Sequential Feedback Scheme Outperforms the Parallel Scheme for Hamiltonian Parameter Estimation

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Measurement and estimation of parameters are essential for science and engineering, where the main quest is to find the highest achievable precision with the given resources and design schemes to attain it. Two schemes, the sequential feedback scheme and the parallel scheme, are usually studied in the quantum parameter estimation. While the sequential feedback scheme represents the most general scheme, it remains unknown whether it can outperform the parallel scheme for any quantum estimation tasks. In this Letter, we show that the sequential feedback scheme has a threefold improvement over the parallel scheme for Hamiltonian parameter estimations on two-dimensional systems, and an order of $O(d+1)$ improvement for Hamiltonian parameter estimation on d-dimensional systems. We also show that, contrary to the conventional belief, it is possible to simultaneously achieve the highest precision for estimating all three components of a magnetic field, which sets a benchmark on the local precision limit for the estimation of a magnetic field.

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A pivotal task in science and technology is to find out the highest achievable precision in measuring and estimating parameters of interest with given resources and design schemes to reach that precision [1–20]. Typically, to estimate some parameters $x=(x_1,x_2,\ldots,x_m)$ encoded in some dynamics ϕ_x, a probe state ρ_0 is prepared which evolves under the dynamics $\rho_0 \xrightarrow{\phi_x} \rho_x$. By performing positive operator valued measurements $\{E_y\}$ on the output state ρ_x, one gets the measurement result y with a probability $p(y|x)=\mathrm{Tr}(E_y\rho_x)$. According to the Cramér-Rao bound the covariance matrix of any unbiased estimator of x is then bounded below by the Fisher information matrix $n\mathrm{Cov}(\hat{x}) \geq I^{-1}(x)$ [21–24], where n is the number of times that the procedure is repeated, $\mathrm{Cov}(\hat{x})$ denotes the covariance matrix of the estimator, and $I(x)$ is the Fisher information matrix, with the ijth entry given by $I_{ij}(x)=\int p(y|x)\{[\partial \ln p(y|x)/(\partial x_i)][\partial \ln p(y|x)/(\partial x_j)]\}dy$ [25]. The Fisher information matrix can be further bounded by the quantum Fisher information matrix (QFIM) $J(\rho_x)$, which gives the quantum Cramér-Rao bound [21,22,26,27] $n\mathrm{Cov}(\hat{x}) \geq I^{-1}(x) \geq J^{-1}(\rho_x)$.

In the multiparameter estimation, the quantum Cramér-Rao bound is usually not achievable even asymptotically [21,22,28–37]. Two tradeoffs have to be considered in multiparameter estimation: the first tradeoff is on the choice of measurements, as the optimal measurements for different parameters are usually incompatible [38]; the second tradeoff is on the choice of the probe states, since the optimal probe states for different parameters are also usually different. These tradeoffs are usually dealt with by specifying a particular figure of merit taken as $\mathrm{Tr}[\mathrm{Cov}(\hat{x})G]$ with $G \geq 0$, then optimizing the measurements and the probe states based on the figure of merit.

Besides the measurements and the probe states, one also needs to optimize the schemes that arrange multiple uses of the dynamics. Two schemes, the sequential feedback scheme and the parallel scheme, as shown in Fig. 1, are usually studied. The sequential feedback scheme represents the most general scheme, which includes the parallel scheme as a special case when taking the controls as SWAP gates. Examples have been found in quantum channel discrimination that the sequential feedback scheme can outperform the parallel scheme for the discrimination of two quantum channels [39,40]. In the quantum parameter estimation, it remains unknown whether the sequential feedback scheme can outperform the parallel scheme. Based on some upper bounds on the precision limit [5,10], it has been shown that the sequential feedback scheme does not lead to higher precision in the single-parameter quantum estimation under several dynamics, including the unitary [3] and dephasing dynamics [7–10]. This has led to a conjecture that in the asymptotical limit, the sequential feedback scheme provides no gains over the parallel scheme for quantum parameter estimation [10].

