Experimental Test of Residual Error-Disturbance Uncertainty Relations for Mixed Spin-½ States

Bülent Demirel,[1] Stephan Sponar,[1] Georg Sulyok,[1] Masanao Ozawa,[2] and Yuji Hasegawa[1,*]
[1]Atominstitut, TU Wien, Stadionallee 2, 1020 Vienna, Austria
[2]Graduate School of Information Science, Nagoya University, Chikusa-ku, Nagoya 464-8601, Japan
(Received 23 December 2015; revised manuscript received 18 April 2016; published 27 September 2016)

The indeterminacy inherent in quantum measurements is an outstanding character of quantum theory, which manifests itself typically in the uncertainty principle. In the last decade, several universally valid forms of error-disturbance uncertainty relations were derived for completely general quantum measurements for arbitrary states. Subsequently, Branciard established a form that is optimal for spin measurements for some pure states. However, the bound in his inequality is not stringent for mixed states. One of the present authors recently derived a new bound tight in the corresponding mixed state case. Here, a neutron-optical experiment is carried out to investigate this new relation: it is tested whether error and disturbance of quantum measurements disappear or persist in mixing up the measured ensemble. The attainability of the new bound is experimentally observed, falsifying the tightness of Branciard's bound for mixed spin states.

DOI: 10.1103/PhysRevLett.117.140402

Quantum measurement, through which a value of a physical property is assigned, has always eluded our consistent, physical understanding [1]. The uncertainty principle proposed by Heisenberg [2] in 1927 states that it is impossible to measure two conjugate observables with arbitrary precision. Exemplary for a preparation uncertainty, Kennard [3] proved the inequality $\Delta q \Delta p \geq \hbar/2$ for standard deviations Δq and Δp for position q and momentum p. Robertson generalized this relation to an arbitrary pair of noncommuting observables A, B for a given quantum state $|\psi\rangle$ replacing the lower limit $\hbar/2$ by the bound $C_{AB} = \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|$ [4]. Heisenberg's initial idea of an uncertainty for successive measurements, referred to as error-disturbance uncertainty relation (EDUR), is, however, not captured by Kennard's relation. A generally valid formulation of EDUR was given only much more recently by Ozawa [5,6] as

$$\epsilon(A)\eta(B) + \epsilon(A)\Delta B + \eta(B)\Delta A \geq C_{AB}. \tag{1}$$

The error $\epsilon(A)$ for measuring an observable A and the disturbance $\eta(B)$ caused on an observable B are defined in Ref. [5] as

$$\begin{aligned}\epsilon(A)^2 &= \mathrm{Tr}[(U^\dagger(\mathbb{1}\otimes M)U - A\otimes\mathbb{1})^2\rho\otimes|\xi\rangle\langle\xi|],\\ \eta(B)^2 &= \mathrm{Tr}[(U^\dagger(B\otimes\mathbb{1})U - B\otimes\mathbb{1})^2\rho\otimes|\xi\rangle\langle\xi|].\end{aligned} \tag{2}$$

The relations in Eq. (2) characterize how the input state ρ of the object system is processed in an apparatus, described by an indirect measurement model with parameters $|\xi\rangle$, U, and M. Here $|\xi\rangle$ is the apparatus's initial state, U is the unitary operator describing the interaction between object and apparatus system, and M is the meter observable of the apparatus [6]. To put it more comprehensively, the error $\epsilon(A)$ quantifies the deviation between the intended measurement and the measurement actually performed in the apparatus. In the same manner the disturbance $\eta(B)$ describes how accurate the value of the observable B can be determined after the approximative A measurement was carried out.

In pursuit of an improvement of relation (1), a stronger inequality

$$\epsilon(A)^2\Delta B^2 + \eta(B)^2\Delta A^2 + 2\epsilon(A)\eta(B)\sqrt{\Delta A^2\Delta B^2 - C_{AB}^2} \geq C_{AB}^2 \tag{3}$$

was introduced by Branciard [7]. The validity of both relations were experimentally tested with neutrons [8–10] and photons [11–14] solely in case of pure states. Other approaches to measuremental uncertainty relations can be found for example in Refs. [15–19].

