Universal Trade-Off Relation between Power and Efficiency for Heat Engines

Naoto Shiraishi
Department of Basic Science, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8902, Japan

Keiji Saito
Department of Physics, Keio University, 3-14-1 Hiyoshi, Yokohama 223-8522, Japan

Hal Tasaki
Department of Physics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan
(Received 4 May 2016; revised manuscript received 4 August 2016; published 31 October 2016)

For a general thermodynamic system described as a Markov process, we prove a general lower bound for dissipation in terms of the square of the heat current, thus establishing that nonvanishing current inevitably implies dissipation. This leads to a universal trade-off relation between efficiency and power, with which we rigorously prove that a heat engine with nonvanishing power never attains the Carnot efficiency. Our theory applies to systems arbitrarily far from equilibrium, and does not assume any specific symmetry of the model.

DOI: 10.1103/PhysRevLett.117.190601

Heat engines have been among central topics of thermodynamics since the seminal work of Carnot [1,2], who established that the efficiency of any heat engine operating with two heat baths cannot exceed the Carnot efficiency η_C. In recent years considerable effort has been devoted to finding thermoelectric materials with higher efficiency [3–6], and to fabricating stochastic cyclic heat engines in small systems [7–12]. It is crucial to develop a fundamental understanding, on the basis of recent progress in nonequilibrium statistical mechanics [13], of heat-to-work conversion mechanisms.

It is known, again since Carnot, that the Carnot efficiency can be achieved in quasistatic processes. But the power, i.e., the work produced in a unit time, of a quasistatic engine vanishes since it takes an infinitely long time to complete a cycle. Then a natural question arises whether there can be an engine with nonvanishing power which attains the Carnot efficiency. This is indeed a special case of a fundamental question of whether there is a universal trade-off relation between energy transfer and dissipation in thermodynamic processes. Note that thermodynamics, which does not have the notion of time scale, cannot answer these questions.

There have been various attempts [14–33] to look for engines with high efficiency and nonvanishing power. In particular Benenti, Saito, and Casati [14] studied the efficiency of thermoelectric transport in the linear response regime, and argued that broken time-reversal symmetry (caused, e.g., by a magnetic field), that leads to nonsymmetrical Onsager matrix, might increase the efficiency; they even suggested that a cycle with nonvanishing power which operates reversibly may be realizable. At this level of argument, the restriction on the Onsager matrix elements imposed by the second law does not prohibit the coexistence of nonvanishing power with the Carnot efficiency. This observation triggered a number of studies on the relation between power and efficiency [15–26,29–33].

Studies based on concrete models mainly within the linear response regime [15–28] have denied the possibility of engines with nonvanishing power and the maximum efficiency, suggesting a general no-go theorem. See Ref. [29] where such a theorem for special models is obtained. There still are a number of attempts, on the other hand, for the realization of such engines [30–33]. No matter what the current "general belief" may be, it is desirable to have decisive conclusions on this fundamental issue without resorting to specific models, approximations, or restrictions (e.g., to the linear response regime).

In this Letter, we present such general and rigorous results. We first prove a general lower bound for dissipation (i.e., entropy production rate) in terms of the square of the total heat current to reservoirs. The bound implies a universal trade-off relation between power and efficiency in heat engines, which, as a corollary, implies that a heat engine with nonvanishing power can never attain the Carnot efficiency.

Our theory applies to *any* heat engine which is described by classical mechanics, and whose interaction with heat baths can be represented by a Markov process. Practically speaking we cover essentially any realistic engines, macroscopic or mesoscopic, except those working in a genuine quantum regime.

Our trade-off relation relies essentially only on the condition that the stochastic dynamics associated with a heat bath leaves the canonical distribution invariant. We thus see that this condition is critical for the no-go theorem for an engine with nonvanishing power and the Carnot efficiency.

To get the present results, it was essential for us to look at this old problem in light of the notion of entropy production, which had been developed in the long and rich history of nonequilibrium statistical mechanics [13]. In particular, the idea of the partial entropy production rate developed for

Markov processes in Refs. [34–36] played an important role. Some of the crucial ideas and techniques in the present Letter appeared in Ref. [37] by two of us (N. S. and K. S.).

Main results.—Consider an arbitrary heat engine which undergoes a cyclic process with period τ. During a cycle, the engine may interact with n external heat baths with finite inverse temperatures $\beta_1, \ldots, \beta_n$ in an arbitrary manner. Let $J_\nu(t)$ be the heat current that flows from the engine to the νth bath at time t. The energy conservation implies that the total work done by the engine is $W = -\sum_{\nu=1}^{n} \int_0^\tau dt J_\nu(t)$. Define the total entropy production in the baths, which is a measure of dissipation in the cycle, by

$$\Delta S := \sum_{\nu=1}^{n} \beta_\nu \int_0^\tau dt J_\nu(t). \tag{1}$$

It satisfies $\Delta S \geq 0$, which is the second law.

