Suppression of Noise-Induced Modulations in Multidelay Systems

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Many physical systems involve time-delayed feedback or coupling. In such delay systems, noise can give rise to undesirable oscillations at frequencies resonant to the delay times. We investigate how an additional feedback term can suppress noise-induced modulations in delay systems with self-feedback that exhibit deterministic oscillatory dynamics. A simple characteristic equation is derived to predict optimal delay times for the prototypical example of a Stuart-Landau oscillator subject to two feedback terms. We then show that a characteristic equation of the same form accurately describes the dominant Floquet modes of more complex oscillatory systems and hence can be used to optimize the suppression of noise-induced modulations. This is shown for mode-locked lasers and FitzHugh-Nagumo oscillators subject to self-feedback.

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The control of dynamical systems via time-delayed feedback has been a very active field over the last few decades and has resulted in a wide range of applications, including chaos control [1,2], the stabilization of steady states [3,4], temporal pattern formation [5], and improving the regularity of oscillatory dynamics [6]. Feedback control is ubiquitous in many areas of science and engineering; examples are found in electronics [7], neuroscience [8], quantum mechanics [9], and optics [6,10–12]. Any physical realization of such systems will inevitably involve some source of noise, which in the combination with time delay can lead to noise-induced oscillations [13] or noise-induced modulations in oscillatory systems [14]. This phenomenon has been observed in various systems involving time delay: fiber ring-cavity lasers [15], passively mode-locked semiconductor lasers subject to optical feedback [16,17], delay-coupled lasers [18], neural networks [19], and gene regulatory networks [20]. Such noise-induced dynamics are generally detrimental to the desired applications; hence, there is a need to control or suppress them. In this Letter we show how this can optimally be achieved by the addition of another feedback loop (Fig. 1) in systems with time-delayed self-feedback which exhibit deterministically stable oscillatory dynamics. We present a simple characteristic equation that appears to be system independent and powerful enough to predict the delay times needed for the optimal suppression of noise-induced modulation of oscillatory dynamics. It therefore has great potential for application in diverse areas. Our results also allow substantially longer delay times to be studied than what is practically possible with computationally restricted numerical tools.

To date, relatively little research has been carried out on the suppression of noise-induced dynamics in systems involving time delay. Most works concentrate on systems in a deterministically stable steady state which are close to a bifurcation leading to oscillatory dynamics [21–23], and they use delayed feedback, either with the aim of suppressing the noise-induced oscillations [13] or stabilizing them [22, 24–26]. In the fiber laser community, experimental studies have reported on the suppression of noise-induced dynamics, referred to as supermode noise, via the addition of a second fiber cavity [15,27–29]. However, their theoretical considerations for the choice of the feedback conditions were based on the Vernier principle [15], which is not sufficient for determining conditions for optimal modulation suppression.

In nondelay systems exhibiting noise-induced oscillations, suppression is possible by adding feedback with the delay time equal to half the period of the noise-induced oscillations [22,25]. However, an oscillatory system with delayed feedback has infinitely many Floquet exponents and a finite number of these can be weakly damped. The most prevalent frequency of the noise-induced modulations will correspond to the Floquet exponent with the smallest modulus of the real part, which, for sufficiently strong feedback, is related to the feedback delay time. In this scenario, because of the multitude of Floquet exponents, the noise-induced modulations cannot be optimally suppressed by simply choosing the second feedback delay time to be equal to half of the modulation period, as this delay time will be resonant with the second mode of the first feedback term. Thus, a different approach is needed and will be addressed in this Letter.

