral properties of these ensembles are still poorly understood.

Of particular importance are *eigenvalue outliers*, which are isolated eigenvalues located outside the continuous (bulk) part of the spectrum [see Fig. 1(a)]. Eigenvalue outliers of sparse non-Hermitian random-matrix ensembles, and their associated eigenvectors, are important for studies on the dynamics of complex systems, and for the evaluation of ranking and inference algorithms on graphs. The stationary state of a stochastic process is given by the left eigenvector associated with an outlier of a Markov matrix, the relaxation time is the inverse of the corresponding spectral gap [2,21], and the large-deviation function of an observable is given by an outlier of a modified Markov matrix [22–26]. Complex dynamical systems, such as neural networks [27–30] or ecosystems [31,32], are often modeled in terms of differential equations coupled through random matrices. The eigenvalue with the largest real part, which is often an outlier, determines the local stability of these systems [33,34]. The PageRank algorithm of Google Search uses the eigenvector associated with the outlier of a Markov matrix to rank pages of the World Wide Web [35,36]. Spectral algorithms detect communities in sparse graphs based on the eigenvectors of outliers in the spectrum of the non-backtracking matrix [18,37,38]. If these outliers exist, then it is possible to detect the communities of the graph. Conversely, if these outliers do not exist, then it is impossible for an algorithm to detect the communities. Quite apart from these applications, the study of outliers of random matrices is also a topic of interest in mathematics [39–41].

In this Letter we present a general theory for the outliers of matrices with a *local tree* structure. We present a set of exact relations for outliers of sparse non-Hermitian random matrices, and for the left- and right-eigenvector elements associated with an outlier. For oriented random matrices or oriented random graphs, i.e., directed graphs that have no bidirected links, we present explicit expressions for the eigenvalue outliers, the spectral gap, and the first two moments of the distribution of eigenvector elements associated with an outlier. Interestingly, we show that the eigenvalue outliers of oriented random matrices and their associated eigenvector moments obey universal expressions.

Outliers of non-Hermitian matrices.—We consider an $n\times n$ random matrix $\boldsymbol{A}_n$ with probability density $p(\boldsymbol{A}_n)$. The matrix $\boldsymbol{A}_n$ has n complex-valued eigenvalues $\lambda_1,\ldots,\lambda_n$, and its empirical spectral distribution is [42]:

$$\mu_{\boldsymbol{A}_n}=\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}, \tag{1}$$

with δ_{λ_j} the Dirac measure, i.e., $\delta_{\lambda_j}(S) = 0$ if $\lambda_j \notin S$ and $\delta_{\lambda_j}(S) = 1$ if $\lambda_j \in S$, with S a Lebesgue-measurable subset of $\mathbb{C}$. We assume that the matrix ensembles considered here are self-averaging, i.e., $\mu_{\boldsymbol{A}_n} \to \mu$ for $n \to \infty$, with μ a deterministic measure. The Lebesgue decomposition theorem [43] states that μ consists of an absolute continuous part $\mu_{\rm ac}$, a singular continuous part $\mu_{\rm sing}$, and a pure point part $\mu_{\rm pp}$. The spectral density $\rho(\lambda)$, also called the density of states, is the probability density function of $\mu_{\rm ac}$ [44]. Its support is the set Ω of all values $\lambda \in \mathbb{C}$ for which $\rho(\lambda) > 0$, and $\partial\Omega$ is the boundary of Ω. The measure $\mu_{\rm pp}$ is discrete; i.e., $\mu_{\rm pp} = n^{-1}\sum_{\alpha\in\mathcal{L}} a_\alpha \delta_{\lambda_\alpha}$, with $\mathcal{L}$ a countable set, and a_α the algebraic multiplicities of the eigenvalues λ_α. The outliers of a random matrix are the eigenvalues λ_α that lie outside the support Ω of the spectral density ($\lambda_\alpha \notin \Omega$). We consider here isolated outliers $\lambda_{\rm isol}$, which are nondegenerate. In Fig. 1(a) we show the eigenvalues of a single instance of a random matrix; the outlier $\lambda_{\rm isol}$ and the boundary $\partial\Omega$ are indicated.