Our main finding is the inequality

$$\left(\int_0^\tau dt \sum_{\nu=1}^{n} |J_\nu(t)| \right)^2 \leq \tau \bar{\Theta} \Delta S, \tag{2}$$

which is proved for a general engine described by a Markov process. Here $\bar{\Theta}$, which depends on the model and state, is always finite and proportional to the size of the engine [38]. For the standard Langevin-type heat baths described by Eq. (7), one has $\bar{\Theta} = 2\bar{\gamma}\,\bar{K}/\bar{\beta}\,\bar{m}$, where $\bar{K}$ denotes the time average of the total kinetic energy of the engine, and $\bar{\beta}$, $\bar{\gamma}$ and $\bar{m}$ are properly averaged inverse temperatures (of the baths) and the damping constant and the mass (of the engine), respectively. See Ref. (11). Note that both the lhs and rhs of Eq. (2) are proportional to the square of the size of the engine. Therefore the inequality is meaningful in the thermodynamic limit as well.

The inequality (2) manifests the fundamental trade-off relation: nonvanishing current inevitably induces dissipation. To see the implication on efficiency of heat engines, consider the case with $n = 2$ and let the inverse temperatures of the baths be β_H and β_L with $\beta_H < \beta_L$. We denote, as usual, by $Q_H > 0$ the heat absorbed by the engine from the bath with β_H, and by $Q_L > 0$ the heat flowed from the engine to the bath with β_L. The work is then $W = Q_H - Q_L$, and the entropy production is $\Delta S = \beta_L Q_L - \beta_H Q_H$. The bound Eq. (2) reduces to $(Q_H + Q_L)^2 \leq \tau \bar{\Theta} \Delta S$.

Let $\eta := W/Q_H$ be the efficiency of the engine, and $\eta_C := 1 - (\beta_H/\beta_L)$ be the Carnot efficiency. Noting a relation in thermodynamics $\eta(\eta_C - \eta) = W\Delta S/\{\beta_L (Q_H)^2\}$ [23], our bound yields a trade-off relation between power and efficiency

$$\frac{W}{\tau} \leq \bar{\Theta} \beta_L \eta (\eta_C - \eta). \tag{3}$$

The averaged power W/τ must vanish as $\eta \uparrow \eta_C$ or (obviously) as $\eta \downarrow 0$. We conclude that an engine with nonvanishing power never attains the maximum efficiency. The bound (3) was discussed numerically in Ref. [20] for thermoelectric phenomena, and derived for Brownian heat engines with time reversal symmetry in Ref. [24], both in the linear response regime. It is proved here for systems arbitrarily far from equilibrium for general models without any specific symmetry.

Setup and the main inequality.—Suppose that there are a heat engine, n heat baths with inverse temperatures $\beta_1, \ldots, \beta_n$, and an external agent who operates on the engine (by, e.g., moving a piston, changing a potential, attaching or detaching heat baths). Although our theorem applies to general Markov processes, we focus on a general classical engine modeled as a system of N particles (with inertia) with arbitrary confining potential and interaction, possibly under magnetic field. Let m_i, $\boldsymbol{r}_i$, and $\boldsymbol{v}_i$ denote the mass, the position, and the velocity, respectively, of the ith particle (with $i = 1, \ldots, N$) [39]. We collectively represent by $X = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)$ the state of the system. We assume that the system is characterized by a set of parameters λ, which does not only determine the dynamics of the system (i.e., engine), but also the way it couples to the baths. We denote by $E^\lambda(X) := \sum_{i=1}^{N} m_i |\boldsymbol{v}_i|^2/2 + U^\lambda(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$ the total energy of the system with parameter λ.

