Higher-Loop Amplitude Monodromy Relations in String and Gauge Theory

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The monodromy relations in string theory provide a powerful and elegant formalism to understand some of the deepest properties of tree-level field theory amplitudes, like the color-kinematics duality. This duality has been instrumental in tremendous progress on the computations of loop amplitudes in quantum field theory, but a higher-loop generalization of the monodromy construction was lacking. In this Letter, we extend the monodromy relations to higher loops in open string theory. Our construction, based on a contour deformation argument of the open string diagram integrands, leads to new identities that relate planar and nonplanar topologies in string theory. We write one and two-loop monodromy formulas explicitly at any multiplicity. In the field theory limit, at one-loop we obtain identities that reproduce known results. At two loops, we check our formulas by unitarity in the case of the four-point $\mathcal{N}=4$ super-Yang-Mills amplitude.

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The search for the fundamental properties of the interactions between elementary particles has been the driving force to uncover basic and profound properties of scattering amplitudes in quantum field theory and string theory. In particular, the color-kinematic duality [1] has led to tremendous progress in the evaluation of loop amplitudes in gauge theories [2–14]. One remarkable consequence of this duality is the discovery of unsuspected kinematic relations between tree-level gauge theory amplitudes [1], generated by a few fundamental relations [15–19].

The monodromies of the open string disc amplitudes [15,16] did provide a rationale for the kinematic relations between amplitudes at tree-level in gauge theory. However, while the color-kinematics duality has been successfully implemented up to the fourth loop order in field theory [3,4], there is not yet a systematic understanding of its validity to all loop orders. It is therefore natural to seek a higher-loop generalization of the string theory approach to these kinematic relations.

In this Letter we generalize the tree-level monodromy construction to higher-loop open string diagrams (world sheets with holes). This allows us to derive a new relation between planar and nonplanar topologies of graphs in string theory. The key ingredient in the construction relies on using a representation of the string integrand with a loop momentum integration. This is crucially needed in order to be able to understand zero mode shifts when an external state jumps from one boundary to another. Furthermore, just like at tree level, the construction does not depend on the precise nature of the scattering amplitude nor the type of theory (bosonic or supersymmetric) considered.

The relations that we obtain in field theory emanate from the leading and first order in the expansion in the inverse string tension α'. At leading order, we find identities between planar and nonplanar amplitudes. At the next order, stringy corrections vanish and we find the loop monodromy relations. They are relations between integrands up to total derivatives, that involve both loop and external momenta. Upon integration, this gives relations between amplitude-like integrals with extra powers of loop momentum in the numerator.

At one loop, our string theoretic construction reproduces the field theory relations of Refs. [20–22]. In observing how the loop momentum factors produce cancellations of internal propagators, we see that Bern-Carrasco-Johansson (BCJ) color-kinematic representations for numerators [1] satisfy the monodromy relation at the integrand level. The generality of our construction leads us to conjecture that our monodromies generate all the kinematic relations at any loop order.

We conclude by showing how our construction extends to higher loops in string theory. In particular, we write the two-loop string monodromy relations. The field theory limit is subtle to understand in the general case, but we provide a proof of concept with an example in $\mathcal{N}=4$ super-Yang-Mills theory at four-point two-loop, which we check by unitarity. We leave the general field theory relations for future work.

Monodromies on the annulus.—One-loop n-particle amplitudes $\mathfrak{A}$ in oriented open-string theory are defined on the annulus. They have a $U(N)$ gauge group and the following color decomposition [23]:

$$\mathfrak{A}(\{\epsilon_i,k_i,a_i\}) = g_s^n\pi^{n-1}\sum_{p=0}^{n}\sum_{\alpha\cup\beta\in\mathfrak{S}_{p,n}} \mathrm{Tr}(\lambda^{a_{\alpha(1)}}\cdots\lambda^{a_{\alpha(p)}})\mathrm{Tr}(\lambda^{a_{\beta(p+1)}}\cdots\lambda^{a_{\beta(n)}})\mathcal{A}(\alpha|\beta). \quad (1)$$

