Higher-Spin Theory of the Magnetorotons

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(Received 24 May 2016; published 15 November 2016)

Fractional quantum Hall liquids exhibit a rich set of excitations, the lowest energy of which are the magnetorotons with dispersion minima at a finite momentum. We propose a theory of the magnetorotons on the quantum Hall plateaus near half filling, namely, at filling fractions $\nu = N/(2N+1)$ at large N. The theory involves an infinite number of bosonic fields arising from bosonizing the fluctuations of the shape of the composite Fermi surface. At zero momentum there are $O(N)$ neutral excitations, each carrying a well-defined spin that runs integer values $2, 3, \ldots$. The mixing of modes at nonzero momentum q leads to the characteristic bending down of the lowest excitation and the appearance of the magnetoroton minima. A purely algebraic argument shows that the magnetoroton minima are located at $q\ell_B = z_i/(2N+1)$, where ℓ_B is the magnetic length and z_i are the zeros of the Bessel function J_1, independent of the microscopic details. We argue that these minima are universal features of any two-dimensional Fermi surface coupled to a gauge field in a small background magnetic field.

DOI: 10.1103/PhysRevLett.117.216403

Interacting electrons moving in two dimensions in a strong magnetic field can form nontrivial topological states: the fractional quantum Hall liquids [1,2]. When the lowest Landau level is filled at certain rational filling fractions, including $\nu = N/(2N+1)$ and $\nu = (N+1)/(2N+1)$ (Jain's sequences), the quantum Hall liquid is gapped, and the lowest-energy mode is a neutral mode. Girvin, MacDonald, and Platzman [3] proposed, based on a variational ansatz, that the neutral excitation has a broad minimum at $q\ell_B \sim 1$ at the Laughlin plateau $\nu = 1/3$. Several years later, the existence of a neutral mode was confirmed experimentally [4]. Later experiments revealed a surprising richness in the structure of the spectrum of neutral excitations. Unexpectedly, the $\nu = 1/3$ state may have more than one branch of excitations [5]. Furthermore, higher in the Jain sequence, i.e., for $\nu = 2/5, 3/7$, etc., the lowest excitation has been found to have a dispersion with more than one minima [6,7]. Various theoretical approaches have been brought to the problem of the magnetoroton [8–12]. Currently, the most common viewpoint is based on the composite fermion picture of the fractional quantum Hall effect, in which the neutral modes are bound states of a composite fermion and a composite hole.

The notion of the composite fermion is tightly connected to the Halperin-Lee-Read (HLR) field theory [13], proposed as the low-energy description of the half filled Landau level. Recently, an analysis of the particle-hole symmetry of the lowest Landau level has lead to a revision of the HLR proposal: the low-energy degrees of freedom is now a Dirac composite fermion coupled to a gauge field [14]. Magnetorotons provide a rare window into the dynamics of a Fermi surface coupled to a gauge field, a long-standing problem of condensed matter physics [15,16].

None of the previous analytical approaches to the magnetoroton can deal with the non-Fermi liquid at $\nu = 1/2$ or even with a composite Fermi liquid with general nonzero values of the Landau parameters. In this Letter, we develop a theory of neutral excitations in the quantum Hall liquid, reliable in the limit $N \to \infty$ in Jain's series $\nu = N/(2N+1)$, where quantum Hall plateaux have been found to up to at least $N = 10$ [17]. In this theory, the neutral excitations are viewed as quantized shape fluctuations of the Fermi surface. This interpretation is quite different from what has been suggested so far and is one with a predictive power. In particular, one can relate the whole dispersion curves of the neutral excitations to the excitation energies at zero momentum. We find that the dispersion curves have deep magnetoroton minima at large N. Remarkably, the momenta at the magnetoroton minima are independent of all microscopic dynamics and are in quantitative agreement with existing experimental data even for small N.

Quantizing the shape of the Fermi surface.—To find the magnetorotons we will first bosonize the Fermi surface. This procedure was studied previously [18–21]. Our approach relies on a commutation algebra of fluctuations of the shape of the Fermi surface, first derived by Haldane [18]. Here we provide a simple semiclassical derivation of this algebra.

