Algorithmic Construction of Local Hidden Variable Models for Entangled Quantum States

Flavien Hirsch,[1] Marco Túlio Quintino,[1] Tamás Vértesi,[2] Matthew F. Pusey,[3] and Nicolas Brunner[1]
[1]*Département de Physique Théorique, Université de Genéve, 1211 Genéve, Switzerland*
[2]*Institute for Nuclear Research, Hungarian Academy of Sciences, P.O. Box 51, H-4001 Debrecen, Hungary*
[3]*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*
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Constructing local hidden variable (LHV) models for entangled quantum states is a fundamental problem, with implications for the foundations of quantum theory and for quantum information processing. It is, however, a challenging problem, as the model should reproduce quantum predictions for all possible local measurements. Here we present a simple method for building LHV models, applicable to any entangled state and considering continuous sets of measurements. This leads to a sequence of tests which, in the limit, fully captures the set of quantum states admitting a LHV model. Similar methods are developed for local hidden state models. We illustrate the practical relevance of these methods with several examples.

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Distant observers performing well-chosen local measurements on a shared entangled state can establish nonlocal correlations, as witnessed by the violation of a Bell inequality [1,2]. Quantum nonlocality is among the most counter-intuitive features of quantum physics, and is a key resource in quantum information processing [3–5].

Initially believed to be two different facets of the same phenomenon, entanglement and nonlocality are now recognized as fundamentally different. Notably, there exist entangled states which cannot give rise to nonlocality considering arbitrary (nonsequential) measurements. The correlations of such states—thus referred to as "local" entangled states—can be perfectly reproduced using a local hidden variable (LHV) model, i.e., using only shared classical resources. This was first demonstrated by Werner [6], who presented a class of entangled states which admit a LHV model for arbitrary projective measurements. This was later extended to more general positive-operator-valued measure (POVM) [7], and other classes of states [8–11]. In particular, several works [12–15] constructed local hidden state models (LHS), a special class of LHV model in which the hidden variable can be understood as a quantum state, naturally associated to the effect of quantum steering [12]. More generally, characterizing local entangled states would deepen our understanding of the relation between entanglement and nonlocality, as well as allow one to distinguish between useful and useless entangled states for nonlocality-based protocols.

Despite these implications, the problem of constructing local models for entangled states remains challenging, as the model should reproduce the quantum statistics for all possible measurements, i.e., a continuous set. So far, LHV (or LHS) models could be constructed for entangled states featuring a high degree of symmetry [11]. Recently, a sufficient condition for a two-qubit state to admit a LHS was discussed [16]. However, for general states, essentially nothing is known, due to the lack of appropriate techniques for discussing the problem.

Here we present a simple and efficient method for constructing LHV and LHS models, applicable to arbitrary local entangled states and considering continuous sets of measurements. The main idea is to map the problem of finding a local model for an entangled state (a seemingly infinite problem) to a finite (and, hence, tractable) problem, namely, to find out whether the correlations resulting from a finite set of measurements on a different entangled state admit a local decomposition. We can define a sequence of tests for determining whether a given entangled state admits a LHV (or LHS) model, which is shown to converge in the limit, and thus give a full characterization of the set of local entangled states (see Fig. 1). The method can be efficiently implemented, and we construct LHV and LHS models for different classes of entangled states. In particular, we

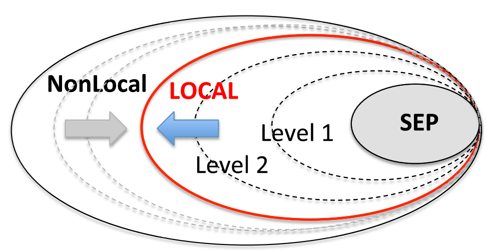

FIG. 1. A method for constructing LHV models for entangled states is discussed. This leads to a sequence of tests, which provide in each level a better approximation of the set of local states (red), a strict superset of the set of separable states (grey region). This is complementary to standard methods, based, e.g., on Bell inequalities, which provide an approximation of the set of local states from outside.

present LHS models for a rank-3 entangled state and for a bound entangled state. We conclude by discussing further possible applications.