The summation over $\mathfrak{S}_{p,n}$ of the external states distributed on the boundaries of the annulus is over all permutations modulo cyclic reordering and reflection symmetry. The quantities k_i, ϵ_i, and λ^a are the external momenta, polarizations and color matrices in the $U(N)$ fundamental representation, respectively. Planar amplitudes are obtained for $p=0$ or $p=n$ with $\mathrm{Tr}(1)=N$. The color-stripped ordered n-gluon amplitude $\mathcal{A}(\alpha|\beta)$ take the following generic form in D dimensions

$$\mathcal{A}(\alpha|\beta) = \int_0^\infty dt \int_{\Delta_{\alpha|\beta}} d^{n-1}\nu \int d^D\ell\, e^{-\pi\alpha' t\ell^2 - 2i\pi\alpha'\ell\cdot\sum_{k=1}^n k_i\nu_i} \times \prod_{1\le r<s\le n} f(e^{-2\pi t}, \nu_r-\nu_s) \times e^{-\alpha' k_r\cdot k_s G(\nu_r,\nu_s)}, \quad (2)$$

where $t\in\mathbb{R}$ is the modulus of the annulus and the ν_i's are the location of the gluons insertions on the string world sheet—one of them is set to it by translation invariance. The loop momentum ℓ^μ is defined as the average of the string momentum ∂X^μ [24];

$$\ell^\mu = \int_0^{\frac{1}{2}} d\nu \frac{\partial X^\mu(\nu)}{\partial\nu}. \quad (3)$$

The domain of integration $\Delta_{\alpha|\beta}$ is the union of the ordered sets $\{\mathfrak{Im}(\nu_{\alpha(1)}) < \cdots < \mathfrak{Im}(\nu_{\alpha(p)})\}$ for $\Re\mathrm{e}(\nu_i)=0$ and $\{\mathfrak{Im}(\nu_{\beta(p+1)}) > \cdots > \mathfrak{Im}(\nu_{\beta(n)})\}$ for $\Re\mathrm{e}(\nu_i)=\frac{1}{2}$.

We will show that the kinematical relations at one loop arise exclusively from shifts in the loop-momentum-dependent part and monodromy properties of the nonzero mode part of the Green's function in Eq. (2)

$$G(\nu_r,\nu_s) = -\log\frac{\vartheta_1(\nu_r-\nu_s|it)}{\vartheta_1{}'(0)}. \quad (4)$$

We refer to the appendix for some properties of the propagators between the same and different boundaries.

The function $f(e^{-2\pi t},\nu_r-\nu_s)$ contains all the theory-dependence of the amplitudes. The crucial point of our analysis is that it does not have any monodromy; therefore, the relations that we obtain are fully generic. This function is a product of partition functions, internal momentum lattice of compactification to D dimensions, and a prescribed polarization dependence [23,25–27]. The latter is composed of derivatives of the Green's function. None of these objects have monodromies: that is why the precise form of f does not matter for our analysis. This property carries over to higher-loop orders.

Local and global monodromies: Let us consider the nonplanar amplitude $\mathcal{A}(1,\ldots,p|p+1,\ldots,n)$, but where we take the modified integration contour $\mathcal{C}$ of Fig. 1 for ν_1. The integrand being holomorphic, in virtue of Cauchy's theorem, the integral vanishes:

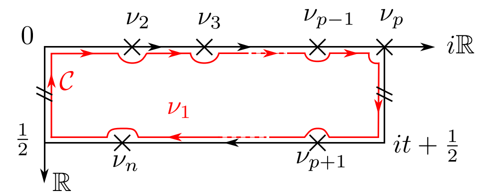


FIG. 1. The ν_1 contour integral (red) vanishes. The two boundaries (black) have opposite orientation.

$$\oint_{\mathcal{C}} d\nu_1 \int_0^\infty d^D\ell\, e^{-\pi\alpha' t\ell^2 - 2i\pi\alpha'\ell\cdot\sum_{k=1}^n k_i\nu_i} e^{-i\pi\alpha'\ell\cdot k_1\nu_1} \times \prod_{r=2}^n f(e^{-2\pi t},\nu_1-\nu_r) e^{-\alpha' k_1\cdot k_r G(\nu_1,\nu_r)} = 0. \quad (5)$$