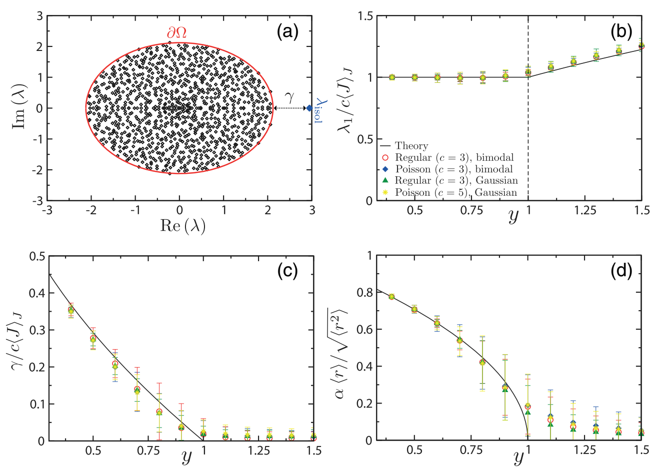

FIG. 1. The outlier $\lambda_{\rm isol}$, the spectral gap γ and the first moment $\langle r\rangle$ of the eigenvector, associated with $\lambda_{\rm isol}$, of oriented adjacency matrices. Direct-diagonalization results of matrices of finite size $n = 1000$ (markers) are compared with our theory for infinite-sized matrices, given by Eqs. (14)–(17) (solid lines). (a) Eigenvalues of one c-regular matrix with Gaussian distributed off-diagonal elements, mean degree $c = 3$ and $y = 0.5$, with $y = \langle J^2\rangle_J/(c\langle J\rangle_J^2)$ the disorder parameter. The boundary $\partial\Omega$ of the support of the spectral density, the spectral gap γ, and the outlier $\lambda_{\rm isol}$ are indicated. (b)–(d) The eigenvalue λ_1 with the largest real part, the spectral gap γ, and the first moment $\langle r\rangle$ of the right eigenvector associated with λ_1, all plotted as a function of y. The eigenvalue λ_1 is an outlier, i.e., $\lambda_1 = \lambda_{\rm isol}$, for $y < 1$, and $\lambda_1 \in \partial\Omega$ for $y > 1$. Results shown are for four different ensembles of oriented matrices. The ensembles are either c-regular or Poissonian with mean connectivity c; nonzero off-diagonal elements are i.i.d. with either a bimodal distribution $p_J(J) = (1-\Delta)\delta(J+1) + \Delta\delta(J-1)$, or a Gaussian distribution with mean $\langle J\rangle_J = 1$; diagonal matrix elements are set to zero. Direct-diagonalization results in subfigures (b)–(d) are from 1000 samples. Error bars represent the standard deviation of the sampled population and $\alpha = \sqrt{\langle (K^{\rm out})^2\rangle_{K^{\rm out}} - c}/c$.

Sparse matrices.—We consider a sparse random and non-Hermitian matrix $\boldsymbol{A}_n$. The matrix elements of $\boldsymbol{A}_n$ are $[\boldsymbol{A}_n]_{jk} = C_{jk}J_{jk}$, with C_{jk} the elements of the adjacency matrix of a random and directed graph [45], and J_{jk} complex-valued weights that determine the dynamics of a process on a graph. A connectivity element C_{jk} is either 0 or 1; if there is a directed link from vertex j to vertex k, then $C_{jk} = 1$, whereas if there is no link between the two vertices, then $C_{jk} = 0$. We set the diagonal elements C_{ii} to one. We consider graph ensembles of finite connectivity, in other words, the outdegrees $K_j^{\rm out} = \sum_{k=1(k\neq j)}^n C_{jk}$, and the indegrees $K_j^{\rm in} = \sum_{k=1(k\neq j)}^n C_{kj}$, are finite and independent of n. Additionally, we consider that the random graph with adjacency matrix C_{jk} is locally treelike [46], which means that a typical neighborhood of a vertex has no cycles of degree three or higher [47]. The ensemble of regular directed graphs [15,17] and the directed Erdös-Rényi (or Poisson) ensemble [14] are examples of ensembles that are locally treelike in the infinite-size limit $n \to \infty$.

