General Method for Constructing Local Hidden Variable Models for Entangled Quantum States

D. Cavalcanti,[1,*] L. Guerini,[1,2] R. Rabelo,[2] and P. Skrzypczyk[3,1]

[1]ICFO-Institut de Ciencies Fotoniques, Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain

[2]Departamento de Matemática, Universidade Federal de Minas Gerais, Caixa Postal 702, 31270-901 Belo Horizonte, Minas Gerais, Brazil

[3]H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL, United Kingdom

(Received 3 December 2015; published 4 November 2016)

Entanglement allows for the nonlocality of quantum theory, which is the resource behind device-independent quantum information protocols. However, not all entangled quantum states display nonlocality. A central question is to determine the precise relation between entanglement and nonlocality. Here we present the first general test to decide whether a quantum state is local, and show that the test can be implemented by semidefinite programing. This method can be applied to any given state and for the construction of new examples of states with local hidden variable models for both projective and general measurements. As applications, we provide a lower-bound estimate of the fraction of two-qubit local entangled states and present new explicit examples of such states, including those that arise from physical noise models, Bell-diagonal states, and noisy Greenberger-Horne-Zeilinger and W states.

DOI: 10.1103/PhysRevLett.117.190401

Introduction.—Entanglement is one of the defining properties of quantum theory, playing a central role in quantum information science. One of the most astonishing consequences of entanglement is that local measurements on composite quantum systems can produce correlations which are impossible to reproduce by any classical mechanism satisfying natural notions of local causality [1]. Such correlations are the key aspect behind the famous nonlocality of quantum theory, and they are witnessed by the violation of Bell inequalities [2]. Witnessing nonlocality certifies the entanglement of the underlying quantum state in a way that makes no assumptions about the functioning of the apparatuses used, a realization that led to the development of the field of device-independent quantum information.

Remarkably, as first shown by Werner [3], although every entangled state needs a quantum channel to be distributed, there exist entangled quantum states whose correlations can be reproduced classically, because they are incapable of displaying nonlocality. More precisely, Werner presented a highly symmetric family of quantum states whose statistics for all possible projective measurements could be reproduced by an ingenious classical model, referred to as a local-hidden-variable (LHV) model. On the one hand, this shows that the relation between entanglement and nonlocality is not straightforward. On the other hand, it shows that not all entangled states are useful for applications in device-independent information processing. Since Werner's original result there have been a number of subsequent results further elucidating the relation between entanglement and nonlocality in terms of finding LHV models for other families of states [4–9] (for a review, see [10]).

Nevertheless, it remains a difficult task to decide whether a given entangled quantum state is nonlocal or not. This lies in the fact that showing that a given state cannot lead to nonlocal correlations requires showing that the statistics of all measurements can be reproduced by a suitable LHV model. Crucially, all constructions to date make use of the symmetries present in the quantum states under scrutiny, and, consequently, they cannot be readily applied to other quantum states. In fact, apart from the very recent sufficient condition for the special case of two-qubits (and one-sided projective measurements) [11], there is no general criterion to test whether a given quantum state is local.

Our main contribution here is to present sufficient conditions for a general quantum state to admit a LHV model, either for projective von Neumann measurements or for general positive-operator-valued measure (POVM) measurements, that can be tested via semidefinite programming (SDP), an efficient form of convex optimization that can be readily implemented in practice. We also show how this method can be modified to provide a means to randomly generate local quantum states. We show the power of these tests by providing a lower-bound estimate on the volume of the set of entangled two-qubit states that possess LHV models for projective and POVM measurements, and by presenting several examples of new local entangled states, including those that would arise from local amplitude-damping noise, two-qubit Bell diagonal states, and three-qubit noisy Greenberger-Horne-Zeilinger (GHZ) and W states. Our method focuses on a particular class of LHV models, known as local-hidden-state (LHS) models, which naturally arise in the context of quantum steering [7,12], a concept closely related to nonlocality.

An advantage of such models is that they automatically imply a LHV model when one of the parties applies POVM measurements. A disadvantage is that there exist entangled states that admit LHV models but do not admit LHS models [7,13]. As we will see in what follows, even with this restriction, our tests are still strong enough to find models for many interesting states.