We assume that the $\nu = 1/2$ state is gapless and has a Fermi surface with the Fermi momentum p_F, related to the external magnetic field B by $p_F^2 = B$. The Fermi liquid is

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characterized by the Fermi velocity v_F and Landau's parameters F_n. The effective mass is defined as $m_* = p_F/v_F$, the Fermi energy scale as $\epsilon_F = v_F p_F$.

In the fractional quantum Hall $\nu = N/(2N+1)$ state, the composite fermions live in a magnetic field $b = B/(2N+1)$, effectively forming an integer quantum Hall state. We are interested in the regime of frequency and momentum of the order of N^{-1} compared to the Fermi energy and momentum. We now propose that all low-energy excitations can be viewed as deformations of the Fermi surface from the circular shape, which we parameterize by a function $p_F(t, \mathbf{x}, \theta)$ that depends on time and space and also on the direction in momentum space θ ($p_y/p_x = \tan\theta$) (see Fig. 1). Furthermore, we decompose the perturbation into different angular momentum channels:

$$p_F(t,\mathbf{x},\theta) = p_F^0 + u(t,\mathbf{x},\theta) = p_F^0 + \sum_{n=-\infty}^{\infty} u_n(t,\mathbf{x}) e^{-in\theta}. \tag{1}$$

In the language of Landau's Fermi liquid theory, the state parameterized by $p_F(t, \mathbf{x}, \theta)$ corresponds to a distribution function $n_{\mathbf{p}}(t, \mathbf{x})$ which is one inside the Fermi line and zero outside the line.

We now derive the commutation relation between the u_ns with the following prescription. If we define an operator F (and similarly G) as

$$F = \int \frac{d\mathbf{x}d\mathbf{p}}{(2\pi)^2} F(\mathbf{x},\mathbf{p}) n_{\mathbf{p}}(\mathbf{x}), \tag{2}$$

where $n_{\mathbf{p}}(\mathbf{x})$ is the quasiparticle distribution function, then we need to impose the condition on the commutation relation so that

$$[F,G] = -i \int \frac{d\mathbf{x}d\mathbf{p}}{(2\pi)^2} \{F,G\}(\mathbf{x},\mathbf{p}) n_{\mathbf{p}}(\mathbf{x}), \tag{3}$$

where the $\{F, G\}$ is the classical Poisson bracket between F and G,

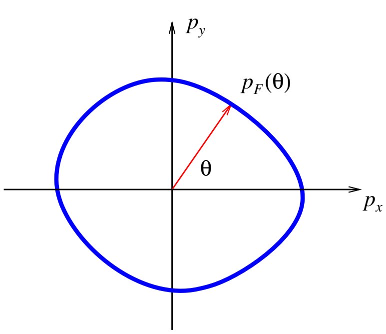

FIG. 1. A deformed Fermi surface.

$$\{F,G\} = \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial x_i} - b\epsilon^{ij}\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}, \tag{4}$$

where we have allowed the composite fermions to be in an external magnetic field b. For Jain's sequences $b = \pm B/(2N+1)$. Restricting $n_{\mathbf{p}}$ to be of the form of the step function (1 inside the Fermi line and 0 outside), F, G, and the right-hand side of Eq. (3) become functionals of the shape of the Fermi surface, and one can easily derive the commutator of the small perturbations u:

$$[u(\mathbf{x},\theta), u(\mathbf{x}',\theta')] = \frac{i(2\pi)^2}{p_F}\left(-n_i(\theta)\frac{\partial}{\partial x_i} + \frac{b}{p_F}\frac{\partial}{\partial\theta}\right) \times [\delta(\mathbf{x}-\mathbf{x}')\delta(\theta-\theta')] + O(u), \tag{5}$$

where $\mathbf{n}(\theta) = (\cos\theta, \sin\theta)$. In terms of u_n, the formula reads

$$[u_m(\mathbf{q}), u_n(\mathbf{q}')] = \frac{\pi}{p_F}\left[\frac{2bm}{p_F}\delta_{m+n,0} + \delta_{m+n,1}q_+ + \delta_{m+n,-1}q_-\right](2\pi)^2\delta(\mathbf{q}+\mathbf{q}') + O(u), \tag{6}$$

where $q_\pm = q_x \pm iq_y$. This commutation relation has been previously derived in Ref. [18] by extending Tomonaga's bosonization method to higher dimensions. Note that the algebra depends only on the size of the Fermi surface p_F but not on any dynamic properties (Fermi velocity, Landau's parameters, etc.).