Each separate portion of the integration corresponds to a different ordering and topology. The portions along the vertical sides cancel by periodicity of the one-loop integral (cf. appendix). We are thus left with the contributions from the boundaries $\Re\mathrm{e}(\nu_1)=0$ and $\Re\mathrm{e}(\nu_1)=\frac{1}{2}$. When exchanging the position of two states on the *same* boundary, the short distance behavior of the Green's function $G(\nu_1,\nu_2)\simeq -\log(\nu_1-\nu_2)$ implies

$$G(\nu_1,\nu_2) = G(\nu_2,\nu_1) \pm i\pi, \quad (6)$$

with $-i\pi$ for a clockwise rotation and $+i\pi$ for a counterclockwise rotation. Thus, on the upper part of the contour in Fig. 1, exchanging the positions of two external states leads to a phase factor multiplying the amplitude

$$\mathcal{A}(12\cdots m|m+1\cdots n) \to e^{i\pi\alpha' k_1\cdot k_2}\mathcal{A}(21\cdots m|m+1\cdots n). \quad (7)$$

On the lower part of the contour in Fig. 1, the phases come with the same sign due to an additional sign from ϑ_2 in Eq. (A2). For external states on different boundaries, the Green's function involves the even function $\vartheta_2(\nu_r-\nu_s)$ and the ordering does not matter (cf. the appendix).

The main difference with the tree-level case arises from the *global* monodromy transformation when a state moves from one boundary to the other, $\nu_1\to\nu_1+\frac{1}{2}$. This produces a new phase $\exp(-i\pi\alpha'\ell\cdot k_1)$ in the integrand

$$\mathcal{A}(12\cdots n) \to \mathcal{A}(2\cdots n|1)[e^{-i\pi\alpha'\ell\cdot k_1}] \coloneqq \int_0^\infty dt \int_{\Delta_{2\cdots n|1}} d^{n-1}\nu \prod_{1\le r<s\le n} f(e^{-2\pi t},\nu_r-\nu_s) e^{-\alpha' k_r\cdot k_s G(\nu_r,\nu_s)} \times \int_0^\infty d^D\ell\, e^{-i\pi\alpha'\ell\cdot k_1} e^{-\pi\alpha' t\ell^2 - 2i\pi\alpha'\ell\cdot\sum_{k=1}^n k_i\nu_i}. \quad (8)$$

On nonorientable surfaces the propagator is obtained by appropriate shifts of the Green's function [Eq. (4)] according to the effects of the twist operators [25]. The local monodromies are the same because they only depend on the short distance behavior of the propagator, and global monodromies are obtained in an immediate generalization of our construction.

Open string relations: We can now collect up all the previous pieces. Paying great care to signs and orientations, according to what was described, the vanishing of the integral along $\mathcal{C}$ gives the following generic relation:

$$\mathcal{A}(1,2,\ldots,p|p+1,\ldots,n)+\sum_{i=2}^{p-1}e^{i\alpha'\pi k_1\cdot k_{2\cdots i}}\mathcal{A}(2,\ldots,i,1,i+1,\ldots,p|p+1,\ldots,n)$$
$$=-\sum_{i=p}^{n}(e^{i\alpha'\pi k_1\cdot k_{p+1\cdots i}}\mathcal{A}(2,\ldots,p|p+1,\ldots,i,1,i+1,\ldots,n)[e^{-i\pi\alpha'\ell\cdot k_1}]) \tag{9}$$

where the bracket notation was defined in Eq. (8) and we set $k_{1\ldots p}\coloneqq\sum_{i=1}^{p}k_i$. In particular, starting from the planar four-point amplitude we find the following formula:

$$\mathcal{A}(1234)+e^{i\pi\alpha' k_1\cdot k_2}\mathcal{A}(2134)+e^{i\pi\alpha' k_1\cdot(k_2+k_3)}\mathcal{A}(2314)$$
$$=-\mathcal{A}(234|1)[e^{-i\pi\alpha'\ell\cdot k_1}]. \tag{10}$$

We also find, starting from a purely planar amplitude

$$(-1)^{|\beta|}\sum_{\gamma\in\alpha\sqcup\sqcup\beta}\prod_{a=1}^{s}\prod_{b=1}^{r}e^{i\pi\alpha'(\alpha_a,\beta_b)}\mathcal{A}(\gamma_1\cdots\gamma_{r+s}n)$$
$$=\mathcal{A}(\alpha_1\cdots\alpha_s n|\beta_r\cdots\beta_1)\left[\prod_{i=1}^{r}e^{-i\pi\alpha'\ell\cdot k_{\beta_i}}\right] \tag{11}$$

where now we integrate the vertex operators with ordered position $\mathfrak{Im}(\nu_{\beta_1})\leq\cdots\leq\mathfrak{Im}(\nu_{\beta_r})$ along the contour of Fig. 1. The sum is over the shuffle product $\alpha\sqcup\sqcup\beta$ and the permutation β of length $|\beta|$, and $(\alpha_i,\beta_j)=k_{\alpha_i}\cdot k_{\beta_j}$ if $\mathfrak{Im}(\nu_{\beta_j})>\mathfrak{Im}(\nu_{\alpha_i})$ in γ and 1 otherwise. The phase factors with external momenta are the same as at tree level: the new ingredients here are the insertions of loop-momentum dependent factors inside the integral.