Preliminaries.—Let us start by more precisely defining LHV and LHS models. Suppose that Alice and Bob apply local measurements defined by measurement operators $\{M_{a|x}\}$ and $\{M_{b|y}\}$ (x and y label measurement choices and a and b outcomes) on a shared state ρ_{AB}. The set of conditional probability distributions they observe is

$$P(a,b|x,y) = \mathrm{tr}[(M_{a|x}\otimes M_{b|y})\rho_{AB}]. \quad (1)$$

The state ρ_{AB} is said to have a LHV model for these measurements if $P(a,b|x,y)$ can be written as

$$P(a,b|x,y) = \int d\lambda q(\lambda)P(a|x,\lambda)P(b|y,\lambda), \quad (2)$$

where λ is the so-called shared local-hidden variable and $\int d\lambda q(\lambda) = 1$. This decomposition can be thought as coming from the following model: a classical variable λ is randomly chosen according to the probability density $q(\lambda)$ and sent to Alice and Bob. Upon receiving λ and choosing their measurement, Alice and Bob output a and b according to the distributions $P(a|x,\lambda)$ and $P(b|y,\lambda)$, respectively. Of particular interest are the cases when the sets $\{M_{a|x}\}$ and $\{M_{b|y}\}$ contain either all projective measurements or all POVM measurements.

A subclass of LHV models is that of LHS models. Let us consider now that only Alice measures ρ_{AB}. The (unnormalized) state on Bob's side, conditioned on Alice having observed the outcome a of measurement x, is

$$\sigma_{a|x} = \mathrm{tr}_A[(M_{a|x}\otimes \mathbb{1}_B)\rho_{AB}], \quad (3)$$

where $\mathrm{tr}[\sigma_{a|x}] = P(a|x)$ is the probability that Alice obtains the outcome a. If these postmeasurement states can be written in the form

$$\sigma_{a|x} = \int d\lambda q(\lambda)P(a|x,\lambda)\rho_\lambda, \quad (4)$$

where $\rho_\lambda \geq 0$, $\mathrm{tr}\rho_\lambda = 1$ for all λ, and $\int d\lambda q(\lambda) = 1$, ρ_{AB} is said to have a LHS model for these measurements. It can be easily checked that if Bob measures his share of the state (4) with any set of POVM measurements, the probability distributions observed will have the form (2). This means that the existence of a LHS model implies a LHV model for arbitrary POVM measurements on Bob's side. Note that LHS models are not as powerful as general LHV models; there exist states that have a LHV model but no LHS model [7,13].

Main results.—The main insight behind the following theorems is to replace the problem of finding a LHS model for a physical state and an infinite set of measurements with that of finding a model for a nonphysical operator and a finite set of measurements. As we will discuss afterwards, this is a huge simplification that will allow us to test for LHS models via SDP.

Theorem 1: (LHS model for projective measurements) Let $\mathcal{M}$ be a finite collection of projective measurements in $\mathbb{C}^{d_A}$. A state ρ_{AB} acting on $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$ admits a LHS model for *all* projective measurements if there exists a unit-trace operator O_{AB} acting on the same Hilbert space, such that O_{AB} admits a LHS model for the measurements in $\mathcal{M}$, and

$$\rho_{AB} = rO_{AB} + (1-r)\frac{\mathbb{1}_A}{d_A}\otimes O_B, \quad (5)$$

where r is the radius of the insphere [14] of the polytope generated by $\mathcal{M}$.

Here we prove this theorem for the case of $d_A = 2$. A proof for arbitrary d_A can be found in Ref. [15].

Proof.—Let $\mathcal{M}$ define a finite set of measurements for Alice given by measurement operators $\Pi_{a|\hat{u}_x} = [\mathbb{1} + (-1)^a\hat{u}_x\cdot\vec{\sigma}]/2$, where $x = 1,\ldots,m_A$, $a,b\in\{0,1\}$, $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ is the vector of Pauli operators and $\hat{u}$ a three-dimensional unit vector. This measurement set can be chosen arbitrarily, for example, in a regular fashion (along the vertices or faces of a regular solid) or at random. Suppose that these measurements, when applied to a given operator O_{AB}, have a LHS description of the form (4)

$$\mathrm{tr}_A[(\Pi_{a|\hat{u}_x}\otimes\mathbb{1}_B)O_{AB}] = \int d\lambda q(\lambda)P(a|\hat{u}_x,\lambda)\rho_\lambda, \quad \forall\ a,x. \quad (6)$$