Gauging the Fermi surface.—The composite fermion is coupled to a dynamical gauge field. A Fermi surface coupled to a gauge field is a long-standing theoretical problem, and the bosonized language allows us to partly address it.

In the bosonic description, the temporal component of the gauge field a_0 is coupled to u_0, and the spatial components are coupled to $u_{\pm 1}$. In the Dirac composite fermion theory, the leading term in action for a_μ is the Maxwell term. If the dynamical gauge field is at infinitely strong coupling, then the constraints $u_0 = u_{\pm 1} = 0$ arise as the result of the equations of motion $\delta S/\delta a_\mu = 0$. The assumption of strong gauge coupling should become better and better in the limit $N \to \infty$. This is due to two reasons. First, the coupling of the composite fermions to the gauge field is set at the Fermi energy ϵ_F and momentum p_F, while the scales of interest for our problem are ϵ_F/N and p_F/N. This gauge coupling is relevant for contact and marginal for Coulomb interactions. Second, at these low energies the Fermi surface is effectively $O(N)$ fermionic species [corresponding to $O(N)$ patches on the Fermi surface in the renormalization group treatment [22,23]], boosting the 't Hooft coupling by an additional factor of N. (The argument is more complicated in the case of the HLR theory with a

Chern-Simons term in the action for a_μ, but the conclusion is the same).

Hamiltonian and equation of motion.—Assuming the composite fermions form a Fermi liquid with Landau's parameters F_n, the Hamiltonian of the system is

$$H = \frac{v_F p_F}{4\pi} \int d\mathbf{x} \sum_{n=-\infty}^{\infty} (1 + F_n) u_n(\mathbf{x}) u_{-n}(\mathbf{x}), \tag{7}$$

where F_n are the Landau parameters. In the case of a marginal Fermi liquid, we may understand by F_n the Landau parameters evaluated at the scale of the energy gap. The Hamiltonian (7) and the commutation relations (6) form our theory of the neutral excitations in the fractional quantum Hall fluid. This theory involves an infinite number of fields u_n, reminiscent of higher-spin relativistic field theories [24,25].

Let us first consider a zero wave number. Then according to Eq. (6) the operators u can be divided into pairs of creation and annihilation operators (u_{-2}, u_2), (u_{-3}, u_3), etc., with u_n for $n > 0$ being the annihilation and with $n < 0$ creation operators. The frequency of the oscillators are

$$\omega_n^{(0)} = n(1 + F_n)\omega_c, \qquad \omega_c = \frac{b}{m_*}. \tag{8}$$

The index n can be interpreted as the spin of the excitation. For example, the contribution of spin-n mode to the spectral density of the density operator is expected to be q^{2n} at small n, so the leading contribution to the spectral weight comes from the $n = 2$ mode. The ordering in energy of the modes depends on F_n; in the simplest scenario $n = 2$ is the lowest mode. Since $\omega_c \sim N^{-1}$, and the cutoff of our theory is $O(N^0)$, one should expect $O(N)$ of these modes (provided that F_n does not increase as a power of n).

If one puts $F_n = 0$ in Eq. (8), one would find $\omega_n^{(0)} = n\omega_c$. This can be interpreted as the energy of creating a pair of a quasiparticle and a quasihole, separated by n Landau-level steps. Note that the naïve lowest mode with $n = 1$ disappears due to the coupling to the dynamical gauge field [26]. As far as we know, Eq. (8) does not have a simple interpretation when the Landau parameters are nonzero.

To find the dispersion relation at finite wave number q, one needs to solve the linearized equation of motion, which can be obtained by taking the commutator with the Hamiltonian (7). In momentum space, choosing $\mathbf{q}$ to point along the x axis, the equation is

$$[\omega - n(1 + F_n)\omega_c]u_n = \frac{v_F q}{2}[(1 + F_{n-1})u_{n-1} + (1 + F_{n+1})u_{n+1}] \tag{9}$$

for $n \geq 2$ and $n \leq -2$ and where by construction $u_{\pm 1} = 0$. The task of finding the spectrum of excitations thus reduces to finding the eigenvalues of a certain tridiagonal matrix. Using Eq. (8), this equation can be rewritten as

$$(\omega - \omega_n^{(0)})u_n = \frac{2N + 1}{2} q\ell_B \left[\frac{\omega_{n-1}^{(0)}}{n - 1} u_{n-1} + \frac{\omega_{n+1}^{(0)}}{n + 1} u_{n+1}\right]. \tag{10}$$