Note that some of our relations involve objects like $\mathcal{A}(2\cdots n|1)$ that seemingly contribute in Eq. (1) only if the state 1 is a color singlet. However, our relations involve color-stripped objects and are, therefore, valid in full generality. Note also that our relations are valid under the t integration; thus, they are not affected by the dilaton tadpole divergence at $t\to 0$ [25].

We have thus shown that the kinematic relations [Eq. (9)] relate planar and nonplanar open string topologies, which normally have independent color structures. This is the one-loop generalization of the string theory fundamental monodromies that generates all amplitude relations at tree-level in string theory [15,16]. Thus, we conjecture our one-loop relations, Eq. (9), written for all the permutations of the external states, generate all the one-loop oriented open string theory relations. Let us now turn to the consequences in field theory.

Field theory relations.—Gauge theory amplitudes are extracted from string theory ones in the standard way. We send $\alpha'\to 0$ and keep fixed the quantity $\alpha' t$ that becomes the Schwinger proper-time in field theory. We also set $\mathfrak{Im}(\nu)=xt$, with $0\leq x\leq 1$. The Green's function of Eq. (4) reduces to the sum of the field theory worldline propagator $x^2-|x|$ and a stringy correction

$$G(\nu)=t(x^2-|x|)+\delta_{\pm}(x)+O(e^{-2\pi t}). \tag{12}$$

for details see Appendix. (In a bosonic open string one would need to keep to the terms of the order $\exp(-2\pi t)$ because of the Tachyon.) At leading order in α', open string amplitudes reduce to the usual parametric representation of the dimensional regulated gauge theory amplitudes [28,29]. (See also Refs. [30–32] for equivalent closed string methods.) All the monodromy phase factors reduce to 1 and from Eq. (11) we recover the well-known photon decoupling relations between nonplanar and planar amplitudes [33], with $\beta^T=(\beta_r,\ldots,\beta_1)$,

$$A(\alpha|\beta^T)=(-1)^{|\beta|}\sum_{\gamma\in\alpha\sqcup\sqcup\beta}A(\gamma). \tag{13}$$

This is an important consistency check on our relations.

At the first order in α' we get contributions from expansion of the phase factors but as well potential ones from the massive stringy mode coming from $\delta_{\pm}(x)$. The analysis of the appendix of Ref. [34] shows that this contributes to next order in α', which, importantly, allows us to neglect it here. Therefore, the field theory limit of Eq. (9) gives a new identity

$$\sum_{i=2}^{p-1}k_1\cdot k_{2\cdots i}A(2,\ldots,i,1,i+1,\ldots,p|p+1,\ldots,n)+\sum_{i=p}^{n}k_1\cdot k_{p+1\cdots i}A(2,\ldots,p|p+1,\ldots,i,1,i+1,\ldots,n)$$
$$=\sum_{i=p}^{n}A(2,\ldots,p|p+1,\ldots,i,1,i+1,\ldots,n)[\ell\cdot k_1]. \tag{14}$$

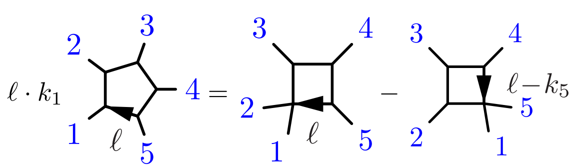
These relations are the one-loop equivalent of the fundamental monodromy identities [17–19] that generates all the amplitude relations at tree level.

