Bifurcations and Singularities for Coupled Oscillators with Inertia and Frustration

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We prove that any nonzero inertia, however small, is able to change the nature of the synchronization transition in Kuramoto-like models, either from continuous to discontinuous or from discontinuous to continuous. This result is obtained through an unstable manifold expansion in the spirit of Crawford, which features singularities in the vicinity of the bifurcation. Far from being unwanted artifacts, these singularities actually control the qualitative behavior of the system. Our numerical tests fully support this picture.

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Understanding synchronization in large populations of coupled oscillators is a question which arises in many different fields, from physics to neuroscience, chemistry, and biology [1]. Describing the oscillators with their phases only, Winfree [2] and Kuramoto [3] have introduced simple models for this phenomenon. The latter model, which features a sinusoidal coupling and an all-to-all interaction between oscillators, has become a paradigmatic model for synchronization, and its very rich behavior prompted an enormous number of studies. The Kuramoto model displays a transition between an incoherent state, where each oscillator rotates at its own intrinsic frequency and a state where at least some oscillators are phase locked. The degree of coherence is measured by an order parameter r, which bifurcates—continuously for symmetric unimodal frequency distributions—from 0 when the coupling is increased or the dispersion in intrinsic frequencies decreases. In order to better fit modeling needs, it has been necessary to consider refined models, including, for instance, citing just a few contributions: more general coupling [4], noise [5], phase shifts bringing frustration [6], delays [7,8], or a more realistic interaction topology [9,10]. In particular, inertia has been introduced to describe the synchronization of a certain firefly [11] and proved later useful to model coupled Josephson junctions [12,13] and power grids [14,15]; recently, an inertial model on a complex network was shown to display a new type of "explosive synchronization" [16]. It has been quickly recognized [17,18] that a strong enough inertia could turn the continuous Kuramoto transition into a discontinuous one with hysteresis. At first sight, a natural adaptation of the original clever self-consistent mean-field approach by Kuramoto [3] seems to explain satisfactorily this observation [17,19]: a sufficiently large inertia induces a bistable dynamical behavior of some oscillators that translates into a hysteretic dynamics at the collective level. However, Fig. 1 makes clear that even a small inertia is enough to trigger a discontinuous transition: this cannot be accounted for by the bistability picture.

In this Letter, we show analytically why any amount of inertia, however small, can act both ways: it can turn discontinuous an otherwise continuous transition and the other way around. These results are obtained through a careful unstable manifold expansion in the spirit of Refs. [21–23] (see, also, Ref. [24] for a very readable discussion of the method), which uses the instability rate of the incoherent state as a small parameter: in the absence of noise, the linearized problem features a continuous spectrum on the imaginary axis, which may induce singularities in the usual expansions. We point out that these singularities related to the continuous spectrum are key for a comprehensive understanding of the bifurcations: they

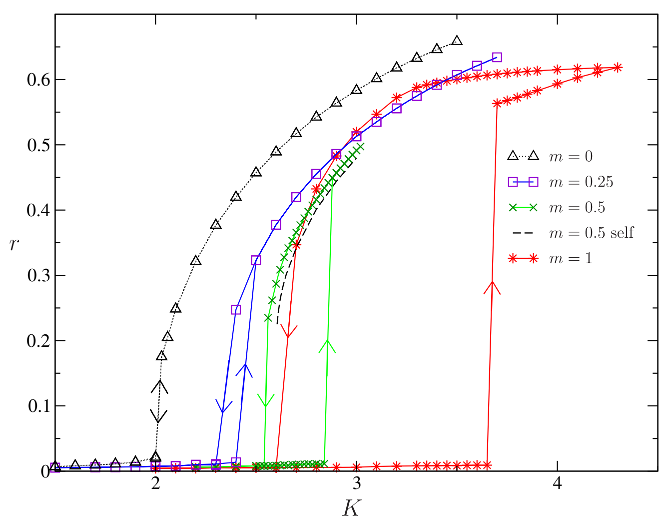

FIG. 1. Asymptotic order parameter r as a function of K for different m ($\alpha = 0$). The arrows indicate the direction of the jumps. Without inertia, the transition is continuous, while a hysteresis appears already for small m. Note the presence of a single branch with $r \neq 0$ for $m = 0.25$, 0.5, while there are two for $m = 1$. The dashed line is the partially synchronized solution given by the self-consistent method (see Ref. [20]) for $m = 0.5$. The frequency distribution is Lorentzian: $g_\sigma(\omega) = (\sigma/\pi)/(\sigma^2 + \omega^2)$ with $\sigma = 1$.