General theory.—We present a theory for the outliers $\lambda_{\rm isol}$ of locally treelike random matrices $\boldsymbol{A}_n$, and their corresponding left and right eigenvectors, which we denote by $\langle l_{\rm isol}|$ and $|r_{\rm isol}\rangle$, respectively. We first write the right and left eigenvectors of a given outlier $\lambda_{\rm isol}$ in terms of the resolvent $\mathbf{G}_n$ of a matrix $\boldsymbol{A}_n$. We define the resolvent $\mathbf{G}_n(\lambda)$ of the matrix $\boldsymbol{A}_n$ as

$$\mathbf{G}_n(\lambda) \equiv (\boldsymbol{A}_n - \lambda\mathbf{1}_n)^{-1}, \qquad (2)$$

with $\lambda \in \mathbb{C}$. The resolvent $\mathbf{G}_n$ is singular at the eigenvalues $\lambda = \lambda_j$ of $\boldsymbol{A}_n$. Indeed, when we apply the eigendecomposition theorem to $\mathbf{G}_n$, we find

$$\mathbf{G}_n = \frac{|r_{\rm isol}\rangle\langle l_{\rm isol}|}{\lambda_{\rm isol} - \lambda} + \sum_{j=2}^{n} \frac{|v_j^{(r)}\rangle\langle v_j^{(l)}|}{\lambda_j - \lambda}, \qquad (3)$$

with $|v_j^{(r)}\rangle$ and $\langle v_j^{(l)}|$, respectively, the right and left eigenvectors associated with λ_j. If we set $\lambda = \lambda_{\rm isol} - i\eta$, with η a small real-valued regularizer, then we have

$$\lim_{\eta\to 0} i\eta\,\mathbf{G}_n(\lambda_{\rm isol} - i\eta) = |r_{\rm isol}\rangle\langle l_{\rm isol}| + \mathcal{O}(\eta). \qquad (4)$$

Since $\lambda_{\rm isol}$ is an outlier, the relation (4) holds, and is well defined for $n \to \infty$.

We compute the elements of the resolvent $\mathbf{G}_n(\lambda - i\eta)$ using the local tree structure of sparse ensembles in the infinite-size limit. The outcome of our procedure is a set of recursive equations for the eigenvector elements $r_j = \langle j|r_{\rm isol}\rangle$ and $l_j = \langle j|l_{\rm isol}\rangle$ (see Supplemental Material [48]):

$$r_j = -g_j \sum_{k\in\partial_j} A_{jk} r_k^{(j)}, \tag{5}$$

$$l_j^* = -g_j \sum_{k\in\partial_j} (l_k^{(j)})^* A_{kj}, \tag{6}$$

with the "neighborhood" ∂_j the set of vertices $k(\neq j)$ for which either $C_{kj} \neq 0$ or $C_{jk} \neq 0$. The variables g_j are the diagonal elements of the resolvent $\mathbf{G}_n$, i.e., $g_j = [\mathbf{G}_n(\lambda - i\eta)]_{jj}$. They solve the equations

$$g_j = \frac{1}{-\lambda + i\eta + A_{jj} - \sum_{k\in\partial_j} A_{jk} g_k^{(j)} A_{kj}}, \tag{7}$$

$$g_j^{(\ell)} = \frac{1}{-\lambda + i\eta + A_{jj} - \sum_{k\in\partial_j\setminus\{\ell\}} A_{jk} g_k^{(j)} A_{kj}}, \tag{8}$$

for $\lambda \notin \Omega$. The random variables $r_j^{(\ell)}$ and $l_j^{(\ell)}$ in Eqs. (5) and (6) solve

$$r_j^{(\ell)} = -g_j^{(\ell)} \sum_{k\in\partial_j\setminus\{\ell\}} A_{jk} r_k^{(j)}, \tag{9}$$

$$(l_j^{(\ell)})^* = -g_j^{(\ell)} \sum_{k\in\partial_j\setminus\{\ell\}} (l_k^{(j)})^* A_{kj}, \tag{10}$$

with $\ell \in \partial_j$, and where the limit $\eta \to 0$ is implicit. An outlier $\lambda_{\rm isol}$ is given by a value λ for which the Eqs. (5)–(10) admit a nontrivial solution, i.e., a solution for which all eigenvector components r_j and l_j^* are neither zero-valued nor infinitely large. The Eqs. (5)–(10) apply to outliers of non-Hermitian matrices with a local tree structure; they extend the equations for the largest eigenvalue of sparse symmetric matrices in Refs. [56–59].

