Two-Jet Rate in e^+e^- at Next-to-Next-to-Leading-Logarithmic Order

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We present the first next-to-next-to-leading-logarithmic resummation for the two-jet rate in e^+e^- annihilation in the Durham and Cambridge algorithms. The results are obtained by extending the ARES method to observables involving any global, recursively infrared and collinear safe jet algorithm in e^+e^- collisions. As opposed to other methods, this approach does not require a factorization theorem for the observables. We present predictions matched to next-to-next-to-leading order and a comparison to LEP data.

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Jet rates and event shapes in electron-positron collisions played a crucial role in establishing QCD as the theory of strong interactions; see, e.g., [1,2]. Nowadays, these observables are still among the most precise tools used for accurate extractions of the main parameter of the theory, the strong coupling constant α_s. These fits rely on comparing precise measurements of distributions to accurate perturbative predictions supplemented with a modeling of nonperturbative effects. Fixed-order predictions up to next-to-next-to-leading order (NNLO) for $e^+e^- \to 3$ jets are available [3–7]. However, they are not reliable in the two-jet limit, where the cross section is dominated by multiple soft-collinear emissions. In this region, terms as large as $\mathcal{O}(\alpha_s^n L^{2n})$ [where $L = \ln(1/v)$] appear to all orders in the integrated distributions of an observable v that vanishes in the two-jet limit. These large logarithms invalidate fixed-order expansions in the coupling constant, and reliable predictions can be obtained only by resumming the logarithmically enhanced terms to all orders in α_s. Double logarithmic terms $\mathcal{O}(\alpha_s^n L^{2n})$ are known to exponentiate (see, e.g., Ref. [8]) and give rise to a well-known Sudakov peak in differential distributions, where most of the data lie. For exponentiating observables, it is customary to define leading logarithms (LL) as terms of the form $\alpha_s^n L^{n+1}$ for the logarithm of the cross section, next-to-leading logarithms (NLL) as $\alpha_s^n L^n$, and next-to-next-to-leading logarithms (NNLL) as $\alpha_s^n L^{n-1}$. For several e^+e^- observables, NNLL predictions (in some cases even beyond) are nowadays available [9–17]. On the contrary, two-jet rates have been described only at NLL accuracy so far [18]. The lack of precise theory predictions close to the peak of the distribution limits the fit range that can be used to extract α_s and results in larger perturbative uncertainties in the latter. Among the existing fits, extractions from the thrust and C parameter [19–21] that rely on the most precise theory predictions show a tension with the world average determination of the coupling [22]. One of the issues is that at LEP energies nonperturbative corrections are sizable, and the separation between perturbative and nonperturbative effects is subtle. Fits of α_s from the two-jet rate have been so far performed based on pure NNLO [23] or NNLO + NLL [24–26] results. Owing to the different sensitivity to nonperturbative effects, an extraction of α_s from NNLO + NNLL predictions for the two-jet rate and from the vast amount of high-precision LEP data [27–31] can shed light on this disturbing tension. The aim of this Letter is to present the first NNLL + NNLO results for this observable.

The two-jet rate is defined through a clustering algorithm based on an ordering v_{ij} and a test variable y_{ij}. In the Durham algorithm [8], the two variables coincide:

$$y_{ij}^{(D)} = v_{ij}^{(D)} = 2\frac{\min\{E_i, E_j\}^2}{Q^2}(1 - \cos\theta_{ij}), \qquad (1)$$

where θ_{ij} is the angle between (pseudo)particles i and j, E_i is the energy of the (pseudo)particle i, and Q is the center-of-mass energy. The clustering procedure selects the pair with the smallest $y_{ij}^{(D)}$. If the latter is smaller than a given $y_{\rm cut}$, the two particles are recombined into a pseudoparticle according to some recombination scheme. Otherwise, the clustering sequence stops, and the number of jets is defined as the number of pseudoparticles left. In the Cambridge algorithm [32,33], the test and ordering variables differ and are defined by

$$y_{ij}^{(C)} = y_{ij}^{(D)}, \qquad v_{ij}^{(C)} = 2(1 - \cos\theta_{ij}). \qquad (2)$$