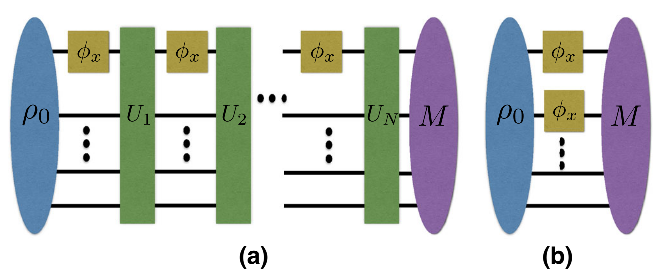

FIG. 1. (a) Sequential feedback scheme. (b) Parallel scheme.

In this Letter, we show that the sequential feedback scheme outperforms the parallel scheme for Hamiltonian parameter estimation; here, we focus on the estimations of small shifts of the parameters around some known values. We first study the estimation of the Hamiltonian for $SU(2)$ dynamics, which is a fundamental problem in quantum parameter estimation [37,41–53] and closely related to the estimation of a magnetic field. It thus has many applications in quantum sensing, data storage, information processing, and magnetic resonance, and it also has implications in quantum gyroscope, quantum reference frame alignments, etc. [41,43–46,54]. By optimizing the general sequential feedback scheme, we obtain the ultimate local precision limit for the estimation of a magnetic field which shows that the sequential feedback scheme outperforms the parallel scheme with a threefold improvement. We also show that the optimal sequential feedback scheme achieves the highest precision for all three parameters of a magnetic field simultaneously, this is contrary to the conventional belief that some tradeoffs have to be made for the estimation of different parameters of a magnetic field. We further show that for the estimation of general Hamiltonian on d-dimensional systems the sequential feedback scheme outperforms the parallel scheme with an order of $O(d+1)$. This sheds light on the comparison between the two schemes. We note that the sequential feedback scheme is also more implementable under many current experimental settings since high-fidelity controls on small systems can now be routinely done, while accurately preparing entangled states with many particles for the parallel scheme is still very challenging.

For the estimation of a two-dimensional Hamiltonian, we consider the Hamiltonian for a spin-1/2 in a magnetic field, which can be written as $H(B,\theta,\phi) = B(\sin\theta\cos\phi\sigma_1 + \sin\theta\sin\phi\sigma_2 + \cos\theta\sigma_3)$; here, $x = (B,\theta,\phi)$ represents the magnitude and the directions of a magnetic field, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. The Hamiltonian can also be written concisely as $H(B,\theta,\phi) = B[\vec{n}(\theta,\phi)\cdot\vec{\sigma}]$, where $\vec{n}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. We are interested in the ultimate local precision limit in estimating $x = (B,\theta,\phi)$, with the aid of ancillary systems. We first consider the scheme without feedback controls, then extend to the general sequential feedback scheme.

We denote $U(x,T) = e^{-iH(B,\theta,\phi)T}$ as the free evolution of the Hamiltonian with T units of time and $U_A(x,T) = U(x,T)\otimes I_A$ as the evolution with an ancillary system with I_A as the identity operator on the ancillary system. Let $\rho_x = U_A(x,T)\rho_{SA}U_A^\dagger(x,T)$ and $\rho_{x+dx} = U_A(x+dx,T)\rho_{SA}U_A^\dagger(x+dx,T)$; here, ρ_{SA} denotes the initial state of system + ancilla and dx represents a small shift of the parameter. The local precision limit of estimating x from the output state ρ_x is related to the Bures distance between ρ_x and ρ_{x+dx} as [21,22,26,27]

$$d^2_{\mathrm{Bures}}(\rho_x,\rho_{x+dx}) = \sum_{ij}\frac{1}{4}J_{ij}(\rho_x)dx_i dx_j; \tag{1}$$

here, the Bures distance d_{Bures} is defined as $d_{\mathrm{Bures}}(\rho_1,\rho_2) = \sqrt{2-2F(\rho_1,\rho_2)}$, with $F(\rho_1,\rho_2) = \sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}$ as the fidelity between ρ_1 and ρ_2, and $J_{ij}(\rho_x)$ is the ijth entry of the QFIM $J(\rho_x)$. Since

$$\max_{\rho_{SA}} d^2_{\mathrm{Bures}}(\rho_x,\rho_{x+dx}) = 2 - 2\min_{\rho_{SA}} F(\rho_{SA}, U'\otimes I_A\rho_{SA}U'^\dagger\otimes I_A), \tag{2}$$

where $U' = U^\dagger(x,T)U(x+dx,T)$, the maximal QFIM is thus related to the minimum fidelity $\min_{\rho_{SA}} F(\rho_{SA}, U'\otimes I_A\rho_{SA}U'^\dagger\otimes I_A)$.