Oriented matrices.—We illustrate our theory on oriented random-matrix ensembles. Oriented matrices contain only directed links, i.e., $C_{jk}C_{kj} = 0$ for all $j \neq k$. For oriented matrices the resolvent Eqs. (7) and (8) simplify and admit the solution

$$g_j = g_j^{(\ell)} = (-\lambda + A_{jj})^{-1}. \tag{11}$$

The eigenvector components are then given by

$$r_j = r_j^{(\ell)}, \quad \text{for all}, \quad \ell \in \partial_j^{\rm in}, \tag{12}$$

$$l_j = l_j^{(\ell)}, \quad \text{for all}, \quad \ell \in \partial_j^{\rm out}, \tag{13}$$

where the random variables $r_j^{(\ell)}$ and $l_j^{(\ell)}$ represent a nontrivial solution to the Eqs. (9) and (10). The "in-neighborhood" $\partial_j^{\rm in}$ is the set of vertices $k(\neq j)$ with $C_{kj} \neq 0$, and the "out-neighbourhood" $\partial_j^{\rm out}$ is the set of vertices $k(\neq j)$ with $C_{jk} \neq 0$.

We derive explicit analytical and numerical results by ensemble averaging the Eqs. (12) and (13). An outlier $\lambda_{\rm isol}$, and its associated eigenvector moments $\langle r^m\rangle = n^{-1}\langle\sum_{j=1}^n r_j^m\rangle$ and $\langle l^m\rangle = n^{-1}\langle\sum_{j=1}^n l_j^m\rangle$, with $m = 1, 2$, follow from a nontrivial solution to the ensemble-averaged equations; the symbol $\langle\ldots\rangle$ denotes here the ensemble average with respect to the distribution $p(\boldsymbol{A}_n)$. Additionally, we can compute the associated ensemble-averaged distribution of eigenvector elements using the population dynamics algorithm [48,60–63]. We illustrate this ensemble-averaging procedure on two paradigmatic examples of sparse matrix ensembles: adjacency matrices and Laplacian matrices of oriented random graphs.

Adjacency matrices.—We consider random adjacency matrices associated with random oriented graphs with a given joint distribution $p_{K^{\rm in},K^{\rm out}}$ of in- and outdegrees [45,64,65]. The off-diagonal weights J_{kj}, with $k \neq j$, are independent and identically distributed (i.i.d.) with distribution p_J, and the diagonal weights J_{jj} are i.i.d. with distribution p_D.

The oriented adjacency matrices we consider here have either exactly one outlier [see Fig. 1(a)] or do not have any outlier. If the outlier exists, we call the random-matrix ensemble gapped. Conversely, if the outlier does not exist, we call the ensemble gapless. If the outlier exists, its value $\lambda_{\rm isol}$ solves [48]

$$\langle(\lambda_{\rm isol} - D)^{-1}\rangle_D = \frac{1}{c\langle J\rangle_J}, \tag{14}$$

with $\langle\cdot\rangle_D$ and $\langle\cdot\rangle_J$ denoting, respectively, the average with respect to the distributions p_D and p_J. The quantity $c = \langle K^{\rm in}\rangle_{K^{\rm in}} = \langle K^{\rm out}\rangle_{K^{\rm out}}$ is the mean degree of the graph, where $\langle\cdot\rangle_{K^{\rm in}}$ and $\langle\cdot\rangle_{K^{\rm out}}$ denote averages with respect to the distributions of in- and outdegrees, respectively. Equation (14) follows from solving the ensemble-averaged version of the Eqs. (5) and (6) for the eigenvector moments. The first two moments of the distribution of right- and left-eigenvector elements read [48]

$$\langle r\rangle^2/\langle r^2\rangle = \frac{\mathcal{Q}}{\langle(K^{\rm out})^2\rangle_{K^{\rm out}} - c}, \tag{15}$$

$$\langle l\rangle^2/\langle l^2\rangle = \frac{\mathcal{Q}}{\langle(K^{\rm in})^2\rangle_{K^{\rm in}} - c}, \tag{16}$$

with $\mathcal{Q} = \langle\langle J\rangle_J^2/|\lambda_{\rm isol} - D|^2\rangle_D^{-1} - c\langle J^2\rangle_J/\langle J\rangle_J^2$. Additionally, we find the support Ω of $\rho(\lambda)$ from a stability analysis around the solution (11) to the resolvent Eqs. (7) and (8); the set Ω contains the values $\lambda \in \mathbb{C}$ with