In particular, using Eq. (13), we obtain the relation between planar gauge theory integrands with linear power of loop momentum

$$A(1\cdots n)[\ell\cdot k_1]+A(21\cdots n)[(\ell+k_2)\cdot k_1]+\cdots \\ +A(23\cdots(n-1)1n)[(\ell+k_{23\ldots n-1})\cdot k_1]=0. \qquad (15)$$

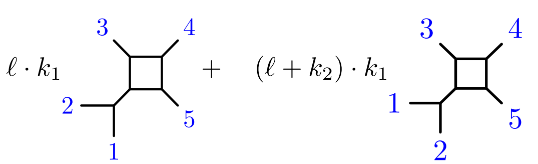
These are the relations derived in Refs. [20–22]: this constitutes an additional check on our formulas.

Let us now analyze the effect of the linear momentum factors at the level of the graphs. At this point we pick any representation of the integrand in terms of cubic graphs only and the field theory limit defines the loop momentum as the internal momentum following immediately the leg n. (This is checked by matching with the usual definition of the Schwinger proper times.) We then rewrite the loop momentum factors as differences of propagators. Hence, each individual graph with numerator n_G produces two graphs with one fewer propagator, e.g.,



(16)

Then, there always exist another graph G' that will produce one of the two reduced graphs as well, with a different numerator $n_{G'}$. In the previous example, it would be the 21345 pentagon for the massive box with 1,2 corner. Finally, reduced graphs also arise directly from string theory, when vertex operators collide [29]. In Eq. (15), these always appear in such combinations of two graphs, say G_1 and G_2;



(17)

The color ordered 3-point vertex is antisymmetric, so $n_{G_1}=-n_{G_2}$ and the $\ell\cdot k_1$ terms cancel. We then realize that the graphs entering the monodromy relations can be organized by triplets of Jacobi numerators $n_G+n_{G'}-n_{G_1}$ times denominator. In a BCJ representation, all these triplets vanish identically and Eq. (14) is satisfied at the integrand level. Thus, any BCJ representation satisfies these monodromy relations, but the converse is not true.

Toward higher-loop relations.—Higher-loop oriented open string diagrams are world sheets with holes, one

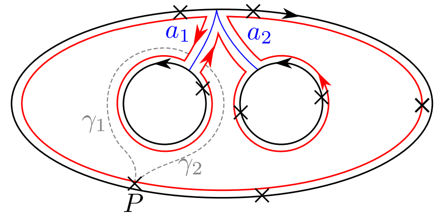


FIG. 2. Two-loop integrand monodromy. Integration over the red contour vanishes. Given the definition of the loop momentum in Eq. (18), parallel integrations along a_1, a_2 cancel only up to a shift in the loop momentum.

for each loop. (We do not consider string diagrams with handles in this work. They lead to nonplanar $1/N^2$ corrections [35].) Just like at one loop, we consider the integral of the position of a string state on a contractible closed contour that follows the interior boundary of the diagram (cf. for instance Fig. 2). The integral vanishes in the absence of insertion of a closed string operator in the interior of the diagram. This constitutes the essence of the monodromy relations at higher loop.

Because the exchange of two external states on the *same* boundary depends only on the local behavior of the Green's function, we have the same *local* monodromy transformation $G(z_1,z_2)=G(z_2,z_1)\pm i\pi$ as at the tree level.

Like at one loop, the *global* monodromy of moving the external state 1 from one boundary to another boundary by crossing the cycle a_I leads to the factor $\exp(-i\alpha'\pi\ell_I\cdot k_1)$. The loop momenta ℓ_I are the zero modes of the string momenta $\ell_I=\int_{a_I}\partial X$ [24]. The string integrand depends on them through the factor

$$\int\prod_{i=1}^{g}d\ell_i e^{\alpha' i\pi\sum_{I,J}\ell_I\ell_J\Omega_{IJ}-2i\pi\alpha'\sum_{I,j}\ell_I\cdot k_j\int_P^{z_j}\omega_I}, \qquad (18)$$

Importantly, the integration path between P and z_j in Eq. (18) depends on a homology class. This implies that this expression has an intrinsic multivaluedness, corresponding to the freedom of shifting the loop momentum by external momenta when punctures cross through the a cycles. (Doing the Gaussian integration reduces to the standard expression of the string propagator, which is single valued on the surface.) Choosing one for each of these contours induces a choice of g cuts on the world sheet along g given a cycles that renders the expression single valued. Our choice to make the a cycle join at some common point also removes the loop momentum shifting ambiguity and give globally defined loop momenta.