Preliminaries.—Consider Alice and Bob sharing an entangled quantum state ρ. Alice performs a set of measurements $\{M_{a|x}\}$ ($M_{a|x} \geq 0$ and $\sum_a M_{a|x} = \mathbb{1}$), and Bob performs measurements $\{M_{b|y}\}$. The resulting statistics is given by

$$p(ab|xy) = \mathrm{Tr}(M_{a|x} \otimes M_{b|y}\rho). \tag{1}$$

The state ρ is said to be local (for $\{M_{a|x}\}$ and $\{M_{b|y}\}$) if distribution (1) admits a Bell local decomposition

$$p(ab|xy) = \int \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda) d\lambda. \tag{2}$$

That is, the quantum statistics can be reproduced using a LHV model consisting of a shared local (hidden) variable λ, distributed with density $\pi(\lambda)$, and local response functions $p_A(a|x,\lambda)$ and $p_B(b|y,\lambda)$. If a decomposition of the form (2) cannot be found, the distribution $p(ab|xy)$ violates (at least) one Bell inequality [2]. In this case, ρ is nonlocal for the sets $\{M_{a|x}\}$ and $\{M_{b|y}\}$.

Another concept of interest is that of a LHS model, associated with quantum steering [12]. Specifically, we say that ρ is "unsteerable" (from Alice to Bob) if

$$p(ab|xy) = \int \pi(\lambda) p_A(a|x,\lambda) \mathrm{Tr}(M_{b|y}\sigma_\lambda) d\lambda. \tag{3}$$

That is, the quantum statistics can be reproduced by a LHS model, where σ_λ denotes the local (hidden) quantum state and $p_A(a|x,\lambda)$ is Alice's response function. If such a decomposition cannot be found, ρ is said to be "steerable" for the set $\{M_{a|x}\}$; note that one would usually consider here a set of measurements $M_{b|y}$ that is tomographically complete, and thus focus the analysis on the set of conditional states of Bob's system

$$\sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\rho), \tag{4}$$

referred to as an assemblage. Note also that any LHS model can be considered as an LHV model. The converse does not necessarily hold, as there exist entangled states which are steerable but nevertheless Bell local [12,15].

The problem of testing the locality or unsteerability of a given entangled state ρ for finite sets of measurements can be solved using existing methods, such as symmetric extensions for quantum states [17], linear and semidefinite programs (SDPs) [2,18,19], and relaxing positivity [20]. Implementable for small number of measurements, these methods become computationally demanding when the number of measurements increases. Nevertheless, they are guaranteed to provide a solution in principle.

The situation is very different when considering continuous sets of measurements, e.g., the set of all projective measurements. Here the methods for finite sets cannot be applied, not even in principle. One must then construct a LHV (or LHS) model explicitly, by exhibiting the distributions $\pi(\lambda)$ and response functions $p_A(a|x,\lambda)$ and $p_B(b|y,\lambda)$. This was achieved for certain classes of entangled states by exploiting their high level of symmetry. However, when considering general states, with less (or no) symmetry, following such an approach is extremely challenging.

In the present Letter, we follow a different path and present a general method for constructing LHV and LHS models for arbitrary states. The method can be efficiently implemented and will be illustrated with examples. Before presenting the main result we start with a simple example, providing the intuition behind our method.

Illustrative example.—Consider the class of Werner states

$$\rho_W(\alpha) = \alpha|\psi^-\rangle\langle\psi^-| + (1-\alpha)\mathbb{1}/4, \tag{5}$$

where $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the singlet state and $\mathbb{1}/4$ is the two-qubit maximally mixed state. In the range $1/3 < \alpha \leq 1/2$, $\rho_W(\alpha)$ is entangled but unsteerable (and, hence, local) for all projective measurements [6]. Werner provided an explicit LHS model by exploiting the high symmetry of the state—$\rho_W(\alpha)$ is invariant under global rotations of the form $U \otimes U$. Here we illustrate the main idea behind our method by rederiving Werner's result, without invoking any symmetry argument.

Consider the set $\mathcal{M}$ of 6 projective qubit measurements with Bloch vectors $\pm\hat{v}_x$ ($x = 1, \ldots, 6$), corresponding to an icosahedron. By performing measurements in $\mathcal{M}$ on the Werner state, Alice prepares for Bob the assemblage

$$\sigma_{\pm|x} = \mathrm{Tr}_A\left(\frac{\mathbb{1} \pm \hat{v}_x \cdot \vec{\sigma}}{2} \otimes \mathbb{1}\rho_W(\alpha)\right), \tag{6}$$

where $\vec{\sigma}$ denotes the vector of Pauli matrices. Using SDP techniques [19], we find that this assemblage admits a LHS model for $\alpha \lesssim 0.54$.