For any $d\times d$ unitary U, we denote $e^{-iE_j^U}$ as the eigenvalues of U with $E_j^U\in(-\pi,\pi]$, $1\le j\le d$. We call E_j^U the eigenangles of U and assume $E^U_{\max} = E_1^U \ge E_2^U \ge \cdots \ge E_d^U = E^U_{\min}$ are arranged in decreasing order. It is known that $\min_{\rho_0} F(\rho_0, U\rho_0U^\dagger) = \cos[(E^U_{\max}-E^U_{\min})/2]$ if $E^U_{\max} - E^U_{\min} \le \pi$ [61]. Denote $C(U) = (E^U_{\max}-E^U_{\min})/2$, and then the equation can be written concisely as $\min_{\rho_0} F(\rho_0, U\rho_0U^\dagger) = \cos C(U)$. Since $E^{U\otimes I_A}_{\max} = E^U_{\max}$ and $E^{U\otimes I_A}_{\min} = E^U_{\min}$, we also have $\min_{\rho_{SA}} F(\rho_{SA}, U\otimes I_A\rho_{SA}U^\dagger\otimes I_A) = \cos C(U)$. (We note that this does not mean the ancillary system is not useful; the role of the ancillary system will be clear later.)

With Eqs. (1) and (2), we can then obtain

$$\sum_{ij} J^{\max}_{ij}dx_i dx_j = 8\{1-\cos C[U^\dagger(x,T)U(x+dx,T)]\}. \tag{3}$$

If $U(x,T)$ is continuous with x, then when dx is sufficiently small, $U^\dagger(x,T)U(x+dx,T)\to I$, $C[U^\dagger(x,T)U(x+dx,T)]\to 0$; thus, up to the second order

$$\sum_{ij} J^{\max}_{ij}dx_i dx_j = 4C^2[U^\dagger(x,T)U(x+dx,T)]. \tag{4}$$

To ensure there exists a QFIM $J(\rho_x)$ that achieves the $J^{\max}$ for all dx, we need to show that the optimal state ρ_{SA} that achieves the maximum Bures distance in Eq. (2) is independent of dx. In the Supplemental Material [55], we showed that any maximally entangled state [which are those states such that the reduced state is completely mixed, i.e., $Tr_A(\rho_{SA}) = 1/2I$] achieves the maximum Bures distance in Eq. (2) for all dx; $J^{\max}$ thus corresponds to the QFIM of any maximally entangled probe state. And the maximal QFIM $J^{\max}$ can be obtained by comparing the coeffcients at both sides of Eq. (4), which is given by (see the Supplemental Material [55])

$$J^{\max} = 4\begin{pmatrix} T^2 & 0 & 0 \\ 0 & \sin^2(BT) & 0 \\ 0 & 0 & \sin^2(BT)\sin^2(\theta) \end{pmatrix}. \quad (5)$$

Furthermore, the projective measurement in the Bell basis saturates the quantum Cramér-Rao bound. In the Supplemental Material [55], we showed that the distribution of the measurement results in the Bell basis is given by $p_1 = \cos^2(BT)$, $p_2 = \sin^2(BT)\cos^2\theta$, $p_3 = \sin^2(BT)\sin^2\theta\cos^2\phi$, and $p_4 = \sin^2(BT)\sin^2\theta\sin^2\phi$, which has the classical Fisher information matrix equal to $J^{\max}$. The quantum Cramér-Rao bound is thus saturable, and $J^{\max}$ sets the local precision limit when the dynamics is evolved for T units of time. This is consistent with previous studies [47,49]; however, our method makes it easy to incorporate feedback controls, as we now show.

For the general sequential feedback scheme as in Fig. 1(a), the total evolution can be written as $U_{FA}(x, Nt) = U_N U_A(x,t)\cdots U_2 U_A(x,t) U_1 U_A(x,t)$, where $U_A(x,t) = e^{-iH(x)t}\otimes I_A$, with $t = (T/N)$, and $U_1, U_2, \ldots, U_N$ denote the feedback controls. It can be shown that $C[U_{FA}^\dagger(x,Nt)U_{FA}(x+dx,Nt)] \le NC[U_A^\dagger(x,t)U_A(x+dx,t)]$, where the equality can be achieved when $U_1 = U_2 = \cdots = U_N = U_A^\dagger(x,t)$ (see the Supplemental Material [55]). In practice, the true value x is not known *a priori*, the estimated value $\hat{x}$ need to be used, and the controls $U_1 = U_2 = \cdots = U_N = U_A^\dagger(\hat{x},t)$ need to be updated adaptively. This, however, does not affect the asymptotical scaling [62–64].