The clustering procedure selects the pair with the smallest $v_{ij}^{(C)}$. If the corresponding $y_{ij}^{(C)}$ is smaller than $y_{\rm cut}$, the two particles are recombined into a pseudoparticle; otherwise, the softer particle becomes a jet. This is commonly referred to as the soft freezing mechanism. The procedure stops when no pseudoparticles are left. The angular-ordered (AO) version of the Durham algorithm [32] works identically to the Cambridge algorithm but without the freezing mechanism. The three-jet resolution parameter y_3 is defined as the minimum $y_{\rm cut}$ that produces two jets. The two-jet rate is the cumulative integral of the y_3 distribution, normalized to the total cross section σ:

$$\Sigma(y_{\rm cut}) = \frac{1}{\sigma}\int_0^{y_{\rm cut}} dy_3 \frac{d\sigma(y_3)}{dy_3}. \tag{3}$$

The resummation technique formulated in Ref. [14] for event shapes does not require the factorization of the singular soft and collinear modes in the observable's definition, but it rather relies on a property known as recursive infrared and collinear (rIRC) safety [34]. In this sense, the all-order treatment does not require a factorization theorem for the observable [35]. In the following, we present an extension of the above method to jet observables and apply it to the two-jet rate in the Durham and Cambridge algorithms.

Let $y_3(\{\tilde p\}, k_1, \ldots, k_n)$ denote a three-jet resolution which depends on all $n+2$ final-state momenta, where $\{\tilde p\}$ indicates the two Born momenta recoiling against the secondary emissions $k_1, \ldots, k_n$. Each parton k_i is emitted off leg $\ell_i = 1, 2$. The essence of the procedure described in Ref. [14] is that the NLL cross section is given by all-order configurations made of partons independently emitted off the Born legs and widely separated in angle [18]. The NNLL corrections are obtained by correcting a *single* parton of the above ensemble to account for all kinematic configurations that give rise to NNLL effects [14]. The two-jet rate at NNLL can be written as

$$\Sigma(y_{\rm cut}) = e^{-R_{\rm NNLL}(y_{\rm cut})}\left[\mathcal{F}_{\rm NLL}(y_{\rm cut}) + \frac{\alpha_s}{\pi}\delta\mathcal{F}_{\rm NNLL}(y_{\rm cut})\right],$$
$$\delta\mathcal{F}_{\rm NNLL}(y_{\rm cut}) = \delta\mathcal{F}_{\rm clust} + \delta\mathcal{F}_{\rm correl} + \delta\mathcal{F}_{\rm sc} + \delta\mathcal{F}_{\rm hc} + \delta\mathcal{F}_{\rm rec} + \delta\mathcal{F}_{\rm wa}, \tag{4}$$

where the physical origin of the various contributions is discussed in the following. The NNLL Sudakov radiator $R_{\rm NNLL}(y_{\rm cut})$ expresses the no-emission probability above $y_{\rm cut}$ and hence embodies the cancellation of infrared and collinear divergences between the virtual corrections to the Born process and the unresolved real emissions as defined in Ref. [14]. As such, it is inclusive over QCD radiation, and it is universal for all observables featuring the same scaling in the presence of a single soft and collinear emission. Since, in the soft-collinear limit, $y_3(\{\tilde p\}, k) = (k_t/Q)^2$, where k_t is the emission's transverse momentum with respect to the emitting quark-antiquark pair, one can obtain $R_{\rm NNLL}(y_{\rm cut})$ from Appendix B of Ref. [14] by setting $a = 2$ and taking the limit $b_\ell \to 0$. All remaining contributions in Eq. (4) arise from resolved real radiation in different kinematical regions. In particular, the terms $\mathcal{F}_{\rm NLL}$, $\delta\mathcal{F}_{\rm sc}$, $\delta\mathcal{F}_{\rm clust}$, and $\delta\mathcal{F}_{\rm correl}$ originate from soft and collinear emissions. The function $\mathcal{F}_{\rm NLL}$ is the only NLL correction to the radiator, and it is defined in terms of soft and collinear gluons independently emitted off the hard legs and widely separated in rapidity. At NLL, the upper rapidity bound is the same for all emissions and approximated by $\ln(1/\sqrt{y_{\rm cut}})$. The soft-collinear term $\delta\mathcal{F}_{\rm sc}$ arises from considering the NNLL effects of the running coupling in the soft matrix element, as well as restoring the exact rapidity bound for a single soft-collinear emission. The two functions $\delta\mathcal{F}_{\rm clust}$ and $\delta\mathcal{F}_{\rm correl}$ account for configurations in which at most two emissions are close in rapidity and produce a pure Abelian clustering correction ($\delta\mathcal{F}_{\rm clust}$) and a non-Abelian correlated ($\delta\mathcal{F}_{\rm correl}$) one. The hard-collinear ($\delta\mathcal{F}_{\rm hc}$) and recoil ($\delta\mathcal{F}_{\rm rec}$) corrections describe configurations where one emission of the ensemble is collinear but hard. In particular, $\delta\mathcal{F}_{\rm hc}$ takes into account the correct approximation of matrix elements in this region, while $\delta\mathcal{F}_{\rm rec}$ describes NNLL kinematical recoil effects in the observable. Finally, the wide-angle correction $\delta\mathcal{F}_{\rm wa}$ encodes configurations in which a single emission of the ensemble is soft and emitted at wide angles.