This analysis can be extended to all projective measurements as follows. Consider qubit POVMs given by $M^{\eta}_{\pm|\hat{v}} = [\mathbb{1} \pm \eta(\hat{v}\cdot\vec{\sigma})]/2$ with $0 < \eta \leq 1$. The corresponding Bloch vectors (with direction $\hat{v}$ and norm η) thus form a "shrunken" Bloch sphere of radius η. Choosing $\eta^* = \sqrt{(5+2\sqrt{5})/15} \approx 0.79$, we obtain a sphere that fits inside the icosahedron. Thus, any noisy measurement $M^{\eta^*}_{\pm|\hat{v}}$ can be expressed as a convex combination of measurements in $\mathcal{M}$ [21]. Because the assemblage (6) (resulting from measurements in $\mathcal{M}$) admits a LHS model for $\alpha \lesssim 0.54$, we get that the assemblage resulting from any possible $M^{\eta}_{\pm|\hat{v}}$ with $\eta \leq \eta^*$ also admits a LHS model. Consequently, the statistics of arbitrary (but sufficiently noisy, i.e., $\eta \leq \eta^*$) measurements performed on the Werner state with $\alpha \lesssim 0.54$ can be simulated. Finally, notice that the statistics of noisy measurements on a given Werner state are equivalent to the statistics of projective measurements on a slightly more noisy Werner state

$$\mathrm{Tr}_A[M^{\eta}_{\pm|\hat{v}} \otimes \mathbb{1}\rho_W(\alpha)] = \mathrm{Tr}_A[M^{1}_{\pm|\hat{v}} \otimes \mathbb{1}\rho_W(\eta\alpha)]. \quad (7)$$

Hence, states $\rho_W(\alpha)$ with $\alpha \lesssim 0.54\eta^* \simeq 0.43$ admit a LHS model for all projective measurements. Note that by starting from a polyhedron with more (but nevertheless finitely many) vertices distributed (sufficiently evenly) over the sphere, the above procedure gives a LHS model for Werner states for $\alpha \to 1/2$, thus converging to Werner's model [21]. This is the optimal LHS model, because $\rho_W(\alpha)$ becomes steerable for $\alpha > 1/2$ [12].

Constructing LHS models.—Based on the idea sketched above, we now present a general method for constructing LHS models for continuous sets of measurements, applicable to any local entangled state. Formally, we will make use of the following result.

Lemma 1: Consider a quantum state χ (of dimension $d \times d$), with reduced states $\chi_{A,B} = \mathrm{Tr}_{B,A}(\chi)$, and a finite set of measurements $\{M_{a|x}\}$, such that the assemblage $\sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\chi)$ is unsteerable. Then the state

$$\rho = \eta\chi + (1-\eta)\xi_A \otimes \chi_B, \quad (8)$$

where ξ_A is an arbitrary density matrix (of dimension d), admits a LHS model for a continuous set of measurements $\mathcal{M}$. The parameter η corresponds to the "shrinking factor" of $\mathcal{M}$ with respect to the finite set $\{M_{a|x}\}$ (and given state ξ_A). Specifically, consider the continuous set of (shrunk) measurements

$$M^{\eta}_a = \eta M_a + (1-\eta)\mathrm{Tr}[\xi_A M_a]\mathbb{1} \quad (9)$$

for any $M_a \in \mathcal{M}$. Then η is the largest number such that all M^{η}_a can be written as a convex combination of the elements of $\{M_{a|x}\}$, i.e., $M^{\eta}_a = \sum_x p_x M_{a|x}$ with $\sum p_x = 1$ and $p_x \geq 0$.

Proof.—The proof is based on the following relation:

$$\mathrm{Tr}_A[M^{\eta}_a \otimes \mathbb{1}\chi] = \mathrm{Tr}_A[M_a \otimes \mathbb{1}\rho]. \quad (10)$$

Because $\sigma_{a|x}$ is unsteerable, it follows that there exists a LHS model for χ and all (shrunk) measurements M^{η}_a. From the above equality, it follows that ρ admits a LHS model for the continuous set of measurements $\mathcal{M}$. □

This allows us to get an explicit protocol for determining whether a given state ρ admits a LHS model.