From Eq. (4), we then have

$$\begin{aligned}\sum_{ij}(J_N^{\max})_{ij}dx_i dx_j &= 4C^2[U_{FA}^\dagger(x,Nt)U_{FA}(x+dx,Nt)] \\ &\le 4N^2C^2[U_A^\dagger(x,t)U_A(x+dx,t)] \\ &= N^2\sum_{ij}(J_1^{\max})_{ij}dx_i dx_j; \quad (6)\end{aligned}$$

thus,

$$J_N^{\max} \le N^2 J_1^{\max} = 4N^2\begin{pmatrix} t^2 & 0 & 0 \\ 0 & \sin^2(Bt) & 0 \\ 0 & 0 & \sin^2(Bt)\sin^2(\theta) \end{pmatrix}. \quad (7)$$

Here, the equality can be saturated asymptotically with the controls $U_1 = U_2 = \cdots = U_N = U_A^\dagger(\hat{x},t) = e^{iH(\hat{x})t}\otimes I_A$. In this case, the feedback controls only act on the system; thus, we can write

$$\begin{aligned}&U_{FA}^\dagger(x,Nt)U_{FA}(x+dx,Nt) \\ &\quad = U'\otimes I_A = e^{ia_{Nt}(x,dx)[\vec{k}_{Nt}(x,dx)\cdot\vec{\sigma}]}\otimes I_A. \quad (8)\end{aligned}$$

For the last equation, we used the fact that any U' can be written as $e^{ia_{Nt}(x,dx)(\vec{k}_{Nt}(x,dx)\cdot\vec{\sigma})}$, where $\vec{k}_{Nt}(x,dx)$ is a unit vector. This has a similar form as the free evolution; thus, by following the same line of argument, one can show that the optimal probe state is any maximally entangled state which, under the optimal feedback scheme, has the QFIM

$$J_N^{\max} = 4N^2\begin{pmatrix} t^2 & 0 & 0 \\ 0 & \sin^2(Bt) & 0 \\ 0 & 0 & \sin^2(Bt)\sin^2(\theta) \end{pmatrix}.$$

In this case, the measurement in the Bell basis also saturates the quantum Cramér-Rao bound $n\mathrm{Cov}(\hat{x}) \ge (J_N^{\max})^{-1}$ (see the Supplemental Material for details [55]); $J_N^{\max}$ thus quantifies the asymptotical precision limit.

To ease comparison with previous results, we rewrite the Hamiltonian as $H = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ with $x_1 = B\sin\theta\cos\phi$, $x_2 = B\sin\theta\sin\phi$, $x_3 = B\cos\theta$. In the asymptotical limit, the estimation is in the vicinity of the actual value; we can thus write

$$\delta\hat{x}_1 = \sin\theta\cos\phi\delta\hat{B} + B\cos\theta\cos\phi\delta\hat{\theta} - B\sin\theta\sin\phi\delta\hat{\phi},$$

$$\delta\hat{x}_2 = \sin\theta\sin\phi\delta\hat{B} + B\cos\theta\sin\phi\delta\hat{\theta} + B\sin\theta\cos\phi\delta\hat{\phi},$$