We first study the prototypical example of a Stuart-Landau oscillator and then compare the results with two specific examples, a passively mode-locked laser and a FitzHugh-Nagumo oscillator. Although these two systems exhibit complex and very different dynamics [30–33], we are able to show that the dependence on the time delay, in

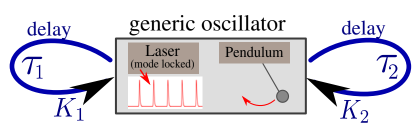

FIG. 1. Oscillatory system with twofold self-feedback.

terms of modulation suppression, is general to all three systems. In this Letter we are exclusively concerned with the influence of the feedback terms on the suppression of the noise-induced modulation of limit cycles. Therefore, we restrict ourselves to feedback conditions that do not qualitatively change the dynamics, i.e., feedback delay times that are integer multiples of the period of the deterministic dynamics. Furthermore, we know that the noise-induced modulations are caused by excitations of eigenmodes of the system and hence are related to the stability of the underlying deterministic systems [22]. We will therefore first investigate the damping rates of the Floquet exponents of the systems in the absence of noise. For this, we perform a linear stability analysis under the assumption of small noise terms since we are considering cases where noise causes a modulation of the dynamics but the underlying oscillatory motion is still preserved.

The Stuart-Landau oscillator subject to two feedback terms is given by

$$\dot{z} = [\lambda_0 + i\omega_0 - (1+i\gamma)|z|^2]z + \sum_{n=1,2} K_n e^{i\theta_n} z(t-\tau_n), \quad (1)$$

where $z \in \mathbb{C}$, λ_0 is the bifurcation parameter exhibiting a supercritical Hopf bifurcation at $\lambda_0 = 0$ in the absence of feedback, ω_0 is the oscillation frequency at $\lambda_0 = 0$, γ is the amplitude-phase coupling (anisochronicity), K_1 and K_2 are the feedback strengths, θ_1 and θ_2 are the feedback phases, and τ_1 and τ_2 are the feedback delay times. We are interested in the oscillatory regime of the system without delay, and we therefore choose $\lambda_0 > 0$ to put the system above the Hopf bifurcation.

This system has solutions of the form $z = re^{i\Omega t}$, where r is the constant amplitude of the oscillations and Ω is the frequency. Entering this ansatz into Eq. (1), the amplitude and the frequency are determined by

$$r^2 = \lambda_0 + K_1 \cos(\theta_1 - \Omega\tau_1) + K_2 \cos(\theta_2 - \Omega\tau_2) \quad (2)$$

and

$$\Omega = \omega - \gamma\lambda_0 + \sum_{n=1,2} K_n \sqrt{1+\gamma^2} \sin[\theta_n - \Omega\tau_n - \arctan(\gamma)]. \quad (3)$$

To calculate the Floquet exponents, λ, of this system, the characteristic equation

$$\left[r^2 + \lambda + \sum_{n=1,2} K_n \cos(\theta_n - \Omega\tau_n)(1 - e^{-\lambda\tau_n})\right]^2 = r^4 - 2\gamma r^2 \sum_{n=1,2} K_n \sin(\theta_n - \Omega\tau_n)(1 - e^{-\lambda\tau_n}) - \sum_{n=1,2} K_n^2 \sin^2(\theta_n - \Omega\tau_n)(1 - e^{-\lambda\tau_n})^2, \quad (4)$$

which is obtained by linearizing the system about its periodic solutions, must be solved. In the case of only one feedback term, this can be done analytically by expanding λ in orders of τ^{-1} [34]. However, the same approach cannot be applied to the dual feedback case, as the order of magnitude of the second delay time can vary in comparison to the first. To simplify Eq. (4), some assumptions are needed.

Since we restrict ourselves to resonant feedback, the delay times are integer multiples n of the period $T_0 = 2\pi/\Omega$ of the oscillator; i.e., $\tau_{1,2} = nT_0$. Furthermore, we restrict the feedback phases by considering that the feedback phase for perfectly resonant feedback is zero. In this case, Eqs. (2) and (3) can be simplified to

$$r^2 = \lambda_0 + K_1 + K_2 \quad \text{and} \quad \Omega = \omega_0 - \gamma r^2, \quad (5)$$

where Ω no longer depends on the feedback delay times. The characteristic equation thus reduces to

$$r^2 = \pm\left[r^2 + \lambda + \sum_{n=1,2} K_n (1 - e^{-\lambda\tau_n})\right]. \quad (6)$$