Protocol 1: The problem is to determine if a target state ρ admits a LHS model for a continuous set of measurements $\mathcal{M}$. Following Lemma 1, we start by picking a finite set $\{M_{a|x}\}$ (with shrinking factor η) and a density matrix ξ_A. Next we solve the following SDP problem:

$$\text{find } q^* = \max q$$

such that

$$\mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\chi) = \sum_\lambda \sigma_\lambda D_\lambda(a|x) \quad \forall\ a, x, \sigma_\lambda \geq 0 \quad \forall\ \lambda$$

$$\eta\chi + (1-\eta)\xi_A \otimes \chi_B = q\rho + (1-q)\frac{\mathbb{1}}{d^2}, \quad (11)$$

where the optimization variables are (i) the positive matrices σ_λ and (ii) a $d \times d$ Hermitian matrix χ [22]. This SDP must be performed considering all possible deterministic strategies for Alice $D_\lambda(a|x)$, of which there are $N = (k_A)^{m_A}$ (where m_A denotes the number of measurements of Alice and k_A the number of outcomes); hence $\lambda = 1, \ldots, N$. If the optimization returns a maximum of $q^* = 1$, then ρ admits a LHS model for all measurements in $\mathcal{M}$. If $q^* < 1$, we have shown that $\rho' = q\rho + (1-q)(\mathbb{1}/d^2)$, with $q \leq q^*$, admits a LHS for $\mathcal{M}$.

The performance of the above protocol depends crucially on the choice of the set $\{M_{a|x}\}$. It must be chosen in a rather uniform manner, over the continuous set $\mathcal{M}$, in order to get a shrinking factor that is as large as possible. Also, the ability of the protocol to detect a larger range of unsteerable states will improve when increasing the number of measurements contained in $\{M_{a|x}\}$. Computing the shrinking factor is in general nontrivial, but we give a general procedure in [23].

Based on Protocol 1, we can define a sequence of tests for unsteerability of a given target state ρ. In the first test, we consider a finite set $\{M_{a|x}\}_1$, with shrinking factor η_1 and apply Protocol 1. We thus get a value of q_1^*. If $q_1^* = 1$, we conclude that ρ admits a LHS model. On the other hand, if $q_1^* < 1$, the test is inconclusive, and we must go to the second level. We construct now a new set $\{M_{a|x}\}_2$, which includes all measurements in $\{M_{a|x}\}_1$ and additional ones. By adding sufficiently new measurements, we get a new shrinking factor $\eta_2 > \eta_1$. Applying Protocol 1 again, we may get a value of $q_2^* > q_1^*$ [24]. If $q_2^* = 1$ we stop; otherwise, we proceed to level 3, and so on.

Clearly, in each new test, the set of measurements considered provides a better approximation to $\mathcal{M}$. Moreover, the sequence of tests will in fact converge in the limit. Indeed, consider any state ρ admitting a LHS model. Then, applying the method to ρ, we will be able to show that there is a state ρ', arbitrarily close to ρ, which admits a LHS model. Specifically, for any $\epsilon > 0$, the state $\rho = (1-\epsilon)\rho + \epsilon(\mathbb{1}/d^2)$ will be detected by going to a sufficiently high level in the sequence of tests (see Supplemental Material [23]).

These ideas can be implemented on a standard computer for small sets of measurements $\{M_{a|x}\}$. For larger sets, the implementation becomes demanding. Nevertheless, the method provides a definite answer in principle.

Constructing LHV models.—These ideas can also be adapted to the problem of constructing LHV models.

Lemma 2: Consider a state χ (of dimension $d \times d$), with reduced states $\chi_{A,B} = \mathrm{Tr}_{B,A}(\chi)$, and finite sets of measurements $\{M_{a|x}\}$, $\{M_{b|y}\}$ such that $p(ab|xy) = \mathrm{Tr}(M_{a|x} \otimes M_{b|y}\chi)$ is local. Then the state

$$\rho = \eta\mu\chi + \eta(1-\mu)\chi_A \otimes \xi_B + \mu(1-\eta)\xi_A \otimes \chi_B + (1-\eta)(1-\mu)\xi_A \otimes \xi_B \quad (12)$$

admits a LHV model for the continuous sets of measurements $\mathcal{M}_{\mathcal{A}}$ for Alice and $\mathcal{M}_{\mathcal{B}}$ for Bob. Here ξ_A, ξ_B are arbitrary density matrices (of dimension d), and η, μ denote the shrinking factors of $\mathcal{M}_{\mathcal{A}}$, $\mathcal{M}_{\mathcal{B}}$ with respect to $\{M_{a|x}\}$, $\{M_{b|y}\}$.