All of the above corrections are obtained following a method close in spirit to an expansion by regions, i.e., by taking the proper kinematical limits in the squared amplitudes, the phase space, and the observable constraint $\Theta[y_{\rm cut} - y_3(\{\tilde p\}, k_1, \ldots, k_n)]$. This leads to the definition of a tailored and simplified version of the observable—in our case, a clustering algorithm—obtained from the exact one by taking the appropriate asymptotic limit in each kinematic region. The NNLL corrections that appear in Eq. (4) have already been derived in the context of event-shape resummations [14], with the exception of the clustering correction $\delta\mathcal{F}_{\rm clust}$, which is absent for event shapes, and the soft-collinear correction $\delta\mathcal{F}_{\rm sc}$, which is generalized in this Letter. In the following, we discuss the algorithms necessary to compute the NLL multiple emission function $\mathcal{F}_{\rm NLL}$ and the new correction $\delta\mathcal{F}_{\rm clust}$. The remaining algorithms are obtained following the same strategy of taking the asymptotic limit in the region considered in each correction. They are reported in Ref. [36] both for the Durham and for the Cambridge algorithm. We will first discuss the case of the Durham algorithm, and we will eventually obtain the Cambridge result as a trivial case of the discussion that follows [37].

We start by recalling the calculation of $\mathcal{F}_{\rm NLL}$, which is determined by an ensemble of soft-collinear, strongly angular-ordered partons emitted independently off the

Born legs. For soft emissions, recoil effects are negligible, and all transverse momenta can be computed with respect to the emitting quark-antiquark pair. For each emission k_i, we define the rapidity fraction with respect to the emitting leg ℓ_i as $\xi_i^{(\ell_i)} = |\eta_i|/\ln(1/\sqrt{y_{\rm cut}})$, where $\ln(1/\sqrt{y_{\rm cut}})$ is the NLL rapidity bound, common to all emissions at this order. For this ensemble, the Durham algorithm is approximated by the following simplified version, $\bar{y}_3^{\rm sc}(\{\tilde{p}\}, k_1, \ldots, k_n)$ [18]: 1. Find the pseudoparticle k_I with the smallest value of $\bar{y}_3^{\rm sc}(\{\tilde{p}\}, k_I) = (k_{tI}/Q)^2$. 2. Considering only pseudoparticles k_j collinear to the same leg ℓ as k_I, find the pseudoparticle k_J which satisfies $\vec{k}_{tJ} \cdot \vec{k}_{tI} > 0$ and has the smallest positive value of $\xi_J^{(\ell)} - \xi_I^{(\ell)}$. 3. If k_J is found, recombine k_I and k_J into a new pseudoparticle k_P with $\vec{k}_{tP} = \vec{k}_{tI} + \vec{k}_{tJ}$ and $\xi_P^{(\ell)} = \xi_J^{(\ell)}$. Otherwise, k_I is clustered with a Born leg and removed from the list of pseudoparticles. 4. If only one pseudoparticle k_P remains, then $\bar{y}_3^{\rm sc}(\{\tilde{p}\}, k_1, \ldots, k_n) = (k_{tP}/Q)^2$; otherwise, go back to step 1. Because of the assumption of strong rapidity ordering between the emissions, this algorithm ensures that $\mathcal{F}_{\rm NLL}$ is free from subleading effects. We point out that, as long as emissions are strongly ordered in rapidity, the clustering history depends only on the rapidity ordering among emissions and not on the actual rapidities.