control the phase diagram in the presence of frustration, as well as the Hamiltonian limit, where the very strong nonlinear effects of the Vlasov equation are recovered. We compare our predictions with large-scale numerical simulations: using a GPU (graphics processing unit) architecture allows us to reach a number of oscillators significantly larger than in most previous works; this is crucial to test with a reasonable precision scaling laws in the vicinity of bifurcations.

We believe that these results should establish the singular expansions in the manner of Crawford as another method of choice to understand the qualitative behavior of Kuramoto-like models, along the original mean-field self-consistent method, the Ott-Antonsen ansatz [25,26] and the bifurcation methods used in Refs. [18,27–29]. Indeed, this method is applicable for the generic distribution function and interaction and provides information on the order of the transition and scaling laws close to the bifurcation.

The model.—Our starting point is the model introduced by Tanaka *et al.* [17], which adds inertia to the original Kuramoto model. It has been since then studied by many authors, often in the presence of noise, and we first discuss some of the theoretical results obtained so far. Reference [17] adapts the original self-consistent Kuramoto method to the presence of inertia, and predicts consistently with the numerics that a large enough inertia makes the transition discontinuous. The small inertia case was apparently not studied. In Refs. [18,27], the authors perform a bifurcation study of the incoherent state in the presence of noise and find a critical inertia beyond which the transition should be discontinuous; their result suggests that a small inertia *can* make a qualitative difference, but the singular nature of the small-noise limit makes an extrapolation to zero noise difficult. We note that a full "phase diagram" compatible with Refs. [18,27] is presented in Ref. [30] (see, also, Ref. [29]). In the following, we also add to the model in Ref. [17] a "frustration" parameter α as in Ref. [6]; this will provide us with a further parameter to make testable predictions. Our resulting model is then the same as Ref. [29], without noise.

Each of the N oscillators in the system has a frequency v_i, with $i \in 1, \ldots, N$ and a phase $\theta_i \in [0, 2\pi[$; it also has a natural frequency ω_i drawn from a frequency distribution g. We assume that g is even [$g(-\omega) = g(\omega)$]. If there is no coupling between the oscillators, the actual frequency v_i tends to the natural frequency ω_i. The dynamical equations for the positions and velocities are

$$\dot{\theta}_i = v_i, \tag{1a}$$

$$m\dot{v}_i = \gamma(\omega_i - v_i) + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i - \alpha). \tag{1b}$$

If the inertia m tends to 0, one recovers the usual Kuramoto model after a suitable change in the time variable,

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i - \alpha). \tag{2}$$

If $\gamma = 0$, there is no restoring force towards the natural frequency, and one obtains for $\alpha = 0$ a Hamiltonian model with an all-to-all coupling and a cosine interaction potential. It is usually called Heisenberg Mean-Field (HMF) model in the literature and has served as a simple paradigmatic model for mean-field Hamiltonian dynamics; see Ref. [31] for a review. We now use rescaled parameters $\tilde{m} = m/\gamma$, $\tilde{K} = K/\gamma$; the Kuramoto limit corresponds to $\tilde{m} \to 0$ and the Vlasov limit to $\tilde{m} \to \infty$, $\tilde{m}/\tilde{K} = \text{cste}$. Our parameters now coincide with those of Ref. [17]. Dropping the $\tilde{}$ for convenience in the $N \to \infty$ limit, the system (1) is described by a kinetic equation for the phase space density $F(\theta, v, \omega, t)$,