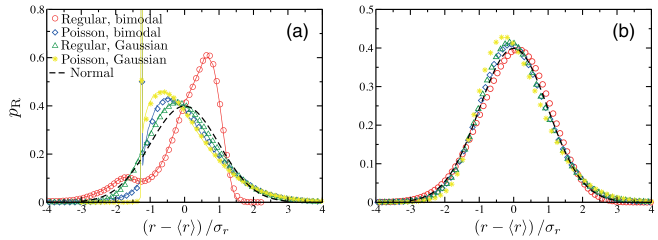

FIG. 2. Probability distribution $p_{\rm R}$ of the right eigenvector elements associated with the outlier of oriented adjacency matrices. The ensembles are the same as in Fig. 1 with disorder parameter $y = 0.4$, and mean connectivity (a) $c = 3$ or (b) $c = 10$. We compare direct-diagonalization results (markers) with population-dynamics results [solid lines in (a)] and with the normal distribution (dashed line). Direct-diagonalization results are for $2e + 4$ matrix samples of size $n = 1000$. In order to illustrate the universality of the distributions at high connectivities, we have rescaled the distributions with their mean $\langle r \rangle$ and their standard deviation σ_r.

$$\left\langle \frac{1}{|\lambda - D|^2} \right\rangle_D^{-1} < c\langle J^2 \rangle_J. \qquad (17)$$

In Fig. 1 we compare the analytical expressions, given by Eqs. (14)–(17), with direct-diagonalization results of matrices of finite size. Results are in good correspondence and converge to the theoretical expressions for large matrix sizes $n \gg 1$ (for which the ensembles become locally treelike).

Equations (14)–(17) imply that outliers of oriented adjacency matrices, and the first moments of their associated eigenvector distributions, are universal. In order to illustrate the universality of outliers, we plot in Fig. 1(b)–1(d), for different matrix ensembles, the eigenvalue outlier, the spectral gap, and the first two moments of the eigenvector distribution, as a function of the disorder parameter $y = \langle J^2 \rangle_J / (c\langle J \rangle_J^2)$. Diagonalization results for different ensembles collapse on one universal curve given by our analytical expressions Eqs. (14)–(17).

A characteristic feature of Fig. 1 is the phase transition from a gapless phase at high disorder, $y > 1$, to a gapped phase at low disorder, $y < 1$. Notice that this phase transition is generic and it also appears in symmetric random matrix ensembles [56–59,66,67].

For large mean connectivities, $c \gg 1$, the distributions of right- and left-eigenvector elements associated with $\lambda_{\rm isol}$ become universal, and from Eqs. (5) and (6), it follows that they are Gaussian. In Fig. 2(b) we illustrate the universal behavior of eigenvector distributions at high connectivities. At low connectivities these distributions are not universal, but direct-diagonalization results are in good correspondence with numerical solutions of Eqs. (5) and (6) using the population dynamics algorithm [see Fig. 2(a)].

Laplacian matrices.—Laplacian matrices generate the dynamics of random walks on graphs. The defining feature of Laplacian matrices is the constraint $J_{jj} = -\sum_{k=1,(k\neq j)}^{n} J_{jk}$ on their diagonal elements. Symmetric Laplacian matrices have been studied in [68,69]. Here we study the spectra of unnormalized Laplacian matrices of oriented graphs with off-diagonal matrix elements $J_{jk} = 1$ and with a given joint degree distribution $p_{K^{\rm in},K^{\rm out}}$ [45,64,65].

