Localized $AdS_5 \times S^5$ Black Holes

Óscar J. C. Dias,[1,*] Jorge E. Santos,[2,†] and Benson Way[2,‡]
[1]STAG research centre and Mathematical Sciences, University of Southampton, SO17 1BJ, Southampton, United Kingdom
[2]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
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According to heuristic arguments, global $AdS_5 \times S^5$ black holes are expected to undergo a phase transition in the microcanonical ensemble. At high energies, one expects black holes that respect the symmetries of the S^5; at low energies, one expects "localized" black holes that appear pointlike on the S^5. According to anti–de Sitter/conformal field theory correspondence, $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory on a 3-sphere should therefore exhibit spontaneous R-symmetry breaking at strong coupling. In this Letter, we numerically construct these localized black holes. We extrapolate the location of this phase transition, and compute the expectation value of the broken scalar operator with lowest conformal dimension. Via the correspondence, these results offer quantitative predictions for $\mathcal{N} = 4$ SYM theory.

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Introduction.—Since its discovery, the duality between type IIB supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with a large N gauge group $SU(N)$ and large 't Hooft coupling remains our best understood example of a gauge-gravity duality [1–4]. This example is also one of the few instances where both sides of the correspondence are explicitly known.

Like other such examples with string theoretic origins, the gravity theory contains a compact space: the S^5. According to the duality, this S^5 is dual to the R symmetry of the gauge theory. Heuristic arguments on the gravity side suggest that black holes can spontaneously break the symmetries of the S^5, an effect which would be dual to R-symmetry breaking [5–7]. However, this aspect of the duality remains largely unexplored.

Let us review these heuristic arguments. When the gravity theory is asymptotically *global* $AdS_5 \times S^5$, the dual gauge theory lives in a background conformal to $\mathbb{R}^{(t)} \times S^3$. Even though the gauge theory itself is scale invariant, the curvature of the S^3 provides a length scale L that can potentially allow for phase transitions to occur. In the gravity theory, L is equal to the AdS length scale and the radius of the S^5.

Meanwhile, black holes are dual to thermal states in the gauge theory. The most well-understood black holes in the gravity theory are those that preserve the symmetries of the S^5, such as AdS_5-Schwarzschild $\times S^5$ ($AdSSchw_5 \times S^5$), which has horizon topology $S^3 \times S^5$. Since the S^5 radius is fixed, its entropy at low energies is determined by the size of the S^3, which gives the entropy scaling $S \sim E^{3/2}$.

However, one expects the existence of spherical black holes that "localize" on the S^5. These are black holes that are small enough to appear pointlike on the S^5 and are affected by the full 10-dimensional geometry. These have horizon topology S^8, so its entropy at low energies scales as $S \sim E^{8/7}$. They would, therefore, compete with $AdSSchw_5 \times S^5$ and dominate at low energies.

As the size of the localized black hole is increased, more of the S^5 gets filled by the black hole. When the entire S^5 gets filled, the horizon changes topology to $S^3 \times S^5$, but contains inhomogeneous deformations in the S^5 directions. Such solutions are called "lumpy" black holes [8].

The existence of lumpy black holes can also be inferred from an instability of $AdSSchw_5 \times S^5$ black holes [9]. When the S^3 horizon radius is much smaller than the S^5 radius, there is a separation of horizon length scales. This separation causes an instability, much like the Gregory-Laflamme instability affecting black strings [10–16]. New solutions often branch off from the critical onset of such instabilities, and these are the lumpy black holes.

Since localized black holes and lumpy black holes only exist for energy scales small compared to the S^5 radius, there must be a phase transition to $AdSSchw_5 \times S^5$ black holes. Because $AdSSchw_5 \times S^5$ respects the S^5 symmetries, but the localized black holes and lumpy black holes do not, this transition spontaneous breaks this symmetry. By the duality, this transition is dual to R-symmetry breaking in the gauge theory.