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial \theta} + \frac{K}{2im}(r_1[F]e^{-i\theta}e^{-i\alpha} - r_{-1}[F]e^{i\theta}e^{i\alpha})\frac{\partial F}{\partial v} - \frac{1}{m}\frac{\partial}{\partial v}((v-\omega)F) = 0, \tag{3}$$

where the usual order parameter is $r = |r_1|$, with

$$r_k[F] = \int F(\theta, v, \omega, t)e^{ik\theta}d\theta dv d\omega. \tag{4}$$

Unstable manifold expansion.—The incoherent stationary solution corresponds to each oscillator running at its natural frequency, with the phases homogeneously distributed: $F(\theta, v, \omega, t) = f^0(v, \omega) = g(\omega)\delta(v - \omega)/(2\pi)$. It is easy to check that f^0 is, indeed, a stationary solution of Eq. (3). Increasing the coupling strength K, f^0 changes from stable to unstable. Our goal is to study the dynamics of Eq. (3) in the vicinity of this bifurcation.

For this purpose, we first decompose Eq. (3) in a linear and a nonlinear part, with $F = f^0 + f$,

$$\frac{\partial f}{\partial t} = \mathcal{L}f + \mathcal{N}[f], \tag{5}$$

with

$$\mathcal{L}f = -v\frac{\partial f}{\partial \theta} - \frac{K}{2im}(r_1[f]e^{-i\theta}e^{-i\alpha} - r_{-1}[f]e^{i\theta}e^{i\alpha})\partial_v f^0 + \frac{1}{m}\frac{\partial}{\partial v}((v-\omega)f), \tag{6}$$

$$\mathcal{N}[f] = -\frac{K}{2im}(r_1[f]e^{-i\theta}e^{-i\alpha} - r_{-1}[f]e^{i\theta}e^{i\alpha})\partial_v f. \tag{7}$$

The precise study of the linear operator $\mathcal{L}$ is an important building block in our nonlinear analysis, and we collect below the main results concerning $\mathcal{L}$ (details are given in the Supplemental Material [20]). Equation (3) is symmetric with respect to rotations $(\theta, v, \omega) = (\theta + \varphi, v, \omega)$; if $\alpha = 0$

and $g(\omega)$ even, it is in addition symmetric with respect to the reflections $(\theta, v, \omega) = -(\theta, v, \omega)$ [22]. In this Letter, we take g even, and we restrict to the case of two unstable eigenvectors. This is generically the case when the following occurs. (i) $\alpha \neq 0$; in this case, there is a complex unstable eigenvalue λ, and $\lambda^\star$ is also an unstable eigenvalue; (ii) $\alpha = 0$, and λ is real; in this case, it is twice degenerate, associated with two eigenvectors. Hence, in both cases, we will build a two-dimensional unstable manifold. We leave for future studies the cases $\alpha = 0$, λ complex, which leads to a four-dimensional unstable manifold [22], as well as noneven $g(\omega)$ distributions. $\mathcal{L}$ is diagonal when expressed in the Fourier basis for the phases. It is then easy to see that the discrete spectrum of $\mathcal{L}$ is associated with the $k = \pm 1$ Fourier modes; that is, the eigenvectors are proportional to $e^{\pm i\theta}$.

Ψ, the eigenvector of $\mathcal{L}$ associated with λ and $k = 1$, is given by $\Psi(\theta, v, \omega) = \psi(v, \omega)e^{i\theta}$, with $\psi(v, \omega) = U_0(\omega)\delta(v - \omega) + U_1(\omega)\delta'(v - \omega)$, and δ' stands for the first derivative with respect to v of the Dirac distribution. The expression for U_0 and U_1 is provided in the Supplemental Material [20]. The dispersion relation from which λ is computed reads

$$\Lambda(\lambda) = 1 - \frac{Ke^{i\alpha}}{2m}\int \frac{g(\omega)}{(\lambda + 1/m + i\omega)(\lambda + i\omega)} d\omega = 0. \tag{8}$$

This dispersion relation can be recovered as the noiseless limit of the one in Ref. [27], as it should. One can also check that the limit $m \to \infty$, $K/m = \text{cst}$ yields the Vlasov dispersion relation with a cosine potential and $g(\omega)$ as the stationary velocity profile; the $m \to 0$ limit yields the standard Kuramoto dispersion relation.