The proof is a straightforward extension of that of Lemma 1. We now have the following protocol.

Protocol 2: The problem is whether a target state ρ admits a LHV model for measurements in $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{B}}$. Following Lemma 2, we take finite sets $\{M_{a|x}\}$ and $\{M_{b|y}\}$ (with shrinking factors η and μ) and density matrices ξ_A and ξ_B. Then we solve the following linear problem:

$$\text{find } q^* = \max q$$

such that

$$\mathrm{Tr}(M_{a|x} \otimes M_{b|y}\chi) = \sum_{\lambda} p_\lambda D_\lambda(ab|xy) \quad \forall\ a, b, x, y$$

$$p_\lambda \geq 0 \quad \forall\ \lambda$$

$$q\rho + (1-q)\frac{\mathbb{1}}{d} = \eta\mu\chi + \eta(1-\mu)\chi_A \otimes \xi_B + \mu(1-\eta)\xi_A \otimes \chi_B + (1-\eta)(1-\mu)\xi_A \otimes \xi_B, \tag{13}$$

where the optimization variables are (i) positive coefficients p_λ and (ii) a $d \times d$ Hermitian matrix χ [22]. Given m_A (m_B) measurements with k_A (k_B) outcomes for Alice (Bob), one has $N = (k_A)^{m_A}(k_B)^{m_B}$ local deterministic strategies $D_\lambda(ab|xy)$, and $\lambda = 1, \ldots, N$.

Again, this leads to a sequence of tests. In the first level, consider finite sets $\{M_{a|x}\}_1$ and $\{M_{b|y}\}_1$, with shrinking factors η_1 and μ_1, and apply Protocol 2. If $q_1^* = 1$, we conclude that ρ admits a LHV model. If $q_1^* < 1$, we proceed to the second level. We construct $\{M_{a|x}\}_2$ and $\{M_{b|y}\}_2$, including all measurements used in the first level plus additional ones. Hence we get better shrinking factors $\eta_2 \geq \eta_1$ and $\mu_2 \geq \mu_1$. Applying Protocol 2, we may get a value of $q_2^* > q_1^*$ [24]. If $q_2^* = 1$ we stop, otherwise we go to level 3, and so on.

Here, the sequence will also converge in the limit. Indeed, consider any local state ρ. There is ρ', arbitrarily close to ρ, which the method will show to have a LHV model (see Supplemental Material [23]). Again, implementations on standard computers is possible for small sets $\{M_{a|x}\}$ and $\{M_{b|y}\}$.

Applications.—We now illustrate the practical relevance of the above methods, by constructing LHS and LHV models for classes of entangled states for which previous methods failed. A nontrivial issue is to obtain the shrinking factor for the sets of measurements that are used. For projective qubit measurements, this can be done efficiently by exploiting the Bloch sphere geometry (see Supplemental Material [23]). Hence, we consider entangled states where (at least) one of the systems is a qubit, and focus primarily on projective measurements.

Consider first the class of two-qubit states

$$\rho(\alpha, \theta) = \alpha|\psi_\theta\rangle\langle\psi_\theta| + (1-\alpha)\mathbb{1}_4/4, \tag{14}$$

that is, partially entangled states $|\psi_\theta\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ mixed with white noise. The state is entangled for $\alpha > [1 + 2\sin(2\theta)]^{-1}$, via partial transposition [25,26]. Using Protocols 1 and 2, we find parameter ranges α, θ where the state is unsteerable and local (see Fig. 2); the details are in Supplemental Material [23]. So far, relevant bounds for the locality of the above state were only given for $\theta = \pi/4$, i.e., for Werner states (5). In this case, we obtain an almost optimal LHS model ($\alpha \simeq 0.495$), and a LHV model that improves Werner's one ($\alpha \simeq 0.554$), but that is below the model of Ref. [9] which achieved $\alpha \simeq 0.659$.