The above algorithm is used whenever emissions are soft collinear and widely separated in angle, even beyond NLL order. In particular, it can be used to compute the NNLL soft-collinear correction $\delta\mathcal{F}_{\rm sc}$. This function is made of two contributions with different physical origins:

$$\delta\mathcal{F}_{\rm sc} = \delta\mathcal{F}_{\rm sc}^{\rm rc} + \delta\mathcal{F}_{\rm sc}^{\rm rap}. \tag{5}$$

The term $\delta\mathcal{F}_{\rm sc}^{\rm rc}$ accounts for NNLL effects in the coupling which have been neglected in $\mathcal{F}_{\rm NLL}$, while the term $\delta\mathcal{F}_{\rm sc}^{\rm rap}$ contains NNLL corrections due to implementing the exact rapidity bound $[|\eta| < \ln(Q/k_t)]$ for a *single* emission k of the soft-collinear ensemble. While the running-coupling correction $\delta\mathcal{F}_{\rm sc}^{\rm rc}$ can be computed using the strongly ordered algorithm defined above, in complete analogy with event-shape observables [14], the rapidity correction $\delta\mathcal{F}_{\rm sc}^{\rm rap}$ requires some care. Since the exact rapidity bound for the emission k $[|\eta| < \ln(Q/k_t)]$ is larger than the NLL bound shared by the other emissions k_i $[|\eta_i| < \ln(1/\sqrt{y_{\rm cut}})]$, the rapidity correction will be nonzero only if the rapidity of emission k is, in magnitude, the largest of all. The rapidity correction is then computed by using the strongly ordered algorithm defined above, with emission k fixed to be the most forward or backward of all [36]. Note that this issue is irrelevant for event shapes, since they are independent of the rapidity fractions, and that the derivation of the rapidity correction given here can be equally applied in that case.

We now turn to the discussion of the NNLL clustering correction $\delta\mathcal{F}_{\rm clust}$, which describes configurations in which at most two of the independently emitted, soft-collinear partons have similar rapidities. We denote by k_a and k_b these two emissions. The function $\delta\mathcal{F}_{\rm clust}$ accounts for the difference between the observable $y_3^{\rm sc}(\{\tilde{p}\}, k_a, k_b, k_1, \ldots, k_n)$, in which k_a and k_b are close in rapidity, and the NLL observable $\bar{y}_3^{\rm sc}(\{\tilde{p}\}, k_a, k_b, k_1, \ldots, k_n)$, in which they are assumed to be far apart. This correction appears whenever the observable depends on the emissions' rapidity fractions; hence, it is absent in the case of event shapes. Its formulation is analogous to the corresponding correction derived for the jet-veto resummation in Ref. [38] and is reported in Ref. [36].

The algorithm that defines $y_3^{\rm sc}(\{\tilde{p}\}, k_1, \ldots, k_n)$ proceeds as the NLL one, with an additional condition to be checked after step 1: 1(b) Let k_{J_a} and k_{J_b} be the pseudoparticles containing the partons k_a and k_b. If these pseudoparticles are close in rapidity (i.e., if neither k_a nor k_b have been recombined with a pseudoparticle with larger $\xi^{(\ell)}$), check whether k_{J_a} and k_{J_b} cluster, i.e., if

$$\min\{E_{J_a}, E_{J_b}\}^2 |\vec{\theta}_{J_a} - \vec{\theta}_{J_b}|^2 < \min\{k_{tJ_a}, k_{tJ_b}\}^2 \tag{6}$$

is satisfied, where $\vec{\theta}_i = \vec{k}_{ti}/E_i$. If so, recombine k_{J_a} and k_{J_b} by adding transverse momenta vectorially and setting the rapidity fraction of the resulting pseudoparticle k_J to $\xi_J^{(\ell)} \simeq \xi_{J_a}^{(\ell)} \simeq \xi_{J_b}^{(\ell)}$. The same algorithm is employed in the computation of the NNLL correlated correction $\delta\mathcal{F}_{\rm correl}$ [14] (see [36] for details). In a similar way, we approximate the original algorithm to compute the remaining NNLL corrections whose definition follows exactly the one given for event shapes [14].