We shall need the projection Π on the unstable eigenspace $\mathcal{V} = \text{Span}(\Psi, \Psi^\star)$. For this purpose, we introduce the adjoint operator $\mathcal{L}^\dagger$ defined through $(f_1, \mathcal{L}f_2) = (\mathcal{L}^\dagger f_1, f_2)$, where $(f_1, f_2) = \iint f_1^\star f_2 d\omega dv d\theta$ is a scalar product. The adjoint eigenvector associated with the eigenvalue λ is $\tilde{\Psi}(\theta, v, \omega) = \tilde{\psi}(v, \omega)e^{i\theta}/2\pi$. We do not know how to compute $\tilde{\psi}(v, \omega)$ in closed form. However, in the following computations, $\tilde{\psi}$ only appears in scalar products with delta functions $\delta(v - \omega)$ and their derivatives; as a consequence, we only need to know $\tilde{\psi}^{(n)}(\omega) \coloneqq \partial_v^n \tilde{\psi}(\omega, \omega)$. The expression for $\tilde{\psi}^{(n)}(\omega)$ is provided in Ref. [20]. Then the projection acts on a function ϕ as $\Pi \cdot \phi = (\tilde{\Psi}, \phi)\Psi + (\tilde{\Psi}^\star, \phi)\Psi^\star$. With this knowledge of the linear part $\mathcal{L}$, we now proceed to the nonlinear analysis. Following Ref. [21], we introduce the unstable manifold $\mathcal{M}$ associated with the stationary solution f^0. $\mathcal{M}$ is the set of functions F that tend to f^0 when $t \to -\infty$ and can be seen as a nonlinear generalization of $\mathcal{V}$. This is a manifold of the same dimension as the linear unstable subspace $\mathcal{V}$, and it is clearly invariant by the dynamics. The tangent space to $\mathcal{M}$ at f^0 is $\mathcal{V}$. Any element ϕ of $\mathcal{M}$ in a neighborhood of f^0 can be written as

$$\phi = A\Psi + A^\star\Psi^\star + H[A, A^\star](\theta, v, \omega). \tag{9}$$

$A\Psi + A^\star\Psi^\star$ is the projection of ϕ on $\mathcal{V}$ according to Π; hence, $\Pi \cdot H = 0$. Furthermore, $H = O((A, A^\star)^2)$. For an initial condition on $\mathcal{M}$ and assuming that the dynamics remain in a region where Eq. (9) is valid, the whole dynamics is parametrized by the function $A(t)$, which is related to r by $r = 2\pi|A| + O(|A|^3)$. Our goal is then to determine the evolution equation for A. H itself is, of course, unknown and has to be determined at the same time as the dynamical equation for A. The strategy is to build an expansion in A and solve order by order. At linear order, only the $k = \pm 1$ Fourier modes in θ enter. Since the nonlinearity is quadratic, only the Fourier modes of order $k = 0, \pm 2$ enter at leading nonlinear order. Hence, we write

$$\begin{aligned} \frac{dA}{dt} &= \lambda A + c_3|A|^2A + O(A^5), \\ H(A, A^\star) &= AA^\star h_{0,0}(v, \omega) + A^2 h_{2,0}(v, \omega)e^{2i\theta} \\ &\quad + \text{c.c.} + \ldots, \end{aligned} \tag{10}$$

where we have used the $A \leftrightarrow -A$ and translation symmetries (see Ref. [20]). Differentiating Eq. (9), we get on one hand, $d\phi/dt = dA/dt(\Psi + A^\star h_{0,0} + 2Ah_{2,0}) + \text{c.c.} + O(A^3)$, and on the other hand, $d\phi/dt = \lambda A\Psi + \text{c.c.} + \mathcal{L}H + \mathcal{N}(A\Psi + A^\star\Psi^\star + H)$. Projecting on $\mathcal{V}$ yields at linear order $dA/dt = \lambda A$. Projecting on $\mathcal{V}^\perp$ then allows us to compute $h_{0,0}$ and $h_{2,0}$. The projection on $\mathcal{V}$ at cubic order then yields c_3. At the expense of increasingly intricate computations, one could go on with this scheme; we have stopped at c_3. We give below the key results of the computation, whereas all details are presented in Ref. [20].