$$\delta\hat{x}_3 = \cos\theta\delta\hat{B} - B\sin\theta\delta\hat{\theta}.$$

It is then easy to get $\delta\hat{x}_1{}^2 + \delta\hat{x}_2{}^2 + \delta\hat{x}_3{}^2 = \delta\hat{B}^2 + B^2\delta\hat{\theta}^2 + B^2\sin^2(\theta)\delta\hat{\phi}^2$. This will be taken as the figure of merit for comparison as it is used in previous studies [37,48], which corresponds to take $G = I$ in $\mathrm{Tr}[\mathrm{Cov}(\hat{x})G]$ under the representation of (x_1, x_2, x_3). We note that the choice of $G = I$ here is just for the purpose of comparison; the precision limit obtained under the feedback scheme is optimal for any G—as the obtained precision saturates the quantum Cramér-Rao bound $n\mathrm{Cov}(\hat{x}) \ge (J_N^{\max})^{-1}$; thus, for any choice of G it also saturates the lower bound $n\mathrm{Tr}[\mathrm{Cov}(\hat{x})G] \ge \mathrm{Tr}[(J_N^{\max})^{-1}G]$. Here, n is the number of times that the procedure is repeated, which accounts for the classical effect; for the following, we will neglect n by assuming that the procedure is repeated with the same (sufficiently large) number of times.

We now compare $\delta\hat{x}_1{}^2 + \delta\hat{x}_2{}^2 + \delta\hat{x}_3{}^2$ obtained from different schemes. Under the optimal sequential feedback scheme, we have $\mathrm{Cov}(\hat{x}) = (J_N^{\max})^{-1}$ with

$$J_N^{\max} = 4N^2\begin{pmatrix} t^2 & 0 & 0 \\ 0 & \sin^2(Bt) & 0 \\ 0 & 0 & \sin^2(Bt)\sin^2(\theta) \end{pmatrix};$$

thus,

$$\delta\hat{x}_1{}^2 + \delta\hat{x}_2{}^2 + \delta\hat{x}_3{}^2 = \delta\hat{B}^2 + B^2\delta\hat{\theta}^2 + B^2\sin^2(\theta)\delta\hat{\phi}^2 = \frac{1}{4N^2}\left[\frac{1}{t^2} + \frac{2B^2}{\sin^2(Bt)}\right]. \tag{9}$$

Under the parallel scheme, the precision has been extensively studied previously [37,41,43–51,65] with the highest precision given by $\mathrm{Cov}(\hat{x}) = 3(J_1^{\max})^{-1}/[N(N+2)]$ [37,48]; here,

$$J_1^{\max} = 4\begin{pmatrix} t^2 & 0 & 0 \\ 0 & \sin^2(Bt) & 0 \\ 0 & 0 & \sin^2(Bt)\sin^2(\theta) \end{pmatrix}.$$

This corresponds to

$$\delta\hat{x}_1{}^2 + \delta\hat{x}_2{}^2 + \hat{x}_3{}^2 = \frac{3}{4N(N+2)}\left[\frac{1}{t^2} + \frac{2B^2}{\sin^2(Bt)}\right]. \tag{10}$$

Comparing Eqs. (9) and (10), we can see that the optimal sequential feedback scheme has a threefold improvement over the optimal parallel scheme.

For a given T, when $N \to \infty$, $t = (T/N) \to 0$, $[B^2/\sin^2(Bt)] \to (1/t^2)$, the precision limit under the optimal sequential feedback scheme thus reaches $\delta\hat{x}_1{}^2 + \delta\hat{x}_2{}^2 + \delta\hat{x}_3{}^2 = 3/(4N^2t^2) = 3/(4T^2)$. Note that for the estimation of a single parameter x_i, the highest precision one can get within T units of time is $\delta\hat{x}_i{}^2 = 1/(4N^2t^2) = 1/(4T^2)$ [3]. It is conventionally believed that for the simultaneous estimation of different parameters of a magnetic field, some tradeoffs have to be made on the probe states and the measurements, achieving the highest precision for all parameters simultaneously is thus not possible. Here we showed that, while the tradeoffs are indeed unavoidable under the parallel scheme, the optimal sequential feedback scheme can achieve the highest precision for all three parameters of a magnetic field simultaneously.

We next show that for the estimation of general Hamiltonian for $SU(d)$ dynamics, the sequential feedback scheme has similar improvement over the parallel scheme.