Beyond this qualitative picture, little is known about this phase transition. Though the critical onset of the instability was located in Ref. [9], later work [8] has demonstrated that lumpy black holes near this onset have *less* entropy than $AdSSchw_5 \times S^5$. Therefore, the onset of the instability cannot be the location of the phase transition, and the transition must be first order. Without further information about the localized black holes, the location of the phase transition, and the expectation value of the scalar operators in the broken phase remain unknown.

In this Letter, we construct these localized black holes numerically. We demonstrate that these solutions entropically dominate over $\mathrm{AdSSchw}_5 \times \mathrm{S}^5$ at small energies, extrapolate the location of the phase transition, and compute the expectation value of a scalar operator in the dual field theory. Through the duality, these results offer new quantitative predictions for $\mathcal{N} = 4$ SYM.

Numerical approach.—The minimal field content in type IIB supergravity that can be asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^5$ consists of a metric g and a self-dual 5-form $F_{(5)} = \mathrm{d}C_{(4)}$. Their equations of motion are

$$E_{\mathrm{MN}} \equiv R_{\mathrm{MN}} - \frac{1}{48} F_{\mathrm{MPQRS}} F_N{}^{\mathrm{PQRS}} = 0, \tag{1a}$$

$$\nabla_M F^{\mathrm{MPQRS}} = 0, \tag{1b}$$

$$F_{(5)} = \star F_{(5)}. \tag{1c}$$

We seek static, topologically S^8 black hole solutions that are asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^5$. Gravitational intuition suggests that the most symmetric of such black holes will have the largest entropy. These have $\mathbb{R}^{(t)} \times SO(4) \times SO(5)$ symmetry, where the full $SO(4)$ symmetry of AdS_5 and the largest subgroup of $SO(6)$ are preserved.

We find these solutions using the DeTurck method [15–17]. This method first requires the choice of a reference metric $\bar{g}$ that has the same symmetries and causal structure as the desired solution. For our localized black holes, the reference metric must contain a horizon, be asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^5$, and contain three axes: one for an S^3, and the "north" and "south" poles of the S^5. Because these requirements lead to an integration domain with five boundaries, we opt to work in two separate coordinate systems, one which is adapted to the horizon, and another which is adapted to asymptotic infinity.

We therefore take the reference metric to be

$$\begin{aligned}
\overline{ds}^2 &= \frac{L^2}{(1-y^2)^2}\left[-\frac{1}{L^2}H_1 dt^2 \right.\\
&\quad \left. + H_2\left(\frac{4dy^2}{2-y^2} + y^2(2-y^2)d\Omega_3^2\right)\right]\\
&\quad + L^2 H_2\left[\frac{16dx^2}{2-x^2} + 4x^2(2-x^2)(1-x^2)^2 d\Omega_4^2\right]\\
&= -M\frac{(\rho^7-\rho_0^7)^2}{(\rho^7+\rho_0^7)^2}dt^2 + L^2 H_2\left[d\rho^2 + \rho^2\left(\frac{4d\xi^2}{2-\xi^2}\right.\right.\\
&\quad \left.\left. + G_1\xi^2(2-\xi^2)d\Omega_3^2 + G_2(1-\xi^2)^2 d\Omega_4^2\right)\right],
\end{aligned} \tag{2}$$

where ρ_0 is a constant, H_1, H_2, M, G_1, and G_2 are scalar functions (to be specified shortly), and the line elements are equated via the coordinate transformation

$$\begin{aligned}
y &= \sqrt{1 - \mathrm{sech}(\rho\xi\sqrt{2-\xi^2})},\\
x &= \sqrt{1 - \sin\left(\frac{1}{2}\rho(1-\xi^2)\right)}.
\end{aligned} \tag{3}$$

The reference metric (2) has the horizon at $\rho = \rho_0$, asymptotic infinity at $y = 1$ ($\rho \to \infty$), the S^3 axis at $y = 0$ ($\xi = 0$), the S^5 north pole at $x = 1$ ($\xi = 1$), and the south pole at $x = 0$.