Note that any set of measurements that can be performed as a convex combination of the measurements in $\mathcal{M}$ also has an LHS description. This is valid, in particular, for noisy von Neumann measurements whose elements are contained within a shrunken Bloch sphere completely contained inside the convex hull of $\mathcal{M}$ (see Fig. 1). This sphere is given by depolarized measurement operators $\Pi^{(r)}_{a|\hat{u}} = r\Pi_{a|\hat{u}} + (1-r)\mathbb{1}_A/2$, where r is the radius of the insphere of the polytope generated by the convex hull of $\mathcal{M}$.

Finally, notice that

$$\mathrm{tr}_A[(\Pi^{(r)}_{a|\hat{u}}\otimes\mathbb{1}_B)O_{AB}] = \mathrm{tr}_A[(\Pi_{a|\hat{u}}\otimes\mathbb{1}_B)\rho_{AB}], \quad (7)$$

assuming that $\rho_{AB} = rO_{AB} + (1-r)\mathbb{1}_A/2\otimes O_B$. That is, applying noisy measurements on an operator O_{AB} is equivalent, at the level of the states prepared for Bob, to applying noise-free measurements on a noisy version of O_{AB}, denoted here as ρ_{AB}. Therefore, if O_{AB} admits a LHS

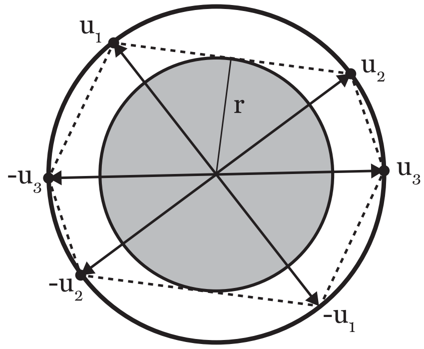

FIG. 1. Diagrammatic representation of the method (restricted to two dimensions for illustrative purposes). The vectors $\pm\hat{u}_x$ are the Bloch vectors corresponding to each of the measurements from the set $\mathcal{M}$. The area enclosed by the dashed lines is the polytope that these measurements form. Any measurement contained inside this polytope can be simulated by appropriately mixing the LHS model that simulates the measurements in $\mathcal{M}$. The shaded circle, of radius r, is the largest circle which is completely contained in the convex hull, and contains all noisy projective measurements $\Pi^{(r)}_{a|\hat{u}}$.

model for the set $\mathcal{M}$, then it also does for the set $\{\Pi^{(r)}_{a|\hat{u}}\}$, which implies that ρ_{AB} admits a LHS model for all projective measurements. □

Note first that the operator O_{AB} need not to be a valid density operator; it can have negative eigenvalues. The requirements on O_{AB} are that it has unit trace, admits a LHS model for the measurements in $\mathcal{M}$, and that it becomes equal to ρ_{AB} when depolarized. Note also that in the case that $\mathcal{M}$ is the (infinite) set of all projective measurements, then this is precisely a brute-force test for the existence of a LHS model. Thus, our method can be seen to provide a sequence of tests (or sufficient conditions), in terms of the set $\mathcal{M}$, for a state to have a LHS model, which in the limit converges to the brute-force test.

To further generalize this result to accommodate general POVMs, we can make use of a result from Ref. [9] that if ρ_{AB} has a LHS model for projective measurements, then the state $\rho'_{AB} = (1/d_A)\rho_{AB} + (1 - 1/d_A)\gamma_A \otimes \rho_B$ has a LHS model for all POVMs, where d_A is the local Hilbert space dimension of Alice and γ_A is an arbitrary state [21]. Combining this result with the above theorem, we obtain the following.

Theorem 2: [LHS for POVM measurements] A state ρ_{AB} acting on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ admits a LHS model for *all* POVMs if there exists an operator O_{AB} that admits a LHS model for $\mathcal{M}$ such that

$$\rho_{AB} = \frac{1}{d_A}\left(rO_{AB} + (1-r)\frac{\mathbb{1}_A}{d_A} \otimes O_B\right) + \frac{d_A - 1}{d_A}\gamma_A \otimes O_B, \tag{8}$$

where γ_A is an arbitrary state.

