No-Enclave Percolation Corresponds to Holes in the Cluster Backbone

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(Received 12 May 2016; revised manuscript received 27 September 2016; published 27 October 2016)

The no-enclave percolation (NEP) model introduced recently by Sheinman *et al.* can be mapped to a problem of holes within a standard percolation backbone, and numerical measurements of such holes give the same size-distribution exponent $\tau = 1.82(1)$ as found for the NEP model. An argument is given that $\tau = 1 + d_B/2 \approx 1.822$ for backbone holes, where d_B is the backbone dimension. On the other hand, a model of simple holes within a percolation cluster yields $\tau = 1 + d_f/2 = 187/96 \approx 1.948$, where d_f is the fractal dimension of the cluster, and this value is consistent with the experimental results of gel collapse of Sheinman *et al.*, which give $\tau = 1.91(6)$. This suggests that the gel clusters are of the universality class of percolation cluster holes. Both models give a discontinuous maximum hole size at p_c, signifying explosive percolation behavior.

DOI: 10.1103/PhysRevLett.117.185701

Recently, Sheinman *et al.* [1] introduced the no-enclave percolation (NEP) model to explain the motor-driven collapse of a model cytoskeletal system studied by Alvarado *et al.* [2]. In the cytoskeletal system, myosin motors act on actin fibers, which contract the system and at the same time reduce the connectivity, driving the system to a critical percolationlike transition. NEP models this by considering random percolation in which clusters collapse, producing solid clusters that represent the gelled regions of the cytoskeletal system. This model has received a great deal of attention [3–13].

In Ref. [1], the authors consider a region surrounded by sites of other clusters and the NEP clusters are composed of all sites within the region. Reversing occupied and vacant sites or bonds, the problem can be thought of as finding the distribution of hole sizes within a single large cluster. The NEP model was found to have a size-distribution exponent $\tau = 1.82(1)$, which is less than the conventional lower bound value 2, and represents a distinct universality class from standard random percolation where $\tau = 187/91$. However, the holes studied in Ref. [1] are not simple percolation holes, but, as we shall argue, holes within the surrounding backbone. In this Letter we derive universal expressions for the scaling behavior for both holes in the backbone and for simple holes in the percolation cluster. We find that the latter gives a scaling consistent with the experimental results of Ref. [1], suggesting this is the appropriate model (i.e., simple holes) to describe the gel system.

After this work was complete, a comment [12] and reply [13] were published discussing the admissibility of having τ less than 2. Here, we show that such exponents are entirely possible for systems with a cutoff in the maximum cluster or hole size, in agreement with Ref. [13] and also with several previous discussions of related systems [14–18].

In Ref. [1] (Supplementary Material), Sheinman *et al.* consider clusters created by bond percolation on the triangular lattice (BTR) at the critical threshold $p_c = 2\sin\pi/18$ [19]. They identify the external boundary by the sites of bordering clusters, and then combine every site within the boundary into the NEP cluster. Because they use the external sites to define the boundary, they effectively close the openings of separation one lattice spacing in their clusters, as shown in Fig. 1. This means that the boundary of the cluster becomes the external accessible hull [20], which has a fractal dimension of $4/3$, rather than the entire hull, which has a fractal dimension of $7/4$. All remaining dual-lattice bonds [dashed lines in Fig. 1(b)] are biconnected, and accordingly they form the backbone of the dual-lattice cluster [21]. Thus, NEP clusters are equivalent

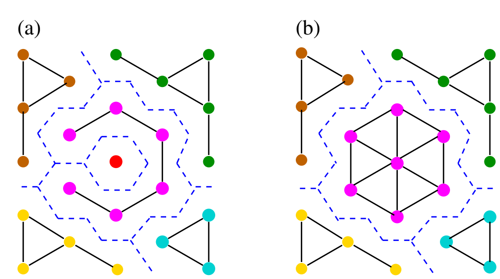

FIG. 1. Diagram illustrating the NEP procedure of Ref. [1]. (a) b-TR clusters (solid lines) and the dual honeycomb lattice (dashed lines). (b) All components within a boundary of neighboring cluster sites become a NEP cluster, and the remaining dashed bonds give the dual-lattice backbone.

to holes within the backbone of the largest dual-lattice cluster.