It now remains for us to specify the functions. Equality of line elements via Eq. (3) implies that only M (which determines H_1), and H_2 remain unspecified. We must choose these so that $H_1 = H_2 = 1$ at $y = 1$ in order to recover $\mathrm{AdS}_5 \times \mathrm{S}^5$ asymptotically. We must also have $M \propto H_2$ at $\rho = \rho_0$ in order to have a regular horizon. (The axes are already manifestly regular.) To satisfy these requirements, we take

$$\begin{aligned}
M &= 1 + \frac{(\rho_0^7-\rho^7)^2}{(\rho^7+\rho_0^7)^2}\sinh^2\left(\rho\xi\sqrt{2-\xi^2}\right),\\
H_3 &= (1+\rho_0^7/\rho^7)^{4/7}.
\end{aligned} \tag{4}$$

This choice has the added benefit that for small ρ_0, the geometry near the horizon is approximately asymptotically flat Schwarzschild in isotropic coordinates.

With a reference metric, the DeTurck method then modifies the Einstein equation (1a) to

$$E_{\mathrm{MN}} - \nabla_{(M}\xi_{N)} = 0, \qquad \xi^M \equiv g^{PQ}[\Gamma^M_{\mathrm{PQ}} - \bar{\Gamma}^M_{\mathrm{PQ}}], \tag{5}$$

where Γ^M_{PQ} and $\bar{\Gamma}^M_{\mathrm{PQ}}$ define the Levi-Civita connections for g and $\bar{g}$, respectively. Unlike Eq. (1a), this equation yields PDEs that are elliptic in character. But after solving these PDEs, we must verify that $\xi^M = 0$ to confirm that (1a) is indeed solved. Local uniqueness properties of elliptic equations guarantee that solutions with $\xi^M = 0$ are distinguishable from those with $\xi^M \neq 0$. The condition $\xi^M = 0$ also fixes all gauge freedom in the metric.

We choose a general ansatz that is consistent with the symmetries. The gauge potential takes the form

$$C_{(4)} = L^3 F dt \wedge d\mathrm{S}_{(3)} + L^4 W d\mathrm{S}_{(4)}, \tag{6}$$

where F and W are unknown functions. The function W can be algebraically eliminated from the equations of motion and determined after the metric and F are known.

For boundary conditions at infinity $y = 1$ ($\rho \to \infty$), we require global $\mathrm{AdS}_5 \times \mathrm{S}^5$ asymptotics. The remaining boundary conditions are determined by regularity. To handle the five boundaries numerically, we divide the integration domain into a number of nonoverlapping warped rectangular regions or "patches" as shown in Fig. 1. The four patches far from the horizon use $\{x, y\}$ coordinates, while the remaining patch uses $\{\rho, \xi\}$ coordinates.

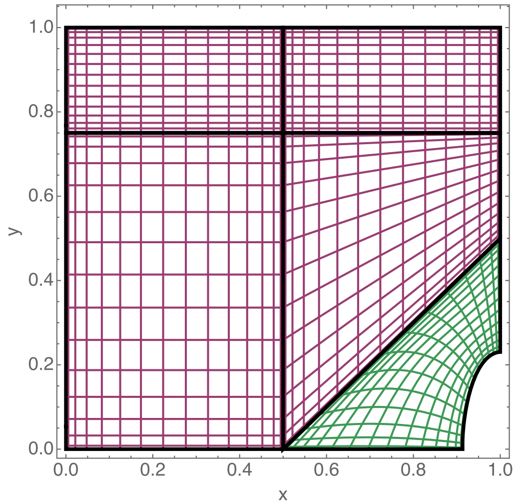

FIG. 1. Integration domain in $\{x, y\}$ coordinates. The green patch near the horizon is mapped from $\{\rho, \xi\}$ coordinates.