The external agent varies the parameters according to a fixed function $\lambda(t)$ of time t. Let $\mathcal{P}_t(X)$ be the probability density to find the system in X at t. It obeys the continuous master equation [40,41]

$$\frac{\partial}{\partial t} \mathcal{P}_t(X) = (\hat{\mathcal{L}}^{\lambda(t)} \mathcal{P}_t)(X). \tag{4}$$

The time evolution operator is decomposed into deterministic and dissipative parts as

$$\hat{\mathcal{L}}^\lambda = \hat{\mathcal{L}}^{0,\lambda} + \sum_{\nu=1}^{n} \sum_{i=1}^{N} \hat{\mathcal{L}}_i^{\nu,\lambda}. \tag{5}$$

Here $\hat{\mathcal{L}}^{0,\lambda}$ is the Liouville operator (see C of Ref. [42]) for the deterministic dynamics described by the Newton equation $m_i \ddot{\boldsymbol{r}}_i(t) = \boldsymbol{F}_i^\lambda(X)$. The force $\boldsymbol{F}_i^\lambda(X)$ consists of $-\boldsymbol{\nabla}_i U^\lambda(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$ and, possibly, some velocity dependent force (such as the Lorentz force). The only assumption is that the resulting time evolution with fixed λ preserves both the phase space volume and the total energy.

The operator $\hat{\mathcal{L}}_i^{\nu,\lambda}$ with $\nu = 1, \ldots, n$ and $i = 1, \ldots, N$ represents the dissipation of the ith particle, i.e., the change in $\boldsymbol{v}_i$, caused by the νth heat bath. The most general expression reads [40]

$$(\hat{\mathcal{L}}_i^{\nu,\lambda} \mathcal{P})(X) := \int dY \{ r_i^{\nu,\lambda}(X, Y) \mathcal{P}(Y) - r_i^{\nu,\lambda}(Y, X) \mathcal{P}(X) \}, \tag{6}$$

where $r_i^{\nu,\lambda}(X, Y) \geq 0$ is the hopping rate from Y to X. It leaves the canonical distribution with β_ν invariant, i.e., $\int dY \{ r_i^{\nu,\lambda}(X, Y) e^{-\beta_\nu E^\lambda(Y)} - r_i^{\nu,\lambda}(Y, X) e^{-\beta_\nu E^\lambda(X)} \} = 0$. Discrete noise in small engines such as the Rayleigh piston and the Brownian motor [43–45] can be represented by Eq. (6) with suitably chosen $r_i^{\nu,\lambda}(X,Y)$, whose explicit form can be found, e.g., in Eq. (2) of Ref. [44]. In the limit where the change in velocity is infinitesimally small, Eq. (6) reduces to

$$\hat{\mathcal{L}}_i^{\nu,\lambda} = \frac{\gamma_\nu(\lambda, \boldsymbol{r}_i)}{m_i}\left(\frac{\partial}{\partial \boldsymbol{v}_i}\boldsymbol{v}_i + \frac{1}{\beta_\nu m_i}\frac{\partial^2}{\partial \boldsymbol{v}_i{}^2}\right), \tag{7}$$

which describes the standard Langevin noise [40]. With Eq. (7), the master equation (4) becomes the Kramers equation. The "damping constant" $\gamma_\nu(\lambda,\boldsymbol{r})$ represents the magnitude of noise from the νth bath. Note that it may depend on $\boldsymbol{r}$, and on t through $\lambda(t)$.

We stress that the above formulation covers essentially any classical heat engines including the Brownian heat engine which was recently realized experimentally [8,9] using a single particle in a harmonic trap [46]. It is also easy to treat overdamped dynamics [47].

The averaged heat current to the νth bath at t is defined in the standard manner (see A of Ref. [42]) as

$$J_\nu(t) \coloneqq -\sum_{i=1}^N \int dX E^{\lambda(t)}(X)(\hat{\mathcal{L}}_i^{\nu,\lambda(t)}\mathcal{P}_t)(X). \tag{8}$$

We then define the total entropy production rate in the system and the baths by

$$\sigma_{\rm tot}(t) \coloneqq \frac{d}{dt}H(\mathcal{P}_t) + \sum_{\nu=1}^n \beta_\nu J_\nu(t), \tag{9}$$

where $H(\mathcal{P}) \coloneqq -\int dX\mathcal{P}(X)\log\mathcal{P}(X)$ is the Shannon entropy of the system.

The core of our theory is the inequality

$$\sum_{\nu=1}^n |J_\nu(t)| \le \sqrt{\Theta(t)\sigma_{\rm tot}(t)}, \tag{10}$$

which is valid for any $\mathcal{P}_t$ satisfying the master equation (4). Here $\Theta(t)$ is a quantity which depends on the model and the state, but is finite and proportional to N. For baths with Eq. (7), we have

$$\Theta(t) = \sum_{i=1}^N\sum_{\nu=1}^n \frac{1}{\beta_\nu}\langle\gamma_\nu(\lambda(t),\boldsymbol{r}_i)|\boldsymbol{v}_i|^2\rangle_t, \tag{11}$$

where $\langle\cdots\rangle_t$ denotes the average with respect to $\mathcal{P}_t$. See B of Ref. [42] for a concrete expression and an upper bound for $\Theta(t)$ for baths with Eq. (6).