As long as $\lambda_0 > 0$ and the delay times are sufficiently large, the largest Floquet exponents are given by the plus sign in Eq. (6):

$$\lambda = -(K_1 + K_2) + K_1 e^{-\lambda\tau_1} + K_2 e^{-\lambda\tau_2}. \quad (7)$$

In this expression there is only implicit dependence on the parameters of the Stuart-Landau system, which enters through the constraints put on the delay times (i.e., $\tau_{1,2} = nT_0$). Here, we also note that the same characteristic equation is obtained for a Pyragas-type feedback scheme, i.e., $K[z(t-\tau) - z(t)]$ [1]. To answer the question of optimal modulation suppression, we solve Eq. (7) numerically and find the Floquet exponents in dependence of the delay times and the feedback strengths.

In Figs. 2(a) and 2(b), the damping rate and the frequency of the dominant Floquet exponent (Re[λ] and Im[λ], respectively), neglecting the neutral mode, are plotted depending on the feedback strengths and τ_2 ($\tau_1 = 500T_0$ and τ_2 ranges from zero to τ_1). Here, we chose $K_1 = K_2$ for simplicity; however, Eq. (7) also holds for $K_1 \neq K_2$. When τ_2 is close to τ_1, the most dominant Floquet exponent is very weakly damped because the frequency of this Floquet exponent is resonant to both feedback terms [$\approx 2\pi/\tau_1$, as indicated by label 1 in Fig. 2(b)]. The same applies if τ_2 is close to zero. For intermediate τ_2 values, peaks in Re[λ] [the white regions in Fig. 2(a)] occur at the values where τ_1 and τ_2 have low common multiples. Here, the frequency of the dominant Floquet exponents corresponds to higher order resonant modes of the first feedback term (integer multiples of $2\pi/\tau_1$). Thus, the dominate frequency of noise-induced modulation can be changed by varying τ_2 since, in the presence of noise, the least damped mode is most strongly excited.

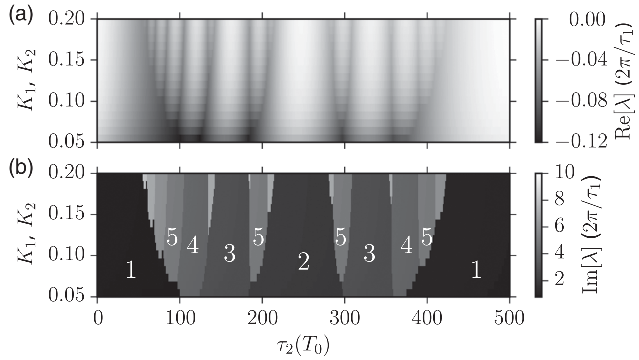

FIG. 2. (a) Real and (b) imaginary parts of the dominant Floquet exponent of the Stuart-Landau system [Eq. (7)], with two feedback terms for $K_1 = K_2$ and $\tau_1 = 500T_0$. Parameters are $\lambda_0 = 2$, $\omega_0 = 5$, and $\gamma = -5$. The integer labels in (b) indicate multiples of $2\pi/\tau_1$.

The frequency component of the dominant Floquet exponents form a Farey tree [35], as is common to systems with competing characteristic times. The number of levels (frequency plateaus) present in the Farey tree—or, in other words, the number of maxima in the damping rate—depends both on the feedback strength and τ_1. The feedback strength dependence can clearly be seen in Fig. 2(b). For increasing values of $K_1 = K_2$ higher order frequency tongues appear, meaning that the number of levels in the Farey tree increases. Although not shown here, increasing τ_1 has, qualitatively, the same effect.

Optimal suppression of the noise-induced modulation of the limit cycle will occur for feedback parameters for which the damping rates are maximized [minimal $\mathrm{Re}[\lambda]$, i.e., the darker regions in Fig. 2(a)], as then the ability of the noise to excite the dominant Floquet modes is reduced [22]. See the Supplemental Material [36] for examples. Comparing various feedback strengths in Fig. 2(a), it is evident that this does not occur for a fixed ratio of τ_1 and τ_2, but that it depends strongly on the feedback strengths. Selecting τ_2 correctly becomes more important the longer τ_1 is, as there are more maxima in the damping rate, meaning that small differences in τ_2 can lead to large differences in the suppression of the noise-induced modulation.