Sheinman *et al.* find that their model yields a size-distribution exponent of $\tau = 1.82(1)$, while their experiments yield $\tau = 1.91(6)$. Thus, their model supports the experimental result that $\tau < 2$, although their value of τ is somewhat low compared to measurements.

In this Letter we consider simple cluster holes as well as backbone holes. We carry out extensive simulations of both for site percolation on the square (SSQ) and triangular lattices (STR), and bond percolation on the square lattice (BSQ). We consider a square $L \times L$ system for the square lattice, and a rhombic $L \times L$ system for the triangular lattice, with $L = 8, 16, \ldots, 16384$, and periodic boundary conditions. For the cluster holes, we occupy the system with sites or bonds with probability p, identify the largest black cluster, remove all remaining sites or bonds, then identify the white clusters by standard cluster search algorithms. For the backbone holes (bond percolation only), we identify the backbone before finding the holes within it. For bond percolation, the holes (white clusters) are constructed on the dual lattice, while for site percolation, they are found on the matching lattice [19]. We carried out from 4×10^5 ($L = 16384$) to 5×10^6 ($L = 8$) runs.

We have not found previous work directly studying holes in percolation clusters. Related problems that have been studied include lacunarity [22,23] and holes in directed percolation [17,18].

The results for the size distribution are shown in Fig. 2. For the backbone holes, we find $\tau_b = 1.82(1)$, which agrees with the NEP simulations of Sheinman *et al.* For cluster holes, we find a value of $\tau_h = 1.949(3)$, which is consistent with the experimental results of Sheinman *et al.* Thus, we argue that their experimental system is more accurately modeled by cluster holes than by backbone holes.

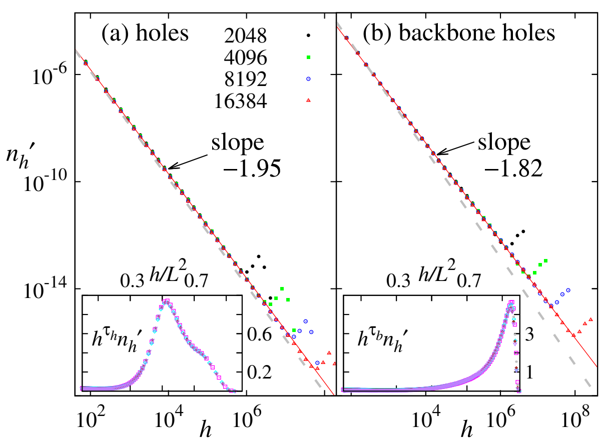

FIG. 2. Scaled number of cluster holes $n_h' = n_h(L, p_c)L^{d-D}$ (a) or backbone holes (b) of size h as a function of h, where $D = d_f$ for cluster holes and $D = d_B$ for backbone holes, showing, respectively, $\tau_h = 1.949(3)$ consistent with the prediction 187/96 of Eq. (3) and $\tau_b = 1.82(1)$ consistent with the prediction 1.822 of Eq. (6), for systems of different size L (see legend). The inset shows $h^\tau n_h'$ vs h/L^2, where τ is τ_h (a) or τ_b (b), demonstrating the accumulation in the distribution due to the largest holes. For (a) we used STR and for (b) BSQ. The dashed gray lines have a slope of -2 for comparison.

To derive a scaling relation for the holes within the percolation cluster, consider the hole-size distribution $n_h(L, p)$, which equals the number of holes per lattice site containing h vacant sites, in the largest cluster of an $L \times L$ system at bond occupation p. At p_c, the total number of cluster holes N is proportional to the number of sites s in the cluster, which scales as $\sim L^{d_f}$ where $d_f = 91/48$ is the cluster fractal dimension [24]. This is demonstrated in Fig. 3(a) for STR, and we have also verified that proportionality for BSQ.