A two-loop example: The generalization of Eq. (9) gives the two-loop integrated relations

$$\sum_{r=1}^{|\alpha|}\left(\prod_{s=1}^{r}e^{i\alpha'\pi k_1\cdot k_{\alpha_s}}\right)\mathcal{A}^{(2)}(\ldots,\alpha_{s-1},1,\alpha_s,\ldots|\beta|\gamma)$$
$$+\sum_{r=1}^{|\beta|}\left(\prod_{s=1}^{r}e^{i\alpha'\pi k_1\cdot k_{\beta_s}}\right)\mathcal{A}^{(2)}(\alpha|\ldots,\beta_{s-1},1,\beta_s,\ldots|\gamma)[e^{-i\alpha'\pi\ell_1\cdot k_1}]$$
$$+\sum_{r=1}^{|\gamma|}\left(\prod_{s=1}^{r}e^{i\alpha'\pi k_1\cdot k_{\gamma_s}}\right)\mathcal{A}^{(2)}(\alpha|\beta|\ldots,\gamma_{s-1},1,\gamma_s,\ldots)[e^{-i\alpha'\pi\ell_2\cdot k_1}]=0. \tag{19}$$

At four points we get

$$\mathcal{A}^{(2)}(1234)+e^{i\pi\alpha' k_1\cdot k_2}\mathcal{A}^{(2)}(2134)+e^{i\pi\alpha' k_1\cdot k_{23}}\mathcal{A}^{(2)}(2314)$$
$$+\mathcal{A}^{(2)}(234|1|.)[e^{-i\pi\alpha'\ell_1\cdot k_1}]+\mathcal{A}^{(2)}(234|.|1)[e^{-i\pi\alpha'\ell_2\cdot k_1}]=0 \tag{20}$$

where $\mathcal{A}^{(2)}(1234)$ etc. are planar two-loop amplitude integrands, and $\mathcal{A}^{(2)}(234|1|.)$, $\mathcal{A}^{(2)}(234|.|1)$ are the two nonplanar amplitude integrands with the external state 1 on the b_I cycle with $I=1$, 2, as Fig. 2. The field theory limit of that relation, at leading order in α', leads to

$$A^{(2)}(1234)+\mathcal{A}^{(2)}(2134)+A^{(2)}(2314)$$
$$+A^{(2)}(234|1|.)+A^{(2)}(234|.|1)=0, \tag{21}$$

where $A_4^{LC}(\cdots)$ are the leading color field theory single trace amplitudes, and with our choice of orientation of the cycles $A^{(2)}(234|1|.)+A^{(2)}(234|.|1)=A_{3;1}(234;1)$ is the double trace field theory amplitude. We recover the relation obtained by the unitarity method in Ref. [36]. For $\mathcal{N}=4$ SYM theory, the graphs are essentially scalar planar and nonplanar double boxes [37], and this relation is easily verified by inspection, thanks to the antisymmetry of the three-point vertex. At order α', we conjecture that the field theory limit yields

$$k_1\cdot k_2A^{(2)}(2134)+k_1\cdot(k_2+k_3)A^{(2)}(2314)$$
$$-A^{(2)}(234|1|.)[\ell_1\cdot k_1]-A^{(2)}(234|.|1)[\ell_2\cdot k_1]=0. \tag{22}$$

These relations are not reducible to KK-like color relations, like these of Ref. [38], just like at tree-level where the BCJ kinematic relation go beyond KK ones. An extension of the one-loop argument [39] indicates that the massive string corrections to the field theory limit of the propagator does not contribute at the first order in α'. A detailed verification of these kinds of identities will be provided somewhere else, but we give below a motivation by considering the two-particle discontinuity in the case of $\mathcal{N}=4$ SYM. The two-particle s-channel cut of the two-loop amplitude is the sum of two contributions, with one-loop and tree-level amplitudes, $A(\cdots)$ and $A^{\rm tree}(\cdots)$ [40], respectively:

$$\mathrm{disc}_sA^{(2)}(2134)=A(\ell,21,-\tilde{\ell})A^{\rm tree}(-\ell,34,\tilde{\ell})$$
$$+A^{\rm tree}(\ell,21,-\tilde{\ell})A(-\ell,34,\tilde{\ell}), \tag{23}$$

where ℓ and $\tilde{\ell}$ are the on-shell cut loop momenta. The s-channel two-particle cut of Eq. (22) gives a first contribution