We now compare the results for the Stuart-Landau system with a passively mode-locked laser. See the Supplemental Material [36] for the equations describing the passively mode-locked laser system with two external feedback cavities [37]. Also in the Supplemental Material [36] are results for the FitzHugh-Nagumo system with two noninvasive feedback terms [33], which are qualitatively similar to the mode-locked laser results presented here. These two systems exhibit complex dynamics in their respective oscillatory regimes, with both producing a pulsed output. Owing to the complexity of these systems, although the existence of some characteristic equation is mathematically proven for the linearization around the periodic orbit [38–40], a simple characteristic equation cannot easily be derived. We therefore use DDE-BIFTOOL [41] to calculate the Floquet exponents. Results of the DDE-BIFTOOL calculations, in dependence of τ_2, are shown for the mode-locked laser in Figs. 3(a) and 3(b), where the real and imaginary parts of the three dominant Floquet exponents are plotted. The Floquet exponents show similar dependence to that found for the Stuart-Landau system in Fig. 2. Despite the more complex dynamics with time-varying amplitudes, fitting of the numerically obtain results shows that the Floquet spectra can be described with a simple characteristic equation in the form of Eq. (7),

$$\lambda = -(K_1^{\mathrm{eff}} + K_2^{\mathrm{eff}}) + K_1^{\mathrm{eff}} e^{-\lambda\tau_1} + K_2^{\mathrm{eff}} e^{-\lambda\tau_2}, \tag{8}$$

now, however, with effective feedback strengths that depend on the system. The delay times are again restricted to integer multiples of the period of the particular dynamical system, $\tau_1 = n_1 T_0$ and $\tau_2 = n_2 T_0$ for $n_1, n_2 \in \mathbb{N}$. In Fig. 3, the white symbols indicate the three dominant Floquet exponents obtained from the DDE-BIFTOOL calculations. Plotted behind these are the results of the fitted characteristic equation, with the red circles indicating the most dominant Floquet exponent. The only fit parameters are the effective feedback strengths, which are $K_1^{\mathrm{eff}} = K_2^{\mathrm{eff}} = 0.0485$ for the mode-locked laser with feedback strengths $K_1 = K_2 = 0.05$. The effective feedback strengths depend on the internal dynamics of the particular system; however, they are independent of the resonant feedback delay lengths; i.e., fitting the numerically obtained Floquet exponents for different τ_1 values yields the same effective feedback strengths (see the Supplemental Material [36] for an example). This almost perfect agreement between numerics and the fit of Eq. (8) shows that, despite the complex and disparate dynamics exhibited by various oscillatory systems, modulation

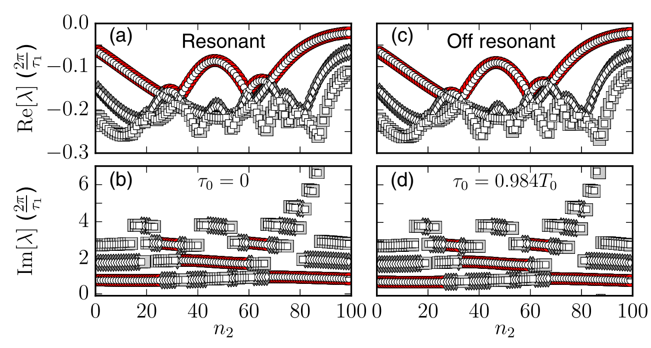

FIG. 3. (a),(c) Real and (b),(d) imaginary parts of the three dominant Floquet exponents of a passively mode-locked laser subject to (a),(b) resonant feedback and (c),(d) off-resonant feedback with $\tau_0 = 0.984T_0$. The white markers indicate the numerically calculated values, while the red and grey markers indicate the results of the fitted characteristic equation. Parameters are $K_1 = K_2 = 0.05$, $\tau_1 = 100T_0$, and $K_1^{\mathrm{eff}} = K_2^{\mathrm{eff}} = 0.0485$ for (a),(b) and $K_1^{\mathrm{eff}} = K_2^{\mathrm{eff}} = 0.0465$ for (c),(d). All other parameters are as in the Supplemental Material [36].