Given an $SU(d)$ dynamics aided with ancillary system $U_A(x,t) = e^{i\sum_{j=1}^{d^2-1} x_j F_j t} \otimes I_A$, here $\{F_j\}$ are traceless self-adjoint matrices and $\mathrm{Tr}(F_jF_k) = \delta_{jk}$; i.e., $\{iF_j\}$ form an orthogonal basis of $su(d)$, and $x = (x_1, x_2, \ldots, x_{d^2-1})$ are the parameters to be estimated. We compare three schemes: (1) the independent scheme, (2) the parallel scheme, and (3) the sequential feedback scheme. The independent scheme is to divide the N uses of the dynamics into d^2-1 groups and use $N/(d^2-1)$ dynamics in each group to estimate one parameter. Under this scheme, the variance of each parameter $\delta x_j^2 \propto 1/\{[N/(d^2-1)]^2t^2\} = (d^2-1)^2/(N^2t^2)$ and the summation of the variance is then $\sum_{j=1}^{d^2-1}\delta x_j^2 \propto [(d^2-1)^3/(N^2t^2)]$. For the parallel scheme, the minimum summation of variance has been obtained previously as $\sum_{j=1}^{d^2-1}\delta x_j^2 = \{[d(d+1)(d^2-1)]/[4N(N+d)t^2]\}$ [48]. For the sequential feedback scheme, we show that (see the Supplemental Material [55]) by taking the maximally entangled state as the probe state and using the optimal feedback control $U_1 = U_2 = \cdots = U_N = U_A^\dagger(\hat{x}, t)$, the quantum Fisher information matrix is given by $(4N^2t^2/d)I$ and the quantum Cramér-Rao bound can be saturated. The summation of variance under the optimal feedback scheme is thus given by $\sum_{j=1}^{d^2-1}\delta x_j^2 = [d(d^2-1)/(4N^2t^2)]$, which has an order of $O(d+1)$ improvement over the parallel scheme and an order of $O(d^3)$ improvement over the independent scheme.

Discussion and conclusion.—The comparison between the sequential feedback scheme and the parallel scheme has been a subject of lasting interest in quantum channel discrimination and quantum parameter estimation. In quantum channel discrimination, Acín [65] and D'Ariano *et al.* [66] studied the optimal parallel scheme for the discrimination between two unitary dynamics. Duan *et al.* [67] then showed the sequential feedback scheme is equivalent to the parallel scheme for the discrimination of unitary dynamics, and then Chiribella *et al.* [39] showed the sequential feedback scheme can outperform the parallel scheme for discriminating quantum channels with memory effects. The optimal sequential scheme has also been obtained for the discrimination of two general quantum channels [68]. For the single-parameter quantum estimation, the sequential feedback scheme is shown to be equivalent to the parallel scheme under unitary [3] and dephasing dynamics [10,69], and it has been conjectured that asymptotically the sequential feedback scheme is equivalent to the parallel scheme [10]. For the multiparameter quantum estimation, Humphreys *et al.* [31] showed the parallel scheme has an order of $O(d)$ improvement over the independent scheme for estimating d parameters with commutating generators; for general unitary dynamics, the optimal parallel scheme has also been studied [37,48], which shows the parallel scheme has similar improvement over the independent scheme.

Prior to this study, a general belief has been that under unitary dynamics the sequential feedback scheme is equivalent to the parallel scheme (while under noisy dynamics, the sequential feedback scheme is believed to be either equivalent to the parallel scheme or can only outperform the parallel scheme for channels with special properties). Here, by showing the sequential feedback scheme has an order of $O(d+1)$ improvement over the parallel scheme for the estimation of d-dimensional Hamiltonian, our study disclosed a unique feature for the multiparameter quantum estimation and deepened the understanding of the relationship between the sequential feedback scheme and the parallel scheme.

Our study also sets a benchmark on the local precision limit for the estimation of a magnetic field, which is of practical importance for many applications. The precision is obtained by optimizing all steps in the procedure of the estimation and thus represents the ultimate precision one can achieve for the estimation of a magnetic field asymptotically. Our study shows that it is possible to achieve the highest precision simultaneously for all three parameters of a magnetic field, contrary to the conventional belief that some tradeoffs have to be made on the precision of different parameters. This opened the possibility and initiated the study of using feedback controls to counteract the tradeoffs in multiparameter quantum estimation. Future research includes finding the ultimate precision at the presence of general noises.

In the Supplemental Material [55], we also discussed the possible extension to Hamiltonian parameter estimation with a prior distribution and showed that the feedback scheme gains over the parallel scheme through adaptive choice of the evolution time [55]. Intuitively, the feedback scheme gains over the parallel scheme by utilizing the information encoded in the prior distribution to design the feedback controls, while under the parallel scheme the information is ignored during the evolution stage. Future research includes quantifying the gain of the feedback scheme exactly under any prior distribution.

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