$$(k_1\cdot\ell_1A^{\rm tree}(\ell_1,12,-\tilde{\ell}_1)+k_1\cdot(\ell_1+k_2)A^{\rm tree}(\ell_1,21,-\tilde{\ell}_1))$$
$$\times A(-\ell_1,34,\tilde{\ell}_1)=0 \tag{24}$$

where ℓ_1 and $\tilde{\ell}_1$ are the cut momenta. This expression vanishes thanks to the monodromy relation between the four-point tree amplitudes in the parenthesis [1,15,16]. The second contribution is

$$(A(1,\ell_2,2,-\tilde{\ell}_2)[k_1\cdot\ell_1]+A(\ell_2,12,-\tilde{\ell}_2)[k_1\cdot(\ell_1+\ell_2)]$$
$$+A(\ell_2,21,-\tilde{\ell}_2)[k_1\cdot(\ell_1+\ell_2+k_2)])A^{\rm tree}(-\ell_2,34,\tilde{\ell}_2)$$
$$=0 \tag{25}$$

where ℓ_1 is the one-loop loop momentum and ℓ_2 and $\tilde{\ell}_2$ are the cut momenta. This expression vanishes thanks to the four-point one-loop monodromy relation [Eq. (15)] in the parenthesis. We believe that this approach has the advantage of fixing some ambiguities in the definition of loop momentum in quantum field theory. And the implications of the monodromy relations at higher-loop in maximally supersymmetric Yang-Mills, by applying our construction to the worldline formalism of Ref. [41], will be studied elsewhere.

Finally, we note that our construction should apply to both the bosonic and supersymmetric string, as far as the difficulties concerning the integration of the supermoduli [42] can be put aside.

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APPENDIX: PLANAR AND NONPLANAR GREEN FUNCTION

The Green function between two external states on the same boundary of the annulus $\Re\mathrm{e}(\nu_r) = \Re\mathrm{e}(\nu_s)$ is given by $\tilde{G}(\nu_r, \nu_s) = -\log \vartheta_1[i\Im\mathrm{m}(\nu_r - \nu_s)|\tau]/\vartheta_1'(0)$ with $\log q = -2\pi t$

$$\frac{\vartheta_1(\nu|\tau)}{\vartheta_1{}'(0)} = \frac{\sin(\pi\nu)}{\pi} \prod_{n\geq 1} \frac{1 - 2q^n \cos(2\pi\nu) + q^{2n}}{(1-q^n)^2} \tag{A1}$$

and between two external states on the different boundaries of the annulus $\Re\mathrm{e}(\nu_r) = \Re\mathrm{e}(\nu_s) + \frac{1}{2}$ is given by $\tilde{G}(\nu_r, \nu_s) = -\log\vartheta_1(\nu_r - \nu_s|\tau)/\vartheta_1{}'(0) = -\log\vartheta_2(i\Im\mathrm{m}(\nu_r - \nu_s)|\tau)/\theta_1{}'(0)$ thanks to the relation between the ϑ functions under the shift $\nu \to \nu + \frac{1}{2}$

$$\vartheta_1\left(\nu + \frac{1}{2}\middle|\tau\right) = \vartheta_2(\nu|\tau), \qquad \vartheta_2\left(\nu + \frac{1}{2}\middle|\tau\right) = -\vartheta_1(\nu|\tau) \tag{A2}$$

where

$$\frac{\vartheta_2(\nu|\tau)}{\vartheta_1{}'(0)} = \frac{\cos(\pi\nu)}{\pi} \prod_{n\geq 1} \frac{1 + 2q^n \cos(2\pi\nu) + q^{2n}}{(1-q^n)^2}. \tag{A3}$$

The periodicity around the loop follows from

$$\begin{aligned} \vartheta_1(\nu + \tau|\tau) &= -e^{-i\pi\tau - 2i\pi\nu}\vartheta_1(\nu|\tau); \\ \vartheta_2(\nu + \tau|\tau) &= e^{-i\pi\tau - 2i\pi\nu}\vartheta_2(\nu|\tau), \end{aligned} \tag{A4}$$

and an appropriate redefinition of the loop momentum.

The string theory correction $\delta_\pm(x)$ to the field theory propagator in Eq. (12) is

$$\delta_\pm(x) = -\log(1 \pm e^{-2i\pi|x|t}). \tag{A5}$$

$\delta_-(x)$ is the contribution of massive string modes propagating between two external states on the same boundary and $\delta_+(x)$ on different boundaries.