suppression occurs in a very similar manner. Differences only occur in how strongly the feedback acts upon the system—i.e., in K^{eff} but not in the form of the characteristic equation. Our results suggest that, using this relatively simple characteristic equation [Eq. (8)], predictions of feedback conditions for optimal suppression of noise-induced modulations can be made for a wide range of systems subject to self-feedback, or coupling, provided that the delay matrix can be written in a diagonal form or that the entries on the diagonal are dominant. This approach also makes it possible to study substantially longer delay times than what is practically possible with numerical tools such as DDE-BIFTOOL, which are restricted due to memory and calculation time requirements.

We have presented results for resonant feedback, i.e., $\tau_{1,2} = n_{1,2}T_0$, where T_0 is the period. It is shown in Ref. [42] that solutions must also exist under modified resonance conditions, i.e., for $\tau_{1,2} = \tau_0 + n_{1,2}T(\tau_0)$, where τ_0 is a small delay ($< T_0$) and $T(\tau_0)$ is the period of the system for feedback with $\tau_1 = \tau_2 = \tau_0$. Under such modified resonance conditions, it can also be possible to describe the dominant Floquet exponents with Eq. (8), as long as the feedback delay times are still close to integer multiples of the resulting period. However, the change in the feedback conditions influences the dynamics of the particular system, resulting in different effective feedback strengths. For example, for the mode-locked laser system with $\tau_0 = 0.984T_0$, the Floquet exponents are described by Eq. (8), with $K_1^{\mathrm{eff}} = K_2^{\mathrm{eff}} = 0.0465$, as can be seen in Figs. 3(c) and 3(d), where the numerically calculated Floquet modes and the results of fitting Eq. (8) are shown. If the delay offset τ_0 results in a significant discrepancy between the delay times and integer multiples of the period, the fit of Eq. (8) becomes less accurate (see the Supplemental Material [36] for examples).

To confirm that the relative modulation suppression can indeed be determined from the Floquet exponents given by Eq. (8), we use the mode-locked laser with $\tau_1 = \tau_0 + 1000T_0$, for $\tau_0 = 0.984T_0$, and compare Floquet spectra (Fig. 4) with numerically calculated power spectra (Fig. 5). The effective feedback strengths used in Fig. 4 are the fitted values from the $n_1 = 100T_0$ case shown in Figs. 3(c) and 3(d). For the calculation of the power spectra, Gaussian white noise $\sqrt{2D}\xi(t)$ with intensity D is added to the system [37]. The τ_2 values used in Figs. 5(a)–5(d) correspond to the positions of the blue vertical lines in Fig. 4. The central peak in all spectra is the third harmonic of the repetition frequency ($f = \Omega/2\pi \approx 40$ GHz) of the laser output. The smaller side peaks are caused by the noise-induced modulation of the oscillatory dynamics. Plotted in green in Fig. 5 is the power spectrum for the case $n_2 = n_1 = 1000$, where the damping rates are very small and frequencies corresponding to all of the larger Floquet exponents are present (note the $1/\tau_1$ spacing of the side peaks). For $n_2 = 90$, the Floquet exponents have comparatively large damping rates, leading to a significant suppression of the side peaks close to the main peak. The second largest Floquet exponent predicted in Fig. 4 has a frequency of about $10/\tau_1$. In agreement with this, the side peaks near the tenth from the center are the least suppressed. At $n_2 = 325$ [Fig. 5(b)], the third harmonic frequency is undamped, while the fundamental and the second harmonic are suppressed. Accordingly, only every third side peak is present in the power spectrum for $n_2 = 325$. Similar comparisons show agreement for the other τ_2 values depicted in Figs. 5(c) and 5(d). The spectra in Fig. 5 show that the lowest power in the side peaks occurs for the example which corresponds to the highest damping rates of the dominant Floquet exponents ($n_2 = 90$). This indicates that the relative damping rates of the dominant Floquet exponents can be used as a guide to select delay times which optimize the suppression of noise-induced modulations. For passively mode-locked lasers, suppressing noise-induced modulations results in a reduction of the timing jitter [32], which is desired for most applications of these devices.