Then, it follows that $n_h(L, p)$ scales as

$$n_h(L, p_c) \sim L^{d_f - d} h^{-\tau_h} f(h/L^d), \tag{1}$$

where the scaling function $f(z)$ cuts off when h is of the order of the size of the largest hole, which is proportional to L^d, where $d = 2$ is the dimensionality. The term at the beginning $L^{d_f - d}$ differs from the usual scaling of cluster size and causes the density of holes of a given size h to go to zero as the system size increases. The form of Eq. (1) is identical to that proposed in Ref. [13], noting that here n_h represents the number of hole sites per lattice site, while in Ref. [13] n_s represents the total number of hole sites, so they differ by a factor of $L^d = M$. Here, we go on to find an exact expression for τ_h, which was not found in Ref. [13].

Next, we need to make some considerations for the case that $\tau_h < 2$, similar to the discussions of Huber *et al.*

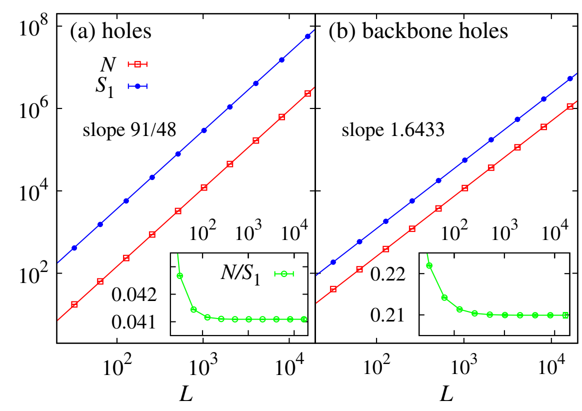

FIG. 3. Log-log plot of the total number N of cluster holes (a) or backbone holes (b) in the largest cluster, and also the size S_1 of the largest cluster (a) or backbone (b), as a function of the size of the system L, with slopes equal to $d_f = 91/48$ (a) and $d_B = 1.6433$ (b), showing that the number of cluster or backbone holes is proportional to the mass of the cluster (a) or backbone (b). For (a) we use STR and for (b) we use BSQ. The inset shows N/S_1, which rapidly goes to a constant ≈ 0.0410907 for cluster holes and ≈ 0.209908 for backbone holes for large L.

[17,18]. Normally, τ has to be greater than 2 so that the size distribution is normalizable: $\sum_{s=1}^{\infty} s n_s \sim \sum^{\infty} s^{1-\tau} < \infty$. However, for systems such as these where there is an upper cutoff to the sum and an asymptotically vanishing number of holes per lattice site, τ can be less than 2. Say $n_h(L, p_c) \sim A h^{-\tau_h}$ with a cutoff $h_{\max}$, then

$$\sum_{h=1}^{h_{\max}} h n_h(L, p_c) \sim \sum_{h=1}^{h_{\max}} A h^{1-\tau_h} + \frac{h_{\max}}{L^d} \sim \frac{A h_{\max}^{2-\tau_h}}{2-\tau_h} + \frac{h_{\max}}{L^d}. \tag{2}$$

This can remain finite as $h_{\max} \to \infty$ if $A \to 0$ with $A \sim h_{\max}^{\tau_h - 2} \sim L^{d(\tau_h - 2)}$, for $\tau_h < 2$. We split off $h_{\max}/L^d$, to allow for a finite fraction of the system to be occupied by the largest hole. Comparing the scaling of A with that of the leading term of Eq. (1), we have $d(\tau_h - 2) = d_f - d$, or

$$\tau_h = 1 + \frac{d_f}{d} = \frac{187}{96} \approx 1.948. \tag{3}$$

This value agrees with our simulation results $\tau_h = 1.949(3)$ for holes, and is close to the experimental value $\tau = 1.91(6)$ found in Ref. [1].