To treat thermodynamic cycles of period τ, we consider the case $\lambda(0)=\lambda(\tau)$, and assume $\mathcal{P}_0=\mathcal{P}_\tau$, which is always realized by running the cycle sufficiently many times. We then define the total entropy production (in the baths) during a cycle by

$$\Delta S \coloneqq \int_0^\tau dt\sigma_{\rm tot}(t) = \int_0^\tau dt\sum_{\nu=1}^n \beta_\nu J_\nu(t), \tag{12}$$

where the contribution from $H(\mathcal{P}_t)$ vanishes because of the cyclicity. It is essential that ΔS is written only in terms of the currents, which are measurable quantities. By integrating Eq. (10) over t, and using the Schwarz inequality, we readily obtain Eq. (2), whose implications have already been discussed, with $\bar{\Theta} \coloneqq \tau^{-1}\int_0^\tau dt\Theta(t)$.

Derivation.—We study the Markov jump process obtained by faithfully discretizing the continuous master equation (4). We prove inequalities corresponding to Eq. (10), from which Eq. (10) follows as continuum limits. The mathematically minded reader should understand that we interpret Eq. (4) as a continuum limit of the master equation (13).

As usual we decompose the whole phase space into small $6N$-dimensional parallelepipeds whose size in the v-directions is ε and that in the r-directions is ε'. Each cell is represented by X at its center.

We now regard X as a discrete variable, and denote by E_X^λ the corresponding energy. The probability $p_{t,X}$ to find the system in X at t obeys the master equation

$$\frac{d}{dt}p_{t,X} = \sum_Y R_{XY}^{\lambda(t)}p_{t,Y}, \tag{13}$$

which is obtained as a discretization of Eq. (4). See C of Ref. [42] for the (standard) discretization procedure.

As in Eq. (5), the transition rate is decomposed as $R_{XY}^\lambda = R_{XY}^{0,\lambda} + \sum_{\nu=1}^n\sum_{i=1}^N R_{XY}^{\nu,i,\lambda}$. To simplify the notation we also write this as $R_{XY}^\lambda = \sum_\mu R_{XY}^{\mu,\lambda}$, where $\mu=0$ or $\mu=(\nu,i)$ with $\nu=1,\ldots,n$ and $i=1,\ldots,N$. The transition rate for each μ satisfies $R_{XY}^{\mu,\lambda}\ge 0$ for $X\ne Y$ and $\sum_X R_{XY}^{\mu,\lambda}=0$. For the deterministic part, we assume that $\sum_Y R_{XY}^{0,\lambda}=0$, which means that the uniform distribution is invariant under $R_{XY}^{0,\lambda}$. This property is always satisfied in the faithful discretization of a dynamics which preserves the phase space volume. For the dissipation of the ith particle from the νth bath, we assume the invariance of the corresponding canonical distribution, i.e., $\sum_Y R_{XY}^{\nu,i,\lambda}e^{-\beta_\nu E_Y^\lambda}=0$.

We decompose the heat current into contributions from each particle as $J_\nu(t)=\sum_{i=1}^N J_{\nu,i}(t)$, where

$$J_{\nu,i}(t) \coloneqq -\sum_{X,Y} E_X^{\lambda(t)}R_{XY}^{\nu,i,\lambda(t)}p_{t,Y} = -\sum_{X,Y} K_X^i R_{XY}^{\nu,i,\lambda(t)}p_{t,Y}, \tag{14}$$

with $K_X^i \coloneqq m_i|\boldsymbol{v}_i|^2/2$. We here noted that the dissipative dynamics changes only the velocity. We also decompose the change in the Shannon entropy $H(p) \coloneqq -\sum_X p_X\log p_X$ as

$$\frac{d}{dt}H(p_t) = -\sum_X \dot{p}_{t,X}\log p_{t,X} = \sum_\mu \eta_\mu(t) \tag{15}$$

with $\eta_\mu(t) \coloneqq -\sum_{X,Y}R_{XY}^{\mu,\lambda(t)}p_{t,Y}\log p_{t,X}$. We then define the entropy production rate for μ by $\sigma_\mu(t) \coloneqq \eta_\mu(t) + \beta_\mu J_\mu(t)$

with $\beta_0 \coloneqq 0$ and $\beta_{(\nu,i)} \coloneqq \beta_\nu$. The total entropy production rate is written as $\sigma_{\rm tot}(t) = \sum_\mu \sigma_\mu(t)$.