We also show that a rank-3 entangled state (i.e., on the boundary of the set of two-qubit quantum states) can admit a LHS model. Specifically, we find that the state $\rho = \sum_{k=1}^3 p_k|\psi_k\rangle\langle\psi_k|$, where $p_1 = 0.4$, $p_2 = 0.05$, and $|\psi_1\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, $|\psi_2\rangle = \sin\theta|00\rangle - \cos\theta|11\rangle$ and $|\psi_3\rangle = |10\rangle$, where $\theta = 10^{-4}\pi$, admits a LHS model.

Next we discuss higher-dimensional states, of the form

$$\rho(\alpha, d) = \alpha|\psi^-\rangle\langle\psi^-| + (1-\alpha)\mathbb{1}_2/2 \otimes \mathbb{1}_d/d, \tag{15}$$

i.e., a two-qubit singlet state $|\psi^-\rangle$ mixed with higher-dimensional noise. The above state is entangled for $\alpha > (1+d)^{-1}$ (via partial transposition). We obtain lower bounds on α (for $d \leq 5$) for the state to admit a LHS model; see Table I.

Moreover, we obtain a LHS model for the bound entangled state of Ref. [27]; see [23]. While these states were conjectured to be local [28], this result represents the first explicit example. Note, however, that this conjecture was recently disproven, as certain bound entangled states can lead to steering [29] and Bell nonlocality [30].

These methods can also be applied to multipartite entangled states. In particular, we could reproduce the result of Ref. [31], constructing a LHV model for a genuine tripartite entangled state.

Finally, we also applied our method considering general POVMs on the two-qubit Werner state (5). In this case, we

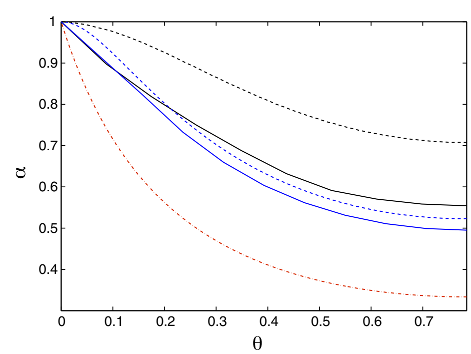

FIG. 2. The state $\rho(\alpha, \theta)$ of Eq. (14) is entangled above the dash-dotted (red) line. Our method guarantees unsteerability below the solid blue line, while the state is steerable above the dashed blue line. Moreover, we can guarantee that the state is local below the solid black line, while it is nonlocal above the dashed black line.

TABLE I. Bounds on the steerability of the states (15), as a function of dimension d. The state is entangled for $\alpha \geq \alpha_{\mathrm{ENT}}$, and unsteerable for $\alpha \leq \alpha_{\mathrm{LHS}}$.

d	2	3	4	5
α_{ENT}	0.33	0.25	0.20	0.16
α_{LHS}	0.49	0.38	0.32	0.28

obtain a LHS model for visibility $\alpha \simeq 0.36 > 1/3$, which shows that the method can be applied in practice for general POVMs [23].

Discussion.—We discussed a procedure for constructing LHS or LHV models, applicable to any local entangled state. The method can be used iteratively, and converges in the limit. We illustrated its practical relevance.

We believe these methods will find further applications, in particular for exploring the relation between entanglement and nonlocality. First, we note that a simplified version of our method was recently used to demonstrate the effect of postquantum steering [32]. More generally, the method can be applied to systems of arbitrary dimension, considering POVMs, and multipartite systems. Here the main technical difficulty consists in obtaining shrinking factors for sets of measurements beyond projective qubit ones.

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Note added.—Recently, we became aware of the related work and complementary results by Cavalcanti and colleagues [33].

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[22] Note that χ does not need to be positive, but should only define a valid assemblage (Protocol 1), or a valid probability distribution (Protocol 2), for the considered sets of measurements $\{M_{a|x}\}$ (and $\{M_{b|y}\}$).
[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.117.190402 for techniques to compute shrinking factors, details about the iterative procedure, proof of the convergence of the method.
[24] Note that value of q_2^* depends on both the shrinking factor η_2 and the visibility (of χ) given the measurements $\{M_{a|x}\}$.

Hence, increasing the shrinking factor does not necessarily imply that $q_2^* > q_1^*$.

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