Note that, while Eq. (1) can apparently be written in the traditional form $n_h \sim h^{-\tau'} g(h/L^d)$ with $\tau' = 2$ and $g(x) = x^{\tau' - \tau_h} f(x) = x^{5/96} f(x)$ [12], the scaling function $g(x)$ goes to zero algebraically as $x \to 0$ rather than to a constant as in typical scaling. The τ_h of Eq. (3) is the proper τ exponent to describe the size distribution of holes.

The scaling relation (3) above is in the form of Mandelbrot's hyperscaling relation [25]

$$\tau = 1 + d_{\text{all}}/d_{\text{object}}, \tag{4}$$

where the objects combine together to make the "all." In normal two-dimensional percolation where the ensemble of fractal clusters fills the nonfractal space, we have $d_{\text{all}} = 2$ and $d_{\text{object}} = 91/48$, yielding $\tau = 187/91 \approx 2.055$. For cluster holes, d_{all} corresponds to the fractal dimension d_f of the largest cluster, which is also the union of the hulls of all the holes, and $d_{\text{object}} = 2$. Equation (3) also follows from the "triplex" formula of Refs. [17,18]

$$\tau = 1 + (D_{\text{num}} - D_{\text{tot}} + D)/D, \tag{5}$$

where $D_{\text{num}} = d_f$ is the fractal dimension of the point set of the clusters, $D = 2$ is the dimension of the objects, and $D_{\text{tot}} = 2$ is the fractal dimension of the union of objects.

The vanishing behavior of percolation cluster holes with $A \to 0$ is an example of the "volatile" fractality discussed by Herrmann and Stanley for backbone blobs, and our scaling (1) is similar in form to Eq. (2) of Ref. [14]. The holes are indeed volatile in that they disappear as they are subsumed into larger holes as L is increased.

In the case of the backbone holes, where the backbone has dimension $d_B = 1.64336(10)$ [14,26,27], a similar argument gives

$$\tau_b = 1 + \frac{d_B}{2} = 1.82168(5), \tag{6}$$

which agrees with our measurements $\tau_b = 1.82(1)$, and also with the numerical results of Sheinman *et al.*, supporting the idea that their NEP clusters are effectively dual-lattice backbone holes. The formula $(\tau_b - 1)(\tau - 1) = 1$ is another instance of the duality noted in Ref. [18], which links τ_b to the τ for backbone blobs [28].

We measured the size of the largest hole in a periodic system of size $L \times L$, where the largest hole can be wrapping, nonwrapping, or a cross-configuration wrapping in both directions. For STR we found the average size of the largest hole to be exactly half the size of the system, while for the other systems we studied it approaches $1/2$ as $L \to \infty$, as shown in Fig. 4. To explain this, observe that the largest white cluster (hole) and the black cluster with every hole but the largest one filled in are identical on the triangular lattice where $p_c = 1/2$, switching black and white, so on the average each is $1/2$ the lattice. For other lattices where there is not perfect self-duality or self-matching, this result holds asymptotically for large L.

Let us consider the "solid" size S_{1s} of the largest black cluster with every hole but the largest filled in. When $p < p_c$, S_{1s} will be $O(1)$, while at p_c it will be $O(L^2)$. For instance, for STR, for $L \to \infty$ one has $S_{1s}/L^2 = 0$ for