Define the dual transition rate [48] by $\tilde{R}^{\mu,\lambda}_{XY} \coloneqq e^{\beta_\nu (E^\lambda_Y - E^\lambda_X)} R^{\mu,\lambda}_{YX}$, which satisfies $\sum_X \tilde{R}^{\mu,\lambda}_{XY} = 0$ because of the condition $\sum_Y R^{\mu,\lambda}_{XY} e^{-\beta_\nu E^\lambda_Y} = 0$. One then has

$$\sigma_\mu(t) = \sum_{X,Y} R^{\mu,\lambda(t)}_{XY} p_{t,Y} \log \frac{R^{\mu,\lambda(t)}_{XY} p_{t,Y}}{\tilde{R}^{\mu,\lambda(t)}_{YX} p_{t,X}} = \sum_{X \neq Y} s\left(R^{\mu,\lambda(t)}_{XY} p_{t,Y}, \tilde{R}^{\mu,\lambda(t)}_{YX} p_{t,X}\right), \tag{16}$$

where the first expression is standard [13] (see F of Ref. [42]), and the second with $s(a,b) \coloneqq a\log(a/b) + b - a$ was introduced in Refs. [34–36], where the summand was named the partial entropy production rate. By using the inequality $s(a,b) \geq c_0 (a-b)^2/(a+b)$ with $c_0 = 8/9$ (see E of Ref. [42]), and defining $\tilde{A}^{\mu,\pm}_{XY} \coloneqq R^{\mu,\lambda(t)}_{XY} p_{t,Y} \pm \tilde{R}^{\mu,\lambda(t)}_{YX} p_{t,X}$, we have

$$\sigma_\mu(t) \geq c_0 \sum_{X \neq Y} \frac{(\tilde{A}^{\mu,-}_{XY})^2}{\tilde{A}^{\mu,+}_{XY}}. \tag{17}$$

For $\mu = (\nu, i)$ we rewrite Eq. (14) as

$$J_\mu(t) = -\sum_{X \neq Y} \Delta K^i_X \tilde{A}^{\mu,-}_{X,Y} = -\sum_{X \neq Y} \Delta K^i_X \sqrt{\tilde{A}^{\mu,+}_{X,Y}} \frac{\tilde{A}^{\mu,-}_{X,Y}}{\sqrt{\tilde{A}^{\mu,+}_{X,Y}}}, \tag{18}$$

where $\Delta K^i_X \coloneqq K^i_X - \langle K^i \rangle_t$. By using the Schwarz inequality and Eq. (17), and noting the relation $\sum_{Y(\neq X)} (\Delta K^i_X)^2 \tilde{R}^{\mu,\lambda}_{YX} = \sum_{Y(\neq X)} (\Delta K^i_X)^2 R^{\mu,\lambda}_{YX}$, which follows from $\tilde{R}^{\mu,\lambda}_{XX} = R^{\mu,\lambda}_{XX}$, we arrive at

$$|J_\mu(t)| \leq \sqrt{\Theta^{(1)}_\mu(t) \sigma_\mu(t)} \tag{19}$$

with

$$\Theta^{(1)}_\mu(t) \coloneqq \frac{1}{c_0} \sum_{X \neq Y} (\Delta K^i_X)^2 A^{\mu,+}_{XY}. \tag{20}$$

Here, we defined $A^{\mu,\pm}_{XY} \coloneqq R^{\mu,\lambda(t)}_{XY} p_{t,Y} \pm R^{\mu,\lambda(t)}_{YX} p_{t,X}$. By summing Eq. (19) over μ, applying the Schwarz inequality, and noting that Eq. (17) implies $\sigma_0(t) \geq 0$, we finally get $\sum_{\nu=1}^n \sum_{i=1}^N |J_{\nu,i}(t)| \leq \sqrt{\Theta(t)\sigma_{\rm tot}(t)}$ with $\Theta(t) = \Theta^{(1)}(t) \coloneqq \sum_{\nu=1}^n \sum_{i=1}^N \Theta^{(1)}_{\nu,i}(t)$. By taking the continuum limit, this implies the desired Eq. (10). For discrete noise, where the rate $r^{\nu,\lambda}_i(X,Y)$ is finite, $\Theta^{(1)}(t)$ remains finite in the continuum limit (see B of Ref. [42]).