Discussion.—Using the reduced dynamics (10) truncated at order A^3 provides essential qualitative information: (i) The bifurcation is subcritical (i.e., with a jump in the order parameter) if and only if $\text{Re}(c_3) > 0$; (ii) in the supercritical case, one obtains the asymptotic order parameter $|A|_\infty = \sqrt{-\lambda_R/\text{Re}(c_3)}$. We have to evaluate c_3 close to the bifurcation point, that is, when $\lambda_R \to 0$. We find for $m > 0$ with $\lambda_I = \text{Im}(\lambda)$ [our hypothesis of a two-dimensional unstable manifold ensures that $\Lambda'(i\lambda_I) \neq 0$]:

$$\text{Re}(c_3) \sim \frac{\pi^3}{2} mK^3 \frac{g(\lambda_I)}{\lambda_R} \text{Re}\left(\frac{e^{i\alpha}}{\Lambda'(i\lambda_I)}\right). \tag{11}$$

From this, the dramatic effect of the inertia m appears clearly: it introduces into c_3 a contribution diverging like $1/\lambda_R$, which is the dominant one: the sign of $s = \text{Re}(e^{i\alpha}/\Lambda'(i\lambda_I))$ controls the type of bifurcation: sub- (resp. super-) critical for $s > 0$ (resp. $s < 0$). For $m = 0$, the next order term, which does not diverge when $\lambda_R \to 0$, is

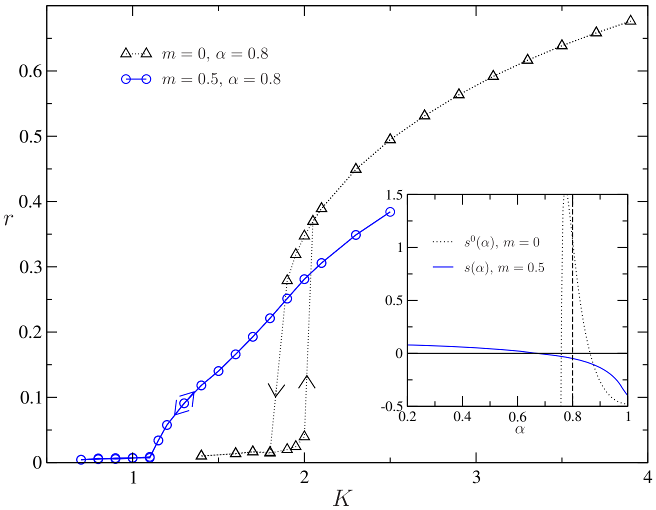

FIG. 2. Asymptotic r as a function of K for $\alpha = 0.8$, with $m = 0$ or $m = 0.5$. The frequency distribution is a superposition of two Lorentzians as in Ref. [32], Fig. 3: $g(\omega) = \tau g_1 + (1-\tau)g_\delta$, with $\tau = 0.8$, $\delta = 0.075$; g is unimodal. The inset shows that $s^0(\alpha = 0.8) > 0$ (hence, discontinuous transition at $m = 0$) and $s(\alpha = 0.8) < 0$ (hence, continuous transition as soon as $m > 0$).