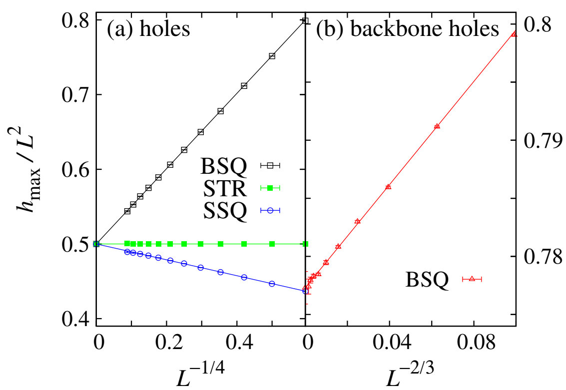

FIG. 4. (a) The largest cluster hole size $h_{\max}$ divided by the number of sites L^2, as a function of $L^{-1/4}$ for various systems, showing that this quantity is exactly $1/2$ for STR, and approaches $1/2$ as $L \to \infty$ for SSQ and BSQ. For BSQ we measure the size of the hole as the number of sites it contains; if we measure the holes by dual-lattice bonds, then $h_{\max}/L^2 = 1/2$ for all L by duality. (b) The fraction of the system filled by the largest backbone hole as a function of $L^{-2/3}$ for BSQ. Here, the fraction approaches $0.7772(4)$ as $L \to \infty$. The scaling in both cases is $L^{d_H - d}$, where the hull dimension $d_H = 7/4$ (holes) and $4/3$ (backbones), because for asymmetric systems the hull contributes to the size of the largest hole $h_{\max}$ proportionally to its length.

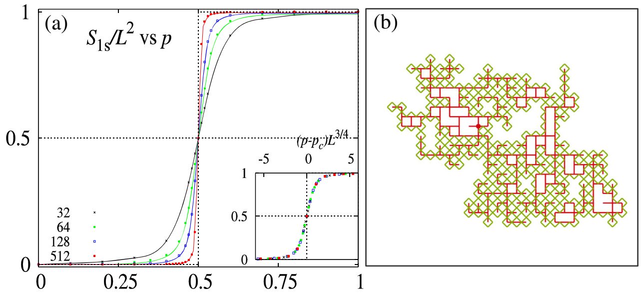

FIG. 5. (a) The solid size S_{1s}/L^2 of the largest cluster on the triangular lattice versus p for different L (see legend). The inset shows the scaling plot S_{1s}/L^2 versus $(p - p_c)L^{1/\nu}$ with $\nu = 4/3$. When $L \to \infty$ this gives a step function indicative of explosive behavior. (b) An example of a hull walk around a percolation cluster of 268 occupied bonds and 239 sites. The 760-step walk (diagonal line segments) connects points on the medial lattice and turns right when encountering an occupied bond and left when encountering a vacant bond, yielding the enclosed area or S_{1s} of 144 square lattice spacings. This is also the size of the hole in the larger dual-lattice cluster.

$p < p_c$, 1/2 for $p = p_c$, and 1 for $p > p_c$, as shown by Fig. 5(a). This implies a discontinuity in the ratio S_{1s}/L^2 [1], which is a signature of an "explosive" phenomenon in percolation, which was first hypothesized in Ref. [29] and shown to in fact occur in later models such as those of Refs. [30–32]. Likewise, the size of the largest hole, $h_{\max}/L^2$, which is equivalent to the black solid cluster switching colors, steps from 1 to 0 at p_c. While the hole problem is based upon standard percolation, they are not equivalent since creating the holes requires a global modification to the system (identifying the holes and removing all internal sites, and in the NEP case, also identifying the backbone). It is an explosive feature contained in standard percolation.

We can also envision creating the solid clusters by an epidemic or Leath kind of growth process, starting with a seed and adding neighbors with a probability p, otherwise blocking neighbors with probability $1 - p$. We could then fill in all the internal blocked sites and holes of the cluster, making it solid. In Fig. 5(b) we show a walk around such a cluster to find the enclosed area or size of the cluster. This enclosed area however is not necessarily a hole within the largest dual-lattice cluster, so the Leath method is not a way to find the hole-size distribution studied here. Also, the hole-area distribution is not the same as the enclosed-area distribution studied in Ref. [33].