Note, however, that unlike in the previous case, which became a brute-force search for the existence of a LHS model for all projective measurements in an appropriate limit, this test provides only a sufficient criteria.

Both theorems can be easily adapted to the case of LHV models by applying the same ideas to Bob's system [22]. That is, one can also choose a set of measurements for Bob, compute the corresponding radius r_B, impose that Alice's and Bob's measurements generate local probability distributions, and locally depolarize according to Alice's and Bob's shrinking factors.

SDP formulation.—We now provide explicit SDP formulations of Theorems 1 and 2. We start by choosing a finite set of measurements $\mathcal{M}$ and calculating r, given by the distance between the closest facet of the polytope generated by $\mathcal{M}$ and the origin, which can easily be computed by standard vertex enumeration algorithms [23,24]. Because $\mathcal{M}$ is finite, we can restrict to a finite set of hidden variables [25] when imposing a LHS model for the operator O_{AB}. Without loss of generality we take $\lambda = \lambda_1, \ldots, \lambda_{m_A}$ to be a m_A-length bit string, which specifies a (deterministic) outcome for each of the m_A measurements of Alice: $a = \lambda_x$ when the measurement along direction $\hat{u}_x$ is performed. There are d^{m_A} distinct deterministic specifications. Thus, according to Theorem 1, the following SDP tests for the existence of a LHS model for projective measurements on the state ρ_{AB}:

$$\begin{aligned}
&\text{given } \rho_{AB}, \mathcal{M}, r\\
&\quad\text{find } O_{AB}, \{\rho_\lambda\}_\lambda\\
&\text{such that } \mathrm{tr}_A[(\Pi_{a|\hat{u}_x} \otimes \mathbb{1}_B)O_{AB}] = \sum_\lambda D_\lambda(a|x)\rho_\lambda, \quad \forall\ a, x\\
&\qquad \rho_\lambda \geq 0, \quad \forall\ \lambda\\
&\qquad rO_{AB} + (1-r)\frac{\mathbb{1}_A}{d_A} \otimes O_B = \rho_{AB},
\end{aligned} \tag{9}$$

where $D_\lambda(a|x) = \delta_{a,\lambda_x}$ are deterministic response functions.

Following Theorem 2, we can substitute the last constraint in the above SDP by Eq. (8) to test, with a given γ_A, for the existence of a LHS model for all POVM measurements. These programs can also be adapted to test families of states $\rho(w)$ that depend linearly on a parameter w (e.g., Werner states): instead of running the feasibility problem (9) one can maximize (or minimize) w subject to the same constraints. This finds the value w^* such that for all $w \leq w^*$, states within the family have a LHS model.

Extensions.—The previous methods extend to multipartite states in a rather straightforward way. In particular, extending $B \to B_1 \otimes \cdots \otimes B_k$, we demand in addition that each ρ_λ in (9) (now an operator on $\mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_k}$) is a fully separable state. This is easily seen to provide a LHV model where each Bob can perform arbitrary POVM measurements. Note that although imposing separability is in general difficult, for the case where Bob holds two qubits, imposing positive partial transpose is sufficient. In the case of higher-dimensional systems, although in principle the above method still applies, the number of measurements necessary to generate a polytope with a large insphere grows quickly with d_A (which is necessary to keep the amount of noise low). This implies that the above SDPs become too costly to be used in practice.

Example 1.—As an illustration of the technique, we first investigate Bell diagonal states, given by $\rho_{\text{Bell}} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, where $|\Psi_i\rangle$ are the four Bell states, $p_i \geq 0$, and $\sum_i p_i = 1$ have LHS models. In this case we adapted the SDP (9) to maximize p_1 provided the same constraints. We find $p_1 \approx 0.4454$, and $p_2 = p_3 = p_4 = (1 - p_1)/3$, which is a Werner state, using $\mathcal{M}$ along the vertices of the rhombicuboctahedron, an Archimedean solid with 24 vertices. Notice that the analytical construction of Werner [3] provides a model for $p_1 \leq 1/2$; thus, with 12 measurements our method already recaptures $\approx$89% of LHS Werner states. We also looked at rank-3 Bell diagonal states, by setting $p_4 = 0$, and found the largest p_1 equal to 0.5664, with the same $\mathcal{M}$.