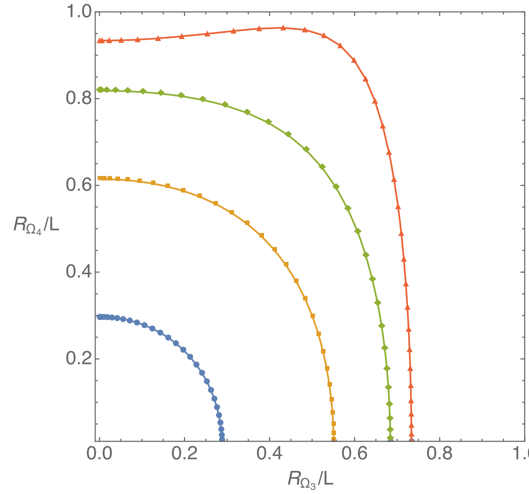

FIG. 2. Radii of the S^3 and S^4 along the horizon. From the bottom left to the top right $T = \{1.90, 0.945, 0.708, 0.538\}$.

We require that the metric g, the form $C_{(4)}$, and their first derivatives match across patch boundaries.

We therefore have a boundary value problem for 7 functions in two dimensions. L drops out of the equations of motion, so the only parameter is ρ_0, which fixes the black hole temperature [18]. We solve the system using a Newton-Raphson algorithm with the reference metric and $F = H_1 y^4(2-y^2)/\sqrt{2}(1-y^2)^4$ at $\rho_0 = 0.1$ as a first seed. We use pseudospectral collocation with transfinite interpolation of Chebyshev grids in each patch, and the linear systems are solved by LU decomposition.

We were able to reach a maximum value of $\rho_0 = 0.85$, with all solutions satisfying $\xi^2 < 10^{-10}$. Our numerical method converges according to standard expectations, and we have also checked that a Komar identity (see, e.g., Ref. [19]) is satisfied within 0.1%. (Details can be found in the Supplemental Material [20].)

Results.—In Fig. 2 we show the radii R_{Ω_3}, R_{Ω_4} of the geometrically preserved S^3 and S^4 along the horizon. This curve at small ρ_0 (high temperatures) is approximated by $R_{\Omega_3}^2 + R_{\Omega_4}^2 \approx 2^{4/7}\rho_0^2 L^2$, implying that the horizon is nearly spherical. At larger ρ_0 (lower temperatures), the horizon is much more deformed.

Now we compute thermodynamic quantities. The temperature T is fixed by ρ_0. The entropy S is found by integrating the horizon area. The energy E is computed using the formalism of Kaluza-Klein holography and holographic renormalization [8,21–28] (technical details can be found in Ref. [8] and the Supplemental Material [20]). By the AdS/CFT dictionary, the 10- and 5-dimensional Newton constant can be converted to the number of colors N of $\mathcal{N} = 4$ SYM theory via $G_{10} = (\pi^4/2)(L^8/N^2)$ and $G_5 = (G_{10}/\pi^3 L^5)$. These thermodynamic quantities numerically satisfy the first law $dE = TdS$ to $< 0.1\%$ error.

In the microcanonical ensemble, the energy is fixed, and the dominant solution has the most entropy. In Fig. 3, we show $\Delta S/N^2$ vs EL/N^2 for various competing solutions, where $\Delta S \equiv S - S_0$ with S_0 being the entropy of $\mathrm{AdSSchw}_5 \times S^5$ [29,30]. Data for the localized black holes as well as a fit are shown by the solid purple line and its points. As a check, we note that this curve is well approximated by 10-dimensional Schwarzschild (the brown dotted line) at low energies.

We see that the localized black holes have the largest entropy among known solutions for $EL/N^2 \lesssim 0.225$. Above this energy, $\mathrm{AdSSchw}_5 \times S^5$ is dominant. The value $EL/N^2 \approx 0.225$ (black dot in Fig. 3) therefore marks the point of a first-order phase transition ($S/N^2 \approx 0.374$ at this energy). We note that while this value is obtained by extrapolating a fit, the error in the fit parameters is under