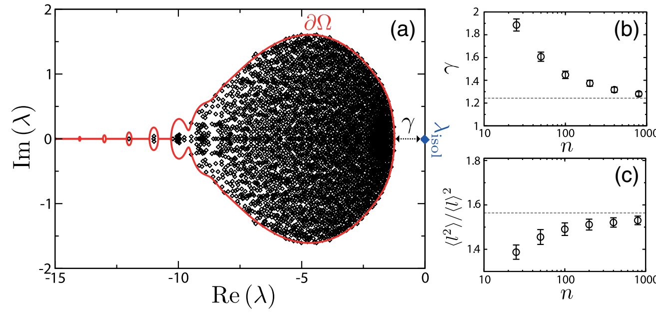

FIG. 3. Results for unnormalized Laplacian matrices associated with oriented Erdös-Rényi random graphs with off-diagonal matrix elements $J_{kj} = 1$ for $k \neq j$. We consider here an ensemble with correlated in- and outdegrees: $p_{K^{\rm in},K^{\rm out}}(k^{\rm in}, k^{\rm out}) = \delta(k^{\rm in}; k)\delta(k^{\rm out}; k)p_{\rm deg}(k)$. The degree distribution $p_{\rm deg}(k)$ is Poissonian, i.e., $p_{\rm deg}(k) = \mathcal{N} e^{-\tilde{c}}\tilde{c}^k/k!$, if $k \geq k_0$, and $p_{\rm deg}(k) = 0$ if $k < k_0$, with $\mathcal{N}$ the normalization constant. Direct-diagonalization results (markers) are compared with analytical results (solid lines) for $k_0 = 2$ and $\tilde{c} = 4$. (a) Spectrum of a single matrix with $n = 4000$. The red line shows the boundary $\partial\Omega$, of the support of the spectral density, which follows from Eqs. (19). (b),(c) The spectral gap γ and the moments $\langle l^2 \rangle / \langle l \rangle^2$ are shown to converge to their theoretical values for $n \to \infty$. Direct-diagonalization results are averages over $1e + 3$ matrices (markers) and theoretical expressions follow from Eqs. (18) and (19) (dashed lines).

Laplacian matrices have an eigenvalue outlier $\lambda_{\rm isol} = 0$, and the associated distribution of right-eigenvector elements reads $p_{\rm R}(r) = \delta(r - 1)$. The associated distribution of left-eigenvector elements $p_{\rm L}(l)$ determines the steady-state statistics of the position of a random walk on the associated graph. From Eqs. (6) we find for the moments of the distribution of left-eigenvector elements (see Supplemental Material [48]):

$$\frac{\langle l^2 \rangle}{\langle l \rangle^2} = \frac{\left\langle \frac{\langle K^{\rm in} \rangle^2 - c}{\langle K^{\rm out} \rangle^2} \right\rangle_{K^{\rm in},K^{\rm out}}}{\langle \frac{c}{K^{\rm out}} \rangle_{K^{\rm out}}^2 - \left\langle \frac{c}{\langle K^{\rm out} \rangle^2} \right\rangle_{K^{\rm out}}}. \qquad (18)$$

We furthermore find that the support Ω of the spectral density is the set of values $\lambda \in \mathbb{C}$ for which either

$$\left\langle \frac{K^{\rm out}}{|\lambda + K^{\rm out}|^2} \right\rangle_{K^{\rm out}} > 1, \quad \text{or} \quad \left\langle \frac{K^{\rm in}}{|\lambda + K^{\rm out}|^2} \right\rangle_{K^{\rm in},K^{\rm out}} > 1. \qquad (19)$$

In Fig. 3(a) we compare the Eqs. (19) for Ω with direct-diagonalization results of Laplacian matrices of finite size. Additionally, in Figs. 3(b) and 3(c) we compare direct-diagonalization results for the spectral gap γ and the ratio of the moments $\langle l^2 \rangle / \langle l \rangle^2$ with the exact expressions (18) and (19) for $n \to \infty$.

Discussion.—We have presented an exact theory for the outliers of random matrices with a local tree structure. Remarkably, for oriented matrices, we find general analytical expressions for the outliers, the associated statistics of eigenvector elements, and the support of the spectral density. These results show that the statistics of outliers of sparse oriented random matrices obey universal expressions. It will be interesting to explore the implications of these results for the dynamics of complex systems with unidirectional interactions, which often appear in biological systems that operate far from thermal equilibrium, for example, neural networks [70,71] or networks of biochemical reactions [72]. Our theory, based on the Eqs. (5)–(10), applies also to sparse nonoriented random matrices, and we illustrate this on the elliptic regular ensemble in the Supplemental Material [48]. Following Refs. [15,16], it is possible to extend our approach to random matrices that contain many short cycles. We expect that studies along these lines will lead to a general theory for the outliers of sparse random matrices.

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