In the limit of Langevin noise with Eq. (7), where $r^{\nu,\lambda}_i(X,Y)$ becomes singular, Eq. (19) becomes meaningless since Eq. (20) diverges. In this case we make use of the additional symmetry $R^{\mu,\lambda}_{XY} e^{-\beta_\nu E^\lambda_Y} = R^{\mu,\lambda}_{YX} e^{-\beta_\nu E^\lambda_X}$ (i.e., the detailed balance condition, see C of Ref. [42]) to derive a stronger bound with a new definition of $\Theta(t)$. With the new symmetry, one easily verifies the standard expression [13] (see F of Ref. [42])

$$\sigma_\mu(t) = \sum_{X,Y} R^{\mu,\lambda(t)}_{XY} p_{t,Y} \log \frac{R^{\mu,\lambda(t)}_{XY} p_{t,Y}}{R^{\mu,\lambda(t)}_{YX} p_{t,X}}, \tag{21}$$

for $\mu \neq 0$. By noting the symmetry between X and Y this can be written as

$$= \frac{1}{2} \sum_{X,Y} \{R^{\mu,\lambda(t)}_{XY} p_{t,Y} - R^{\mu,\lambda(t)}_{YX} p_{t,X}\} \log \frac{R^{\mu,\lambda(t)}_{XY} p_{t,Y}}{R^{\mu,\lambda(t)}_{YX} p_{t,X}}. \tag{22}$$

By using the inequality $(a-b)\log(a/b) \geq 2(a-b)^2/(a+b)$ (see E of Ref. [42]), we find that

$$\sigma_\mu(t) \geq \sum_{X \neq Y} \frac{(A^{\mu,-}_{XY})^2}{A^{\mu,+}_{XY}}. \tag{23}$$

Again by using the symmetry, Eq. (14) is rewritten as

$$J_\mu(t) = -\sum_{X \neq Y} K^i_X A^{\mu,-}_{XY} = -\frac{1}{2} \sum_{X \neq Y} (K^i_X - K^i_Y) A^{\mu,-}_{XY} = -\frac{1}{2} \sum_{X \neq Y} (K^i_X - K^i_Y) \sqrt{A^{\mu,+}_{XY}} \frac{A^{\mu,-}_{XY}}{\sqrt{A^{\mu,+}_{XY}}}, \tag{24}$$

which leads to Eq. (19) with $\Theta^{(1)}_\mu(t)$ replaced by $\Theta^{(2)}_\mu(t)$:

$$\Theta^{(2)}_\mu(t) \coloneqq \frac{1}{4} \sum_{X \neq Y} (K^i_X - K^i_Y)^2 A^{\mu,+}_{XY} = \frac{1}{2} \sum_{X \neq Y} (K^i_X - K^i_Y)^2 R^{\mu,\lambda(t)}_{XY} p_{t,Y}. \tag{25}$$

The continuum limit, which is now finite, is readily evaluated as in D of Ref. [42], and we get Eq. (11) with $\Theta(t) = \Theta^{(2)}(t) \coloneqq \sum_{\nu=1}^n \sum_{i=1}^N \Theta^{(2)}_{\nu,i}(t)$.

Discussion.—We have proved that the power of a classical Markovian heat engine must vanish as its efficiency approaches the Carnot bound. The essence was the trade-off relation (2) which shows that any heat flux inevitably induces dissipation. In Ref. [33], attainability of nonvanishing power and the Carnot efficiency is discussed with the Onsager matrix in the classical regime. Our result denies the possibility of realizing this abstract proposal as a Markov process. The clarification of proposals based on quantum systems [30–32] is the next challenge. Toward this direction, extensions of the present results to the case where the engine exchanges quantum particles with particle baths will be discussed in Ref. [47]. A quantum cyclic heat engine will also be considered in Ref. [49].

From a theoretical point of view, the most basic result of ours is the inequality (10), which states for each moment that $\sigma_{\rm tot}(t)$ is strictly positive whenever there is nonvanishing heat current. We must note that $\sigma_{\rm tot}(t)$, which involves the change in the Shannon entropy, may not be a physically observable quantity. But if we are able to interpret $\sigma_{\rm tot}(t)$ as a measure of instantaneous dissipation, the bound (10) can be regarded as a more fundamental trade-off relation between heat current and dissipation, which is valid in any thermodynamic processes. See Ref. [50] for a related observation. It is interesting to apply the relation to transient processes.

When the state of the engine is close to equilibrium, the relation (10) may be understood as follows. In order to have nonvanishing current J between the engine and a bath, there should be a difference $\Delta\beta$ in their inverse temperatures. Then the current J induces the entropy production rate $\sigma \sim \Delta\beta J$. Now if the current satisfies the linear response $J \simeq \kappa\Delta\beta$, we have $\sigma \sim J^2/\kappa$. The bound (10), which is $J^2 \lesssim \Theta\sigma$, then reads $\kappa \lesssim \Theta$. Thus, at least everything is close to equilibrium, our trade-off relation boils down to an upper bound on the heat conductivity. In fact, in the close-to-equilibrium regime, we can show [47] $\kappa \simeq \Theta^{(2)}$ for $\Theta^{(2)}$ of Eq. (11) or Eq. (25).