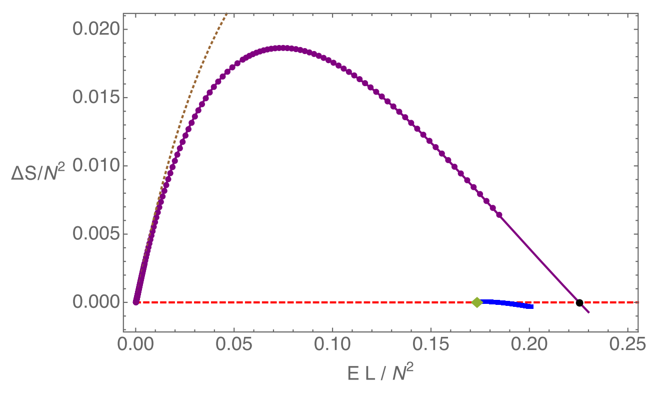

FIG. 3. Microcanonical phase diagram: entropy with respect to that of $\mathrm{AdSSchw}_5 \times S^5$ versus energy. The dashed red line is the $\mathrm{AdSSchw}_5 \times S^5$ phase, which is dynamically unstable for $EL/N^2 \lesssim 0.173$ [9] (green diamond). The blue squares are the lumpy black hole phase [8]. The solid purple curve and its points describe the localized black holes and a fit of their data. The brown dotted line is the lowest-order 10-dimensional Schwarzschild approximation. There is a first-order phase transition at $EL/N^2 \approx 0.225$ (black dot).

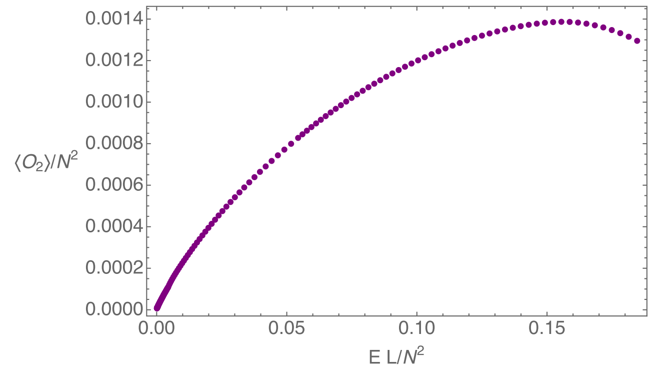

FIG. 4. Dimension 2 scalar condensate versus energy.

0.03%. Furthermore, other extrapolation methods like polynomial interpolation are within 2% of this value.

We also point out that $\mathrm{AdSSchw}_5 \times \mathrm{S}^5$ black holes are unstable for $EL/N^2 \lesssim 0.173$ [9] (green diamond in Fig. 3), and we have data for localized black holes in this energy range. The fact that localized black holes have more entropy than $\mathrm{AdSSchw}_5 \times \mathrm{S}^5$ for these energies indicates that they are a plausible end point to the instability.

The lumpy black holes are shown by the blue squares in Fig. 3. They are subdominant. As mentioned in the introduction, they are connected to the onset of the instability (green diamond). We expect this phase to eventually join up with the localized black hole phase.

Recall from the introduction that localized black holes are dual to a thermal state where the R symmetry of $\mathcal{N} = 4$ SYM theory has been spontaneously broken [in our case down to $SO(5)$]. This results in a condensation of an infinite tower of scalar operators with increasing conformal dimension. The lowest conformal dimension is 2, and the associated scalar operator is

$$\mathcal{O}_2 = \frac{2}{g_{\mathrm{YM}}^2}\sqrt{\frac{5}{3}}\mathrm{Tr}\left[(X^1)^2 - \frac{1}{5}\sum_{i=2}^{6}(X^i)^2\right], \tag{7}$$

where X^i the are the six real scalars of $\mathcal{N} = 4$ SYM theory in the vector representation of $SO(6)$ and g_{YM} is the coupling constant (see, e.g., Ref. [31] for the action of $\mathcal{N} = 4$ SYM theory). The expectation value $\langle \mathcal{O}_2 \rangle$ in the broken phase can be found from the supergravity solution through the formalism of Kaluza-Klein holography [8,21–28] (some details are also in the Supplemental Material [20]). We show $\langle \mathcal{O}_2 \rangle$ for a range of energies in Fig. 4. Because the symmetry breaking transition is first order, $\langle \mathcal{O}_2 \rangle$ will have a nonzero value at the phase transition.