Remarkably, Eq. (10) determines completely the dispersion curves from their starting points at $q = 0$. Thus, we speculate that Eq. (10) is valid even when the $\nu = 1/2$ state is a non-Fermi liquid. For small q, the equation can be solved perturbatively over q. For example, for the $n = 2$ mode we find

$$\frac{\omega_2(q)}{\omega_2^{(0)}} = 1 - \frac{(2N + 1)^2}{24(1 - \omega_2^{(0)}/\omega_3^{(0)})}(q\ell_B)^2 + O(q^4). \tag{11}$$

If the spin-2 mode is the lightest one, then its dispersion curve bends down when we go to finite q. Equation (11) relates the curvature at $q = 0$ of the lowest mode and the ratio of the energies of the spin-3 and spin-2 modes and is one prediction of the theory.

It is intriguing that Ref. [5] found two modes at $\nu = 1/3$. While it is tempting to identified them with spin-2 and spin-3 excitations, it is unclear if such an identification can be made at such a low value of N, $N = 1$.

The magnetoroton minima.—For $Nq\ell_B \sim 1$ one has to solve the full system of equations, Eq. (9) or (10), to find the dispersion curves. In Fig. 2, we plot a typical result. We note that the energy of the lowest mode goes to zero at a finite momentum. We now show analytically that this always happens at an infinitely strong gauge coupling. We need to solve Eq. (10) with $\omega = 0$ and the boundary conditions $u_1 = 0$ and $u_n \to 0$ when $n \to \infty$. The solution to this recursion relation, which satisfies the boundary condition $u_n \to 0$ when $n \to \infty$, is

$$u_n = \frac{(-1)^n}{1 + F_n} J_n\left(\frac{p_F q}{b}\right). \tag{12}$$

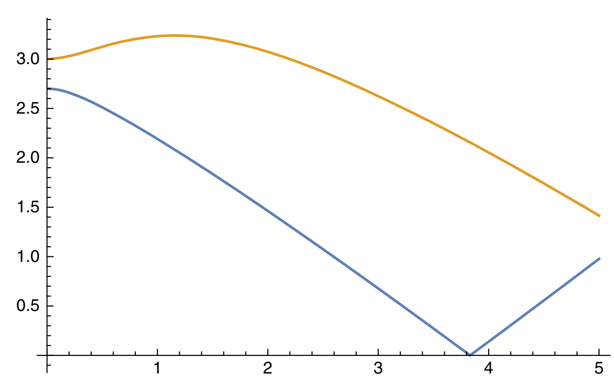

FIG. 2. The dispersion curves for the lowest two modes for $F_2 = 0.35$, $F_n = 0$ with $n \geq 3$. The horizontal axis is $(2N + 1)q\ell_B$, and the vertical axis is the energy in units of ω_c. The cusp at zero energy is an artifact of the infinite N limit.

The boundary condition $u_1 = 0$ requires $J_1(p_F q/b) = 0$. The latter occurs at $q = z_i b/p_F$, where z_i are the zeros of the Bessel function J_1. One can write this as

$$q\ell_B = z_i \frac{b}{p_F^2} = z_i \frac{b}{B} = \frac{z_i}{2N+1} \tag{13}$$

for the filling fractions $\nu = N/(2N+1)$ and $\nu = (N+1)/(2N+1)$.

The fact that the energy of an excitation is exactly zero is an artifact of the strong gauge coupling approximation, which we have argued to occur at infinite N; when the hard constraints on $u_0 = u_{\pm 1} = 0$ are relaxed, these zeros of the dispersion relation should become minima. The values of the energy at the minima are smaller by a power of N compared to the energy scale of the excitations at $q = 0$ ($\omega_n^{(0)}$) but are nevertheless nonzero [27]. This is confirmed in a more detailed treatment of the composite fermions, taking into account the density-density Coulomb interaction [28]. On the other hand, the strict $N = \infty$ limit of infinitely strong gauge coupling allows us to determine analytically the locations of the minima of the dispersion curves. Here we find a surprising result that the positions of the minima on the momentum axis do not at all depend on the parameters appearing in the Hamiltonian [29].