It is a pleasure to thank Takashi Hara, for discussions and his essential contribution to the present work. We also thank Kay Brandner, Tatsuhiko Koike, Takashi Mori, and Hiroyasu Tajima for useful discussions, and Shin-ichi Sasa and Yohei Nakayama for useful comments on the Letter. The present work was supported by Grant-in-Aid for JSPS Fellows No. 26-7602 (N. S.) and JSPS Grants-in-Aid for Scientific Research No. JP26400404 (K. S.) and No. JP16H02211 (K. S. and H. T.).

[1] S. Carnot, *Reflections on the Motive Power of Fire and on Machines Fitted to Develop that Power* (Bachelier, Paris, 1824).
[2] H. B. Callen, *Thermodynamics and an Introduction to Thermostatics*, 2nd ed. (John Wiley & Sons, New York, 1985).
[3] G. Mahan, B. Sales, and J. Sharp, Thermoelectric materials: New approaches to an old problem, Phys. Today **50**, 42 (1997).
[4] A. Majumdar, Thermoelectricity in Semiconductor Nanostructures, Science **303**, 777 (2004).
[5] M. S. Dresselhaus, G. Chen, M. Y. Tang, R. G. Yang, H. Lee, D. Z. Wang, Z. F. Ren, J.-P. Fleurial, and P. Gogna, New Directions for Low-Dimensional Thermoelectric Materials, Adv. Mater. **19**, 1043 (2007).
[6] G. J. Snyder and E. R. Toberer, Complex thermoelectric materials, Nat. Mater. **7**, 105 (2008).
[7] P. G. Steeneken, K. Le Phan, M. J. Goossens, G. E. J. Koops, G. J. A. M. Brom, C. van der Avoort, and J. T. M. van Beek, Piezoresistive heat engine and refrigerator, Nat. Phys. **7**, 354 (2011).
[8] V. Blickle and C. Bechinger, Realization of a micrometre-sized stochastic heat engine, Nat. Phys. **8**, 143 (2011).
[9] A. Martinez, E. Roldan, L. Dinis, D. Petrov, J. M. R. Parrondo, and R. A. Rica, Brownian Carnot engine, Nat. Phys. **12**, 67 (2015).
[10] M. Ribezzi-Crivellari and F. Ritort, Free-energy inference from partial work measurements in small systems, Proc. Natl. Acad. Sci. U.S.A. **111**, E3386 (2014).
[11] J. V. Koski, V. F. Maisi, J. P. Pekola, and D. V. Averin, Experimental realization of a Szilard engine with a single electron, Proc. Natl. Acad. Sci. U.S.A. **111**, 13786 (2014).
[12] J. Rosnagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. S. Kaler, and Kilian Singer, A single-atom heat engine, Science **352**, 325 (2016).
[13] U. Seifert, Stochastic thermodynamics, fluctuation theorems, and molecular machines, Rep. Prog. Phys. **75**, 126001 (2012).
[14] G. Benenti, K. Saito, and G. Casati, Thermodynamic Bounds on Efficiency for Systems with Broken Time-Reversal Symmetry, Phys. Rev. Lett. **106**, 230602 (2011).
[15] K. Brandner and U. Seifert, Multi-terminal thermoelectric transport in a magnetic field: Bounds on Onsager coefficients and efficiency, New J. Phys. **15**, 105003 (2013).
[16] B. Sothmann and M. Büttiker, Magnon-driven quantum-dot heat engine, Europhys. Lett. **99**, 27001 (2012).
[17] K. Brandner, K. Saito, and U. Seifert, Strong Bounds on Onsager Coefficients and Efficiency for Three-Terminal Thermoelectric Transport in a Magnetic Field, Phys. Rev. Lett. **110**, 070603 (2013).
[18] V. Balachandran, G. Benenti, and G. Casati, Efficiency of three-terminal thermoelectric transport under broken time-reversal symmetry, Phys. Rev. B **87**, 165419 (2013).
[19] J. Stark, K. Brandner, K. Saito, and U. Seifert, Classical Nernst Engine, Phys. Rev. Lett. **112**, 140601 (2014).
[20] K. Brandner and U. Seifert, Bound on thermoelectric power in a magnetic field within linear response, Phys. Rev. E **91**, 012121 (2015).
[21] R. Sánchez, B. Sothmann, and A. N. Jordan, Chiral Thermoelectrics with Quantum Hall Edge States, Phys. Rev. Lett. **114**, 146801 (2015).
[22] V. Holubec, An exactly solvable model of a stochastic heat engine: optimization of power, power fluctuations and efficiency, J. Stat. Mech. (2014) P05022.
[23] R. S. Whitney, Most Efficient Quantum Thermoelectric at Finite Power Output, Phys. Rev. Lett. **112**, 130601 (2014).
[24] K. Brandner, K. Saito, and U. Seifert, Thermodynamics of micro-and nanosystems driven by periodic temperature variations, Phys. Rev. X **5**, 031019 (2015).
[25] K. Proesmans and C. Van den Broeck, Onsager Coefficients in Periodically Driven Systems, Phys. Rev. Lett. **115**, 090601 (2015).
[26] K. Proesmans, B. Cleuren, and C. Van den Broeck, Linear stochastic thermodynamics for periodically driven systems, J. Stat. Mech. (2016) P023202.
[27] K. Sekimoto and S.-i. Sasa, Complementarity relation for irreversible process derived from stochastic energetics, J. Phys. Soc. Jpn. **66**, 3326 (1997).
[28] E. Aurell, K. Gawędzki, C. Mejía-Monasterio, R. Mohayaee, and P. Muratore-Ginanneschi, Refined second law of thermodynamics for fast random processes, J. Stat. Phys. **147**, 487 (2012).