needed; the bifurcation is then controlled by $s^0 = \mathrm{Re}(\Lambda''(i\lambda_I)/\Lambda'(i\lambda_I))$: sub- (resp. super-) critical for $s_0 > 0$ (resp. $s_0 < 0$) (this generalizes to $\alpha \neq 0$ a result of Ref. [23]; see Ref. [20].) Hence, any small m may either turn a supercritical bifurcation at $m = 0$ into a subcritical one, or the other way around, turn a subcritical bifurcation at $m = 0$ into a supercritical one. While the first direction illustrated in Fig. 1 was anticipated in Refs. [29,30], the second direction is unexpected. Figure 2 provides an example with a unimodal $g(\omega)$ and $\alpha \neq 0$. Furthermore, in the supercritical case, we predict the scaling law for the asymptotic order parameter $|A|_\infty \propto \lambda_R$, and this is also observed. If the distribution g is unimodal, we note that $s(\alpha = 0) > 0$, so the bifurcation is always subcritical. Finally, Eq. (11) makes clear that both the standard first order Kuramoto ($m = 0$, $\alpha = 0$) and Vlasov ($m = \infty$, $K/m = \mathrm{cst}$) limits are singular. In the first case, the divergent term vanishes, and the bifurcation is controlled by the sign of s^0. One recovers the already known results: for a symmetric unimodal g, $s^0(\alpha = 0) < 0$ and the bifurcation is supercritical, with standard scaling $|A|_\infty \propto \sqrt{\lambda_R}$. In the second case, Eq. (11) diverges when $m, K \to \infty$. Redoing the computations in this limit indeed yields (for $\alpha = 0$) $c_3 \propto -(1/\lambda_R^3)$, as found in Ref. [21]. This leads to the "trapping scaling" well known in plasma physics $|A|_\infty \propto \lambda_R^2$.

Numerics.—We present in this section precise numerical simulations that fully support the above analysis. The time-evolved system is obtained via a GPU parallel implementation of a fourth order Runge-Kutta scheme for Eq. (5) [30]. The order parameter is computed by its standard discrete definition [3]. For every simulation, we take $N = 65\,536$, and a time step of $\Delta t = 10^{-3}$. The asymptotic order parameter r is the average of $|r_1|(t)$ for $t \in [1500, 2000]$. In order to test our prediction on the type of bifurcation, we start from an unsynchronized state (drawing positions θ_i uniformly on a unit circle). The ω_i are sampled according to g, and the initial velocities are $v_i = \omega_i$. We let the system evolve until $t = 2000$ and measure the averaged order parameter. Then we vary the coupling constant $K \to K + \Delta K$ with $\Delta K = 0.1$ or 0.2 (or smaller close to transitions) and reiterate the procedure; at some point, the bifurcation towards synchronization is observed. When K is large enough, we apply the same procedure in the other direction, $K \to K - \Delta K$. Thus, we are able to distinguish clearly a subcritical bifurcation (with a characteristic hysteresis cycle) from a supercritical bifurcation (with no hysteresis). In Fig. 1, we see how the hysteretic cycle depends on the inertia m. For $m = 1$, there are two branches with $r \neq 0$: these correspond to the bistable behavior of the single oscillator dynamics in a range of ω (see Ref. [17]); for $m = 0.5$ and $m = 0.25$, the single oscillator dynamics is not bistable in the transition region, and, accordingly, there is only one branch with $r \neq 0$. The bifurcation remains, nevertheless, clearly subcritical. In Fig. 2, inertia induces a supercritical transition; Eq. (11) also correctly predicts the linear scaling of the saturated state in this case. Finally, we note that in the subcritical regime, the numerically observed K_c is sometimes lower than the prediction (3); this is presumably related to strong finite size effects [33], especially in the presence of inertia [15].

Conclusions.—We have constructed an unstable manifold expansion for models of synchronization with inertia and frustration, circumventing the problem of the continuous spectrum on the imaginary axis. The singularities appearing in the expansion may at first sight seem harmful, but they actually control the system's behavior in the vicinity of the bifurcation and allow useful qualitative and quantitative predictions. In particular, while synchronization models tend to present complicated phase diagrams for which it is difficult to develop an intuition [29,32], we have obtained simple criteria determining the character of the transition. We note that since the unstable manifold is not attractive, the reduced description could be valid only for specific initial conditions; numerics show that its validity is much wider than what might have been expected. We remark that the bifurcation diagram of the standard Kuramoto model (in particular, without inertia) has been established rigorously very recently [34,35]. It is tempting to relate this mathematical success to the absence of singularities in the corresponding unstable manifold expansion, although the exact relationship between these facts is still unknown. Finally, the versatility of the method suggests that, beyond the synchronization models, it can be adapted to many different situations featuring bifurcations with a continuous spectrum, the most important physical example being probably the bifurcations of the Vlasov equation.

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