We now show that the robustness of the locations of the magnetoroton minima is due to them being determined by the commutator algebra (6) but not by the Hamiltonian. In fact, at the values of q set by Eq. (13), there exists a pair of operators $\hat{O}$ and $\hat{O}^\dagger$, which commutes with all u_n (and consequently with the Hamiltonian) to leading order in u:

$$\hat{O} = \sum_{n=2}^{\infty} (-1)^n J_n\left(\frac{p_F}{b} q\right) u_n. \tag{14}$$

In other words, if one defines the commutator matrix C_{mn} as

$$[u_m(\mathbf{q}), u_{-n}(\mathbf{q}')] = C_{mn}(2\pi)^2 \delta(\mathbf{q} + \mathbf{q}') \tag{15}$$

for m, $n > 0$, where

$$C_{mn} = \frac{2\pi b}{p_F^2} \begin{pmatrix} 2 & z & 0 & 0 & \dots \\ z & 3 & z & 0 & \dots \\ 0 & z & 4 & z & \dots \\ 0 & 0 & z & 5 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \qquad z = \frac{2N+1}{2} q\ell_B, \tag{16}$$

then at the momenta (13) the matrix C has a zero eigenvalue. Across these momenta, the role of creation and annihilation operators is exchanged for one pair of operators. It is not difficult to show that any Hamiltonian quadratic in u's needs to have a zero eigenvalue when such an exchange occurs.

The positions of the magnetoroton minima (13) and their complete independence of the details of the Hamiltonian are the central result of this Letter. In the past, model calculations have shown that the positions of the magnetoroton minima depend very weakly on the interactions (see, e.g., Ref. [30]), but the fundamental reason behind this fact was not understood.

It is worth remembering, however, that our derivation requires $q\ell_B \ll 1$, which means that z_i in Eq. (13) should be one of the first $o(N)$ roots of J_1. However, the values found in Eq. (13) seem to fit the existing data quite well even for relatively large $q\ell_B$. Limiting ourselves to the range explored in Ref. [7], $q\ell_B \lesssim 1.2$, our prediction for the locations of the magnetoroton minima is summarized in the following table (experimental values extracted from Ref. [7] in parentheses):

	$n = 1$	$n = 2$	$n = 3$
$\nu = 2/5$	0.77 (0.86)		
$\nu = 3/7$	0.55 (0.52)	1.00 (1.06)	
$\nu = 4/9$	0.43 (0.40)	0.78 (0.85)	1.13 (1.25)

All these values are surprisingly close (within 15% or less) to existing experimental [7] and numerical [10] results, despite the smallness of N and the large values of the $q\ell_B$ under discussion. Even for $N = 1$, the calculated position of the magnetoroton $q\ell_B = 1.28$ is in good agreement with the original estimate of Ref. [3]. We interpret the agreement as confirming the validity of the interpretation of the low-lying neutral excitations as shape fluctuations of the Fermi surface.

Since the locations of the magnetoroton minima depend only on the commutator algebra, which originates from the kinematics of the Fermi surface rather than from the Hamiltonian, we expect the minima would survive even in the non-Fermi-liquid regime of short-ranged electron-electron interactions.

In summary, the universal momenta at the magnetoroton minima (13), along with the existence of multiple branches of neutral excitations, each with a distinct value of the spin at $q = 0$, are the main predictions of this Letter. These predictions should be valid in any system described by a Fermi surface coupled to a dynamical gauge field in a small background magnetic field.

The authors thank Jainendra Jain, David Mross, Subir Sachdev, and Paul Wiegmann for discussions. This work is supported, in part, by U.S. DOE Grant No. DE-FG02-13ER41958. D. T. S. is supported, in part, by a Simons Investigator Grant from the Simons Foundation. S. G. is supported, in part, by European Research Council under the European Unions Seventh Framework Program (FP7/2007-2013), ERC Grant Agreement No. STG 279943, "Strongly

Coupled Systems." Additional support was provided by the Chicago MRSEC, which is funded by NSF through Grant No. DMR-1420709, and by the ARO MURI Grant No. 63834-PH-MUR.

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