Our studies are not limited to measurements of physical quantities on a pure single quantum system [20], but are rather concerned with statistical ensembles of a quantum system reflecting actual circumstances. All information of physical importance is, thus, attributed to a statistical state, represented by a so-called density matrix [21]. There is no uniqueness of the representation of a mixed state as a convex sum of pure states [22]; i.e., the same mixed-state density matrix can be obtained with different *blends* for that [23,24]; experiments can distinguish the difference in mixture but no evidence can be found in different generation methods of the mixture. All *as-if realities* consisting in blending are not accessible, turning to be virtual [24]. Nevertheless, (phase) mixture occurring due to dephasing in double-slit experiments can easily wash out interference

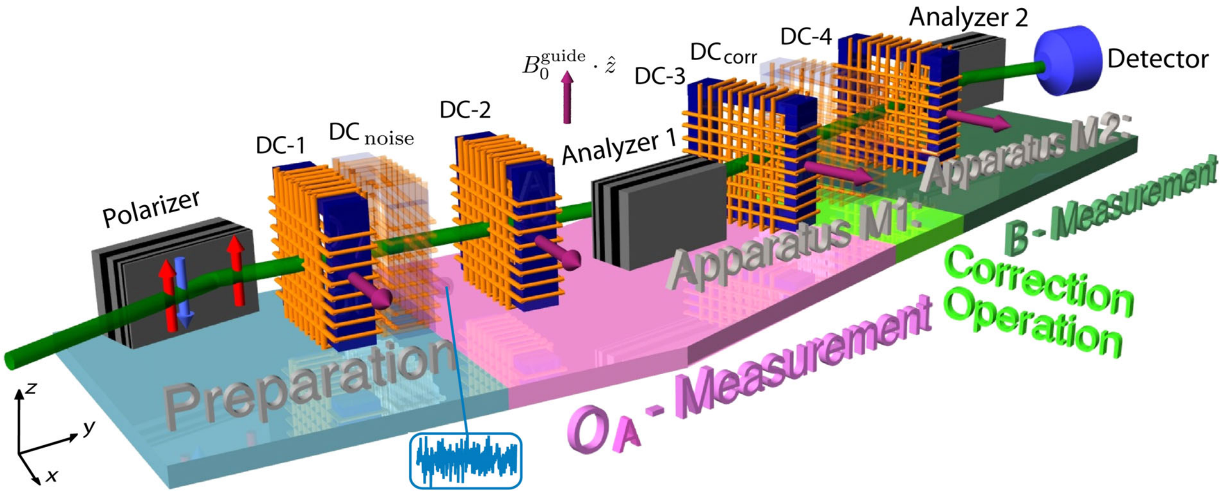

FIG. 1. Illustration of the experimental setup. The neutron polarimeter setup consists of three stages. (i) Preparation (blue region): a monochromatic neutron beam is polarized in the $+z$ direction by passing through a supermirror spin polarizer. In the coil (DC-1) the directions and mixture of the input states are adjusted. (ii) Apparatus $M1$, consisting of a projective O_A measurement (pink region) and a correction operation (light green region): the first measurement is carried out by analyzer 1 together with the coils (DC-3/4) followed by a unitary rotation of the output state of the O_A measurement. (iii) Apparatus $M2$, measuring B (dark green region): the second measurement carried out by the coil (DC-4) together with the analyzer 2 is fixed to make a B measurement. Transparent coils are virtual, in practice other dc coils fulfill their tasks.

fringes; i.e., quantum interference vanishes for mixed states and quasiclassical behavior can emerge in certain circumstances [25,26].

An extension of the Robertson bound C_{AB} in the EDURs to $C'_{AB} = \frac{1}{2}|\mathrm{Tr}([A,B]\rho)|$ is not stringent for mixed states and entails a disappearance of uncertainty for totally mixed ensembles represented by ρ [27]. Improvement of the bound was put forward by Ozawa [28] who showed that C_{AB} in Eq. (3) can be replaced by a stronger quantity D_{AB} defined as $D_{AB} = \frac{1}{2}\mathrm{Tr}(|\sqrt{\rho}[A,B]\sqrt{\rho}|)$. This new parameter coincides with the Robertson bound C_{AB} when ρ is a pure state, but makes the EDUR in