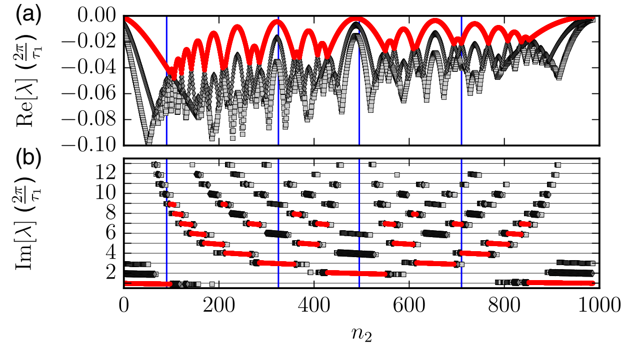

FIG. 4. (a) Real and (b) imaginary parts of the dominant Floquet exponents of the mode-locked laser system as predicted from the fitted characteristic equation (8), depending on n_2, where $\tau_2 = \tau_0 + n_2T_0$. The vertical blue lines indicate the parameters for the power spectra shown in Fig. 5. Parameters are $K_1^{\mathrm{eff}} = K_2^{\mathrm{eff}} = 0.0465$, $\tau_0 = 0.984T_0$, and $\tau_1 = \tau_0 + 1000T_0$.

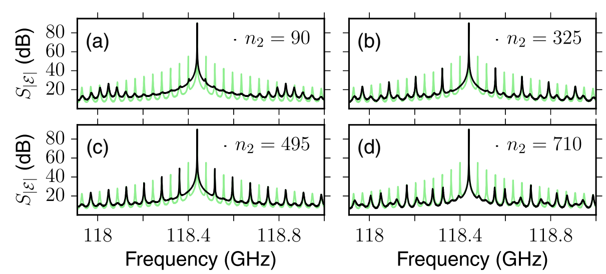

FIG. 5. Power spectra $S_{|\mathcal{E}|}$ of the output of the mode-locked laser system for various feedback delay times $\tau_2 = \tau_0 + n_2T_0$ (corresponding to the vertical lines in Fig. 4). In green is the spectrum for $\tau_2 = \tau_1 = \tau_0 + 1000T_0$. Parameters are $K_1 = K_2 = 0.05$, $\tau_0 = 0.984T_0$, and $\tau_1 = \tau_0 + 1000T_0$, and $n_1 = 90$ for (a), $n_1 = 325$ for (b), $n_1 = 495$ for (c), and $n_1 = 710$ for (d). All of the other parameters are as in the Supplemental Material [36].

In conclusion, we have investigated how the suppression of noise-induced modulations in oscillatory systems subject to feedback depends upon an additional feedback term. For the prototypical Stuart-Landau system, we have derived a simple characteristic equation for the Floquet exponents in the case of resonant feedback. We have shown that the dominant Floquet exponents of more complex oscillatory systems under resonant self-feedback can also be described by the same simple characteristic equation. The local dynamics has been found to enter only implicitly through the effective feedback strengths. We have found that optimal feedback conditions for oscillation suppression do not depend trivially on some fixed ratio of the delay times, but rather on how strongly the feedback acts on the system and influences its stability. We believe that the results we have presented in this Letter can be applied to a wide range of dynamical systems and add insight into the mechanism on how noise-induced modulations can be suppressed in an optimal way in systems subject to delayed feedback. General derivations of Eq. (8) and characteristic equations for other delayed-coupling schemes, as well as the derivation of analytic expressions for K^{eff} are challenging tasks and warrant further investigation.

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