The considerations made so far for the Durham case can be straightforwardly adapted to any other rIRC jet algorithm. In particular, for the Cambridge algorithm, the NNLL logarithmic structure is much simpler. In this case, the ordering variable (2) depends only on the angular distance between emissions. Since at NLL all partons are well separated in rapidity, there will be no clustering between the emissions, and each of them will be recombined with one of the Born legs in an angular-ordered way. One therefore obtains the trivial result $\mathcal{F}_{\rm NLL}(\lambda) = 1$. The same arguments imply that the NNLL corrections $\delta\mathcal{F}_{\rm sc} = \delta\mathcal{F}_{\rm hc} = 0$ [36]. Moreover, both the recoil and the wide-angle corrections admit a simple analytic form given that the emission emitted either at wide angles or collinearly will never cluster with any of the other soft-collinear emissions. As a consequence, the contribution from this emission factorizes with respect to the remaining ensemble [36]. The same property applies to the clustering and correlated corrections which can be entirely formulated in terms of the clustering condition between two soft-collinear emissions [36], analogously to the jet-veto resummation [38]. We note that the freezing condition present in the Cambridge algorithm does not play a role at

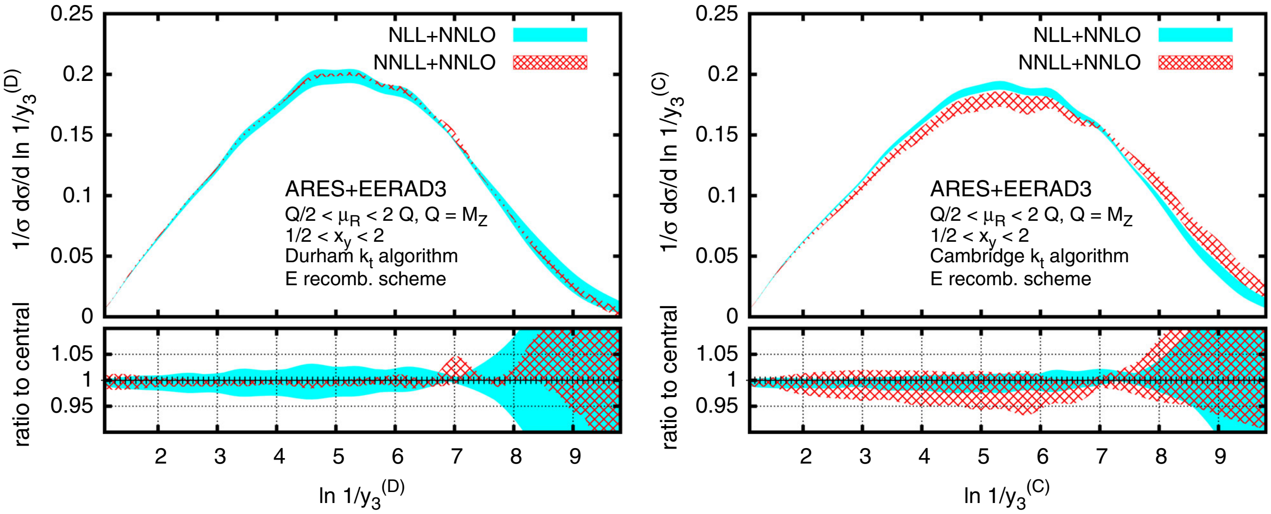

FIG. 1. Differential distributions for the three-jet resolution in the Durham (left) and Cambridge (right) algorithms. The plots show both the NLL + NNLO (blue, solid band) and the NNLL + NNLO (red, hatched band) results.

NNLL. Therefore, the AO version of the Durham algorithm coincides with the Cambridge algorithm at this order, while the two differ at NNLO.