Example 2.—As a more physical example, we consider an initial maximally entangled state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ undergoing independent local amplitude damping given by the evolution $\rho(t) = \sum_{i,j} E_i \otimes E_j \rho(0) E_i^\dagger \otimes E_j^\dagger$, defined by the Kraus operators $E_0 = |0\rangle\langle 0| + \sqrt{1 - e^{-\gamma t}}|1\rangle\langle 1|$ and $E_1 = \sqrt{e^{-\gamma t}}|0\rangle\langle 1|$. This noise model is used to describe spontaneous decay of two-level systems [26] and is particularly relevant for atomic Bell experiments [27,28]. While the evolved state becomes separable only asymptotically (i.e., for $t \to \infty$), we found it to have a LHS model for all $\gamma t \gtrsim -\ln 0.60$.

Example 3.—Finally, we consider noisy 3-qubit GHZ and W states given by $\rho(p) = p|\psi\rangle\langle\psi| + (1 - p)\mathbb{1}/8$, where $|\psi\rangle = |\text{GHZ}\rangle := (|000\rangle + |111\rangle)/\sqrt{2}$ or $|\psi\rangle = |W\rangle := (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. These states are fully separable for $p \leq 0.2$ and $p \leq 0.2096$, respectively. With $\mathcal{M}$ corresponding to the rhombicuboctahedron, we found that these states admit LHS models for projective measurements for $p \leq 0.232$ and $p \leq 0.228$, respectively.

Generating entangled states with LHS models.—A complementary problem to the one of deciding if a target state is local is the generatation of local entangled states. Furthermore, it is also interesting to generate local states that contain as much entanglement as possible. To this end, we make use of the concept of entanglement witnesses.

Entanglement witnesses are Hermitian operators W for which a negative expectation value for the state ρ_{AB}, $\text{tr}[W\rho_{AB}] < 0$, certifies that it is entangled. As shown in Refs. [17,29], if W has additional appropriate structure, the absolute value of this negative expectation value also provides a lower bound on the amount of entanglement of ρ_{AB}, i.e., $E(\rho_{AB}) \geq -\text{tr}[W\rho_{AB}]$. Finally, such entanglement witnesses themselves can be obtained through simple SDPs, where by imposing the different constraints on W we obtain estimators for different entanglement quantifiers $E(\rho_{AB})$ [15,17,29].

We now propose a method to generate entangled states with LHS models and high entanglement. We start with a given witness W (obtained via SDP). As before, we choose a set of measurements $\mathcal{M}$ and compute the radius of the insphere r. We now search for the state which maximally violates the witness and has a LHS model for projective measurements by solving the following SDP:

$$\begin{aligned} &\min_{O_{AB},\rho_\lambda} \ \text{tr}\left[W\left(rO_{AB} + (1-r)\frac{\mathbb{1}_A}{d_A} \otimes O_B\right)\right] \\ &\text{such that } \text{tr}_A[(\Pi_{a|\hat{u}_x} \otimes \mathbb{1}_B)O_{AB}] = \sum_\lambda D_\lambda(a|x)\rho_\lambda, \quad \forall\ a, x \\ &\qquad \rho_\lambda \geq 0, \quad \forall\ \lambda, \qquad \text{tr}[O_{AB}] = 1, \\ &\qquad rO_{AB} + (1-r)\frac{\mathbb{1}_A}{d_A} \otimes O_B \geq 0. \end{aligned} \tag{10}$$

If the solution of this SDP is negative, then the minimizing operator $\rho^*_{AB} = rO^*_{AB} + (1-r)(\mathbb{1}_A/d_A) \otimes O^*_B$ is an entangled state which has a LHS model: entanglement is guaranteed by the violation of the witness and the fact it has a LHS model is imposed by the constraints of the SDP.