[1] Z. Bern, J. J. M. Carrasco, and H. Johansson, Phys. Rev. D **78**, 085011 (2008).
[2] Z. Bern, J. J. M. Carrasco, and H. Johansson, Phys. Rev. Lett. **105**, 061602 (2010).
[3] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, Phys. Rev. D **82**, 125040 (2010).
[4] R. H. Boels, B. A. Kniehl, O. V. Tarasov, and G. Yang, J. High Energy Phys. 02 (2013) 063.
[5] C. R. Mafra and O. Schlotterer, Fortschr. Phys. **63**, 105 (2015).
[6] J. Nohle, Phys. Rev. D **90**, 025020 (2014).
[7] Z. Bern, S. Davies, T. Dennen, Y.-t. Huang, and J. Nohle, Phys. Rev. D **92**, 045041 (2015).
[8] S. Badger, G. Mogull, A. Ochirov, and D. O'Connell, J. High Energy Phys. 10 (2015) 064.
[9] C. R. Mafra, O. Schlotterer, and S. Stieberger, J. High Energy Phys. 07 (2011) 092.
[10] C. R. Mafra and O. Schlotterer, J. High Energy Phys. 10 (2015) 124.
[11] D. Chester, Phys. Rev. D **93**, 065047 (2016).
[12] A. Primo and W. J. Torres Bobadilla, J. High Energy Phys. 04 (2016) 125.
[13] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, Phys. Rev. Lett. **115**, 121603 (2015).
[14] J. J. M. Carrasco and H. Johansson, J. Phys. A **44**, 454004 (2011).
[15] N. E. J. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, Phys. Rev. Lett. **103**, 161602 (2009).
[16] S. Stieberger, arXiv:0907.2211.
[17] B. Feng, R. Huang, and Y. Jia, Phys. Lett. B **695**, 350 (2011).
[18] H. Johansson and A. Ochirov, J. High Energy Phys. 01 (2016) 170.
[19] L. de la Cruz, A. Kniss, and S. Weinzierl, J. High Energy Phys. 09 (2015) 197.
[20] R. H. Boels and R. S. Isermann, Phys. Rev. D **85**, 021701 (2012).
[21] R. H. Boels and R. S. Isermann, J. High Energy Phys. 03 (2012) 051.
[22] Y.-J. Du and H. Luo, J. High Energy Phys. 01 (2013) 129.
[23] M. B. Green, J. H. Schwarz, and E. Witten, Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies and Phenomenology (Cambridge University Press, Cambridge, 1988).
[24] E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60**, 917 (1988).
[25] M. B. Green and J. H. Schwarz, Nucl. Phys. **B198**, 441 (1982).
[26] C. R. Mafra and O. Schlotterer, J. High Energy Phys. 08 (2014) 099.
[27] S. He, R. Monteiro, and O. Schlotterer, J. High Energy Phys. 01 (2016) 171.
[28] M. B. Green, J. H. Schwarz, and L. Brink, Nucl. Phys. **B198**, 474 (1982).
[29] Z. Bern, Phys. Lett. B **296**, 85 (1992).
[30] Z. Bern and D. A. Kosower, Phys. Rev. Lett. **66**, 1669 (1991).
[31] Z. Bern and D. A. Kosower, Nucl. Phys. **B362**, 389 (1991).
[32] Z. Bern and D. A. Kosower, Nucl. Phys. **B379**, 451 (1992).
[33] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
[34] M. B. Green and P. Vanhove, Phys. Rev. D **61**, 104011 (2000).

[35] N. Berkovits, M. B. Green, J. G. Russo, and P. Vanhove, J. High Energy Phys. 11 (2009) 063.

[36] B. Feng, Y. Jia, and R. Huang, Nucl. Phys. **B854**, 243 (2012).

[37] Z. Bern, J. S. Rozowsky, and B. Yan, Phys. Lett. B **401**, 273 (1997).

[38] S. G. Naculich, Phys. Lett. B **707**, 191 (2012).

[39] P. Tourkine and P. Vanhove, Classical Quantum Gravity **29**, 115006 (2012).

[40] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein, and J. S. Rozowsky, Nucl. Phys. **B530**, 401 (1998).

[41] P. Dai and W. Siegel, Nucl. Phys. **B770**, 107 (2007).

[42] E. Witten, arXiv:1209.5461.