We tested our results by subtracting the derivative of the second-order expansion of Eq. (4) from the $\mathcal{O}(\alpha_s^2)$ distributions obtained with the generator EVENT2 [39], finding agreement [36]. Moreover, we applied the method to both the inclusive-k_t [40,41] and the flavor-k_t [42] algorithms, finding also perfect agreement with EVENT2 at $\mathcal{O}(\alpha_s^2)$ [43].

We illustrate the impact of our calculation by matching the NNLL two-jet rate (4) to the $\mathcal{O}(\alpha_s^3)$ result obtained with the program EERAD3 [44] for both the Durham and the Cambridge algorithms. Figure 1 shows the matched differential distributions for the three-jet resolution parameter, defined in (3), at NNLL + NNLO and NLL + NNLO. The results are obtained at $Q = M_Z$, using the coupling $\alpha_s(M_Z) = 0.118$ and the E recombination scheme. To impose unitarity, following Ref. [14], we employ the modified logarithms

$$\ln\frac{1}{y_3} \to \ln\left[1 + \left(\frac{x_y}{y_3}\right) - \left(\frac{x_y}{y_{3,\max}}\right)\right], \qquad (7)$$

in such a way that the x_y dependence is N^{3}LL. This also ensures that the distribution vanishes at the kinematical endpoint $y_{3,\max}$, taken from the NNLO result. Furthermore, the variation of x_y probes the size of subleading logarithmic effects. Our theoretical uncertainties are obtained by varying, one at a time, x_y and the renormalization scale μ_R by a factor of 2 in either direction around the central values $x_y = 1$ and $\mu_R = Q$ and taking the envelope of these variations.

For the Durham algorithm, as expected, we observe a significant reduction of the theory error when going from NLL to NNLL. On the contrary, for the Cambridge algorithm, NNLL corrections are quite large, and the NNLL uncertainty is larger than the NLL one, which in turn seems to be underestimated. This effect can be understood by observing that the NLL prediction for the Cambridge algorithm does not contain any information about multiple emission effects, since no clustering occurs at this order and $\mathcal{F}_{\rm NLL} = 1$. These effects appear only at NNLL, explaining the sizable numerical corrections. It follows that the NLL theory uncertainty as estimated in Fig. 1 is unable to capture large subleading effects. A similar phenomenon was already observed in the resummation for the jet-veto efficiency [38].

To conclude, in Fig. 2 we compare our NNLL + NNLO prediction to the data taken by the L3 Collaboration at LEP2 [30] at $Q = 206$ GeV. At this high center-of-mass energy, the impact of hadronization effects, which are not included in our calculation, is moderate. Overall, we find

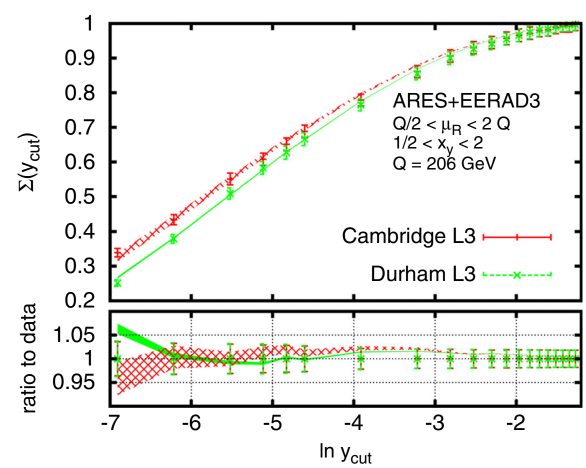

FIG. 2. Comparison of NNLL + NNLO predictions for the two-jet rates to data from the L3 Collaboration [30].

good agreement with data down to the lowest values of $y_{\rm cut}$. Owing to the small residual perturbative uncertainties, our calculation shows promise for a precise determination of the strong coupling using e^+e^- data measured at LEP.

In this Letter, we have presented a general method for final-state resummation at NNLL order for global rIRC safe observables that vanish in the two-jet limit, where a single family of large logarithms is resummed. We derived explicit results for the two-jet rate in e^+e^-. The computer code ARES used to obtain the results presented here can be made available upon request to the authors.

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