Once we find an example of a LHS entangled state ρ^*_{AB}, we can iterate this procedure and find new examples with more entanglement: we find the entanglement witness W^* that is optimal for the state ρ^*_{AB} and use W^* in the SDP (10) to find a new state ρ^{**}_{AB}, which is generally more entangled according to the chosen quantifier. This procedure can then be iterated until it converges [30]. Note that different quantifiers of entanglement have different properties, and, thus, exploring a number of different quantifiers can provide LHS states with different properties. Finally, as before, we can adapt (10) according to Eq. (8) to find examples of entangled states with LHS models for all POVM measurements.

Using this method, we generated a large list of bipartite entangled states that have LHS models for projective and POVM measurements [31]. In Ref. [15] we analyze these examples in terms of their entanglement content and other relevant parameters. Finally, by using entanglement witnesses that detect genuine multipartite entanglement [20], we were also able to obtain new examples of genuine tripartite entangled 3-qubit states with LHS models for projective measurements. To the best of our knowledge, only two examples were previously known [32,33].

Estimating the volume of LHS states.—The previous programs can be directly applied to provide a lower bound on the relative volume of the set of entangled states that admit LHS models. We uniformly sampled 2×10^4 2-qubit states according to the Hilbert-Schmidt and Bures measures, for which we obtained $\approx$23% and $\approx$7% separable states, respectively, in good accordance with the values of

24.2% and 7.3% obtained from geometrical arguments [34]. We then applied the above SDPs to estimate how many of the entangled states admit LHS models [35]. With the measurements $\mathcal{M}$ chosen to be the vertices of the icosahedron ($r \approx 0.79$), we obtain that $\gtrsim 25\%$ of the entangled states sampled according to the Hilbert-Schmidt measure admit LHS models, while $\gtrsim 7\%$ admit LHS models using the Bures measure. We were not able to obtain any entangled state admitting LHS models for POVMs by applying the same technique with measurements given by the icosahedron. A better estimation of the volume of the set of local states could be obtained, both for projective measurements and POVMs, by considering more measurements in the set $\mathcal{M}$.

Discussion.—Not all entangled quantum states exhibit nonlocality—the strongest signature of their inseparability. Understanding the relation between nonlocality and entanglement is an important problem, and it has been notoriously difficult to find general-purpose methods for determining which entangled states are local. In this Letter we have presented a criterion for a state to admit a LHS model for projective or general measurements. Although LHS models are only a subset of general LHV models, we have demonstrated the power of our criteria by finding new physical examples of multipartite entangled states that are local.

We also showed how our work naturally provides a method to generate examples of entangled local states; we used it to give the first estimate of the relative volume of the set of entangled 2-qubit states that admit LHS models, showing that a significant fraction of them in fact do so. As a consequence, our results provide a lower bound for the fraction of states that are useless as resources for any device-independent quantum information processing task.

Our method works particularly well for projective measurements on 2- or multiqubit states, becoming equal to a brute-force search in the appropriate limit. In [22] it was further shown how a variant of the above methods provides necessary and sufficient criteria for general POVM measurements in the limit. It would be interesting, in future work, to build upon the general methods presented here to provide practical tests for higher-dimensional and multipartite systems.

We thank M. T. Quintino, F. Hirsch, and N. Brunner for discussions. We especially thank F. Hirsch for pointing out the possibility of using nonphysical operators in the methods developed. L. G. is thankful to the Lepant Institute of Quantum Information and Decoherence for the warm hospitality. This work was supported by the Beatriu de Pinós fellowship (Grant No. BP-DGR 2013), the Spanish MINECO (Severo Ochoa Grant No. SEV-2015-0522 and FOQUS Grant No. FIS2013-46768-P), the Generalitat de Catalunya (Grants No. SGR875 and No. SGR875), the ERC AdG NLST, the ERC CoG QITBOX, CNPq, CAPES, and the Brazilian program Science Without Borders.

Note added.—We recently became aware of a complementary work by F. Hirsch *et al.* [22].