For completeness, less us discuss the canonical ensemble where the temperature is fixed and the solution with lowest free energy $F = E - TS$ dominates. There is the first-order Hawking-Page phase transition [29] between large $\mathrm{AdSSchw}_5 \times \mathrm{S}^5$ black holes at higher temperatures and thermal $\mathrm{AdS}_5 \times \mathrm{S}^5$ at lower. In the gauge theory, this is dual to a confinement or deconfinement transition [30]. All other known solutions, namely, the localized and lumpy black holes, never dominate the canonical ensemble. (The Supplemental Material [20] contains a phase diagram.)

Discussion.—To summarize, we have numerically constructed asymptotically global $\mathrm{AdS}_5 \times \mathrm{S}^5$ localized black holes in type IIB supergravity. These black holes are topologically S^8 and are more entropic than any other known solution at low energies. At higher energies near $EL/N^2 \approx 0.255$, there is a first-order phase transition to $\mathrm{AdSSchw}_5 \times \mathrm{S}^5$ black holes. By the AdS/CFT correspondence, these localized black holes are dual to a thermal state of $\mathcal{N} = 4$ super Yang Mills (with a large N gauge group and large ‘t Hooft coupling) theory with spontaneously broken R symmetry. The scalar sector with the broken symmetry contains a dimension-2 operator with an expectation value shown in Fig. 4 and preserves a $SO(5)$ subgroup of the $SO(6)$ R symmetry.

As a test of AdS/CFT, it would be desirable to reproduce these results from the gauge theory side. So far, lattice tests of AdS/CFT rely on finite temperature, and have been restricted to the canonical ensemble [32–38]. However, there has been recent progress in first-order phase transitions in several ensembles [39–41].

The completion of the phase diagram in Fig. 3 can be conjectured from other systems with Gregory-Laflamme instabilities [11–15,42,43] (see reviews in Refs. [14,16]). We expect the lumpy black hole to meet with the localized black holes in the space of solutions. For this to happen without violating the first law, there must be a cusp somewhere in the S/N^2 vs EL/N^2 curve. These families must meet at a topological transition point, which would be a solution with a naked singularity. Analogous systems with Gregory-Laflamme instabilities suggest that this topological transition point is closer to the lumpy black hole side of the curve. That is, that the cusp would be a topologically S^8 black hole.

Let us now comment on dynamical evolution. Entropy arguments suggest that the evolution of unstable $\mathrm{AdSSchw}_5 \times \mathrm{S}^5$ black holes would proceed towards the most dominant solution, which are the localized S^8 black holes. This entails a violation of cosmic censorship, much like in the evolution of the black string [44] or black ring [45]. Whether or not the evolution proceeds in this way, and the implications for $\mathcal{N} = 4$ SYM theory if cosmic censorship is violated remain important open problems. Interestingly, there is a range of energies $0.173 \lesssim EL/N^2 \lesssim 0.225$ where $\mathrm{AdSSchw}_5 \times \mathrm{S}^5$ is subdominant in entropy but nevertheless dynamically stable. In the field theory, this means that the time scale for spontaneous symmetry breaking at these energies is exponentially suppressed in N, compared to those at lower energies.

Many localized solutions dual to $\mathcal{N} = 4$ SYM theory states remain to be studied. In global $\mathrm{AdS}_5 \times \mathrm{S}^5$, there are localized solutions that break more symmetries, but these are likely less entropic than the ones preserving $SO(5)$. There are other localized solutions arising from higher harmonics of the Gregory-Laflamme instability [8,46]. In particular, the next harmonic leads to double S^8 black holes and $S^4 \times S^4$ 'black belts' [8]. However, these require a delicate balancing of forces and are likely unstable. Rotational effects remain largely unexplored except for the onset of the Gregory-Laflamme instability for equal spin black holes [16]. Beyond global $AdS_5 \times S^5$, there is freedom to choose a different gauge theory background than one conformal to $\mathbb{R}^{(t)} \times S^3$. This can yield novel physics like plasma balls and boundary black holes (see Ref. [47] for a review), but none of these studies have included the effects of localization.

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*ojcd1r13@soton.ac.uk
†jss55@cam.ac.uk
‡bw356@cam.ac.uk

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