Finally, we consider another place that holes appear: when sections of a boundary are broken off. Consider the largest white hole (site percolation) and add a layer of blocked sites at the internal boundary, as shown in Fig. 6. This breaks up the white cluster into many additional holes. We find that the total number of holes is proportional to the hull length L^{d_H}, and the corresponding $\tau_{h'}$ is predicted by Eq. (4) with $d_{\rm all} = d_H = 7/4$:

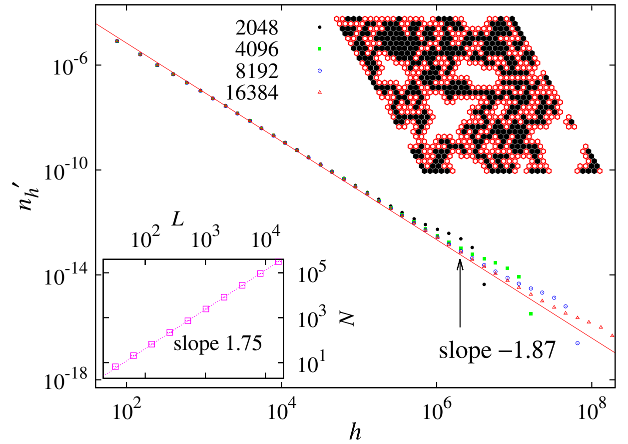

FIG. 6. Main plot: the rescaled number of holes $n'_h(L, p_c) = n_h L^{d-d_H}$ along the boundary of the largest white cluster broken up by an internal layer of blocked white sites (red hexagons in the inset figure) versus h, for systems of different size L (legend). The slope agrees with the prediction 15/8 of Ref. (7). Inset plot: the total number of boundary holes N as a function of L on a log-log plot, with a slope of $d_H = 7/4$.

$$\tau_{h'} = 1 + \frac{d_H}{2} = \frac{15}{8} = 1.875 \qquad (7)$$

as shown in Fig. 6. Again, $\tau < 2$.

The appearance of large holes is very much a phenomenon of a planar lattice, and these considerations do not apply in higher dimensions. On the other hand, they should apply to any critical two-dimensional system, such as the random cluster or q-state Potts model [34,35], where $\tau = 1 + d_f/2 = 2 - (6-g)(g-2)/(16g) < 2$ for $0 < q \le 4$, where $g = 4 - 2/\pi \cos^{-1}(q/2 - 1)$, $2 \le g \le 4$ [36,37]. Note that one can also study the backbone holes of this model. This is an interesting problem for future research.

Thus, we have shown that the universality class of the NEP model is that of holes in backbone percolation, which has an exponent of $\tau_b = 1 + d_B/2 \approx 1.822$ of Eq. (6). If we interpret the clusters instead to be simple cluster holes, then we would have another universality class with $\tau_h = 1 + d_f/2 = 187/96$ of Eq. (3). The general expression for τ, Eq. (4), is further supported by the case of holes cut from the hull, which gives $\tau_{h'} = 1 + d_H/2 = 15/8$ of Eq. (7). Likewise, one can conceive of removing the outer layer of the largest backbone hole, which would yield another case with $\tau_{b'} = 1 + d_{H_b}/2 = 5/3$. We are not aware if any of these universality classes, all with $\tau < 2$ and all associated with explosive percolation behavior, have been discussed before.

The scaling behavior of percolation holes appears to model the properties of the experimental gel clusters seen in Ref. [1]. This suggests that in the experiment the steric effect causes clusters fully surrounded by others to be absorbed, and a nearest-neighbor separation can disable this effect.

Y. D. and H. H. thank the National Natural Science Foundation of China for support under Grant No. 11275185. Y. D. acknowledges the Ministry of Education (China) for the Fundamental Research Funds for the Central Universities under Grant No. 2340000034. R. M. Z. thanks the University of Science and Technology of China for its hospitality, where this Letter was written. The authors acknowledge helpful discussions with G. Huber, G. Pruessner, and M. Weigel.

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