*daniel.cavalcanti@icfo.es

[1] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[2] J. S. Bell, Physics **1**, 195 (1964).
[3] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[4] J. Barrett, Phys. Rev. A **65**, 042302 (2002).
[5] A. Acín, N. Gisin, and B. Toner, Phys. Rev. A **73**, 062105 (2006).
[6] M. L. Almeida, P. Pironio, J. Barrett, G. Tóth, and A. Acín, Phys. Rev. Lett. **99**, 040403 (2007).
[7] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).
[8] S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, J. Opt. Soc. Am. B **32**, A40 (2015).
[9] F. Hirsch, M. T. Quintino, J. Bowles, and N. Brunner, Phys. Rev. Lett. **111**, 160402 (2013).
[10] R. Augusiak, M. Demianowicz, and A. Acín, J. Phys. A **47**, 424002 (2014).
[11] J. Bowles, F. Hirsch, M. T. Quintino, and N. Brunner, Phys. Rev. A **93**, 022121 (2016).
[12] E. Schrödinger, Math. Proc. Cambridge Philos. Soc. **32**, 446 (1936).
[13] M. T. Quintino, T. Vértesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, and N. Brunner, Phys. Rev. A **92**, 032107 (2015).
[14] The insphere of a polytope is the largest centered sphere contained in it.
[15] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.117.190401, which includes Refs. [16–20], for a proof of Theorem 1 for arbitrary d_A and an analysis of the entangled LHS states obtained via SDP (10) in terms of their entanglement content and other relevant parameters.
[16] R. A. Bertlmann and P. Krammer, J. Phys. A **41**, 235303 (2008).
[17] F. G. S. L. Brandão, Phys. Rev. A **72**, 022310 (2005).
[18] A. C. Doherty, P. A. Parrilo, and F. M Spedalieri, Phys. Rev. Lett. **88**, 187904 (2002).
[19] A. C. Doherty, P. A. Parrilo, and F. M Spedalieri, Phys. Rev. A **69**, 022308 (2004).
[20] B. Jungnitsch, T. Moroder, and O. Gühne, Phys. Rev. Lett. **106**, 190502 (2011).
[21] In fact, Ref. [9] presents a more general construction that works for LHV models. We only need the weaker result stated, which is implicit in their construction.
[22] F. Hirsch, M. T. Quintino, T. Vértesi, M. F. Pusey, and N. Brunner, following Letter, Phys. Rev. Lett. **117**, 190402 (2016).
[23] T. Christof and A. Loebel, Porta: Polyhedron Representation Transformation Algorithm, http://porta.zib.de, 1997.
[24] D. Avis, Lrs, Version 6.1, http://cgm.cs.mcgill.ca/avis/C/lrs.html, 2015.
[25] M. F. Pusey, Phys. Rev. A **88**, 032313 (2013).

[26] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*, 10th ed. (Cambridge University Press, New York, 2011).

[27] J. Hofmann, M. Krug, N. Ortegel, L. Gérard, M. Weber, W. Rosenfeld, and H. Weinfurter, Science **337**, 72 (2012).

[28] S. Pironio, A. Acín, S. Massar, A. B. de La Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Nature (London) **464**, 1021 (2010).

[29] J. Eisert, F. G. S. L. Brandão, and K. M. R. Audenaert, New J. Phys. **9**, 46 (2007).

[30] Typically, this is convergence up to numerical precision. Note also that this can occur at a local maximum.

[31] The data will be maintained at https://git.io/vV7Bu.

[32] G. Tóth and A. Acín, Phys. Rev. A **74**, 030306 (2006).

[33] J. Bowles, J. Francfort, M. Fillettaz, F. Hirsch, and N. Brunner, Phys. Rev. Lett. **116**, 130401 (2016).

[34] P. B. Slater, Phys. Rev. A **75**, 032326 (2007).

[35] All the SDPs implemented in this work used MATLAB and the packages CVX, [36,37] and QETLAB [38].

[36] M. Grant and S. Boyd, CVX: Matlab Software for Disciplined Convex Programming, Version 2.1, http://cvxr.com/cvx, 2014.

[37] M. Grant and S. Boyd, in *Recent Advances in Learning and Control*, Lecture Notes in Control and Information Sciences, edited by V. Blondel, S. Boyd, and H. Kimura (Springer-Verlag Limited, Berlin, 2008), pp. 95–110.

[38] N. Johnston, A. Cosentino, and V. Russo, QETLAB: A MATLAB Toolbox for Quantum Entanglement, Version 0.8, 2016, http://doi.org/10.5281/zenodo.44637.