[29] O. Raz, Y. Subaşı, and R. Pugatch, Geometric Heat Engines Featuring Power that Grows with Efficiency, Phys. Rev. Lett. **116**, 160601 (2016).
[30] M. Mintchev, L. Santoni, and P. Sorba, Thermoelectric efficiency of critical quantum junctions, arXiv:1310.2392.
[31] A. E. Allahverdyan, K. V. Hovhannisyan, A. V. Melkikh, and S. G. Gevorkian, Carnot Cycle at Finite Power: Attainability of Maximal Efficiency, Phys. Rev. Lett. **111**, 050601 (2013).
[32] M. Campisi and R. Fazio, The power of a critical heat engine, Nat. Commun. **7**, 11895 (2016).
[33] M. Ponmurugan, Attainability of maximum work and the reversible efficiency from minimally nonlinear irreversible heat engines, arXiv:1604.01912.
[34] N. Shiraishi and T. Sagawa, Fluctuation theorem for partially masked nonequilibrium dynamics, Phys. Rev. E **91**, 012130 (2015).
[35] N. Shiraishi, S. Ito, K. Kawaguchi, and T. Sagawa, Role of measurement-feedback separation in autonomous Maxwell's demons, New J. Phys. **17**, 045012 (2015).
[36] N. Shiraishi, T. Matsumoto, and T. Sagawa, Measurement-feedback formalism meets information reservoirs, New J. Phys. **18**, 013044 (2016).
[37] N. Shiraishi and K. Saito, Incompatibility between Carnot efficiency and finite power in Markovian dynamics, arXiv:1602.03645.
[38] More precisely, the size and the kinetic energy in this paragraph mean those of the parts of the engine that interact with the baths.
[39] With a trivial extension, one can include a, "particle", representing a macroscopic body (such as a ratchet) which forms apparatus constructing the engine.
[40] N. G. Van Kampen, *Stochastic Process in Physics and Chemistry*, 3rd ed. (Elsevier, New York, 2007).
[41] K. Sekimoto, *Stochastic Energetics* (Springer, New York, 2010).
[42] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.117.190601 for some technical points.
[43] A. Siegel, Differential-Operator Approximations to the Linear Boltzmann Equation, J. Am. Phys. **1**, 378 (1960).
[44] C. Van den Broeck, R. Kawai, and P. Meurs, Microscopic Analysis of a Thermal Brownian Motor, Phys. Rev. Lett. **93**, 090601 (2004).
[45] A. Fruleux, R. Kawai, and K. Sekimoto, Momentum Transfer in Nonequilibrium Steady States, Phys. Rev. Lett. **108**, 160601 (2012).
[46] Extension to heat baths with time-dependent temperatures is trivial.
[47] N. Shiraishi, K. Saito, and H. Tasaki (in preparation).
[48] $\tilde{R}_{XY}$ may be identified with $\bar{R}_{\bar{Y}\bar{X}}$, where the bars indicate time reversal. Although the latter is standard (see, e.g., Ref. [40]), it is not necessary to consider time reversal explicitly for our purpose.
[49] N. Shiraishi, H. Tajima, and K. Saito (in preparation).
[50] C. Maes, F. Redig, and M. Verschuere, No Current Without Heat, J. Stat. Phys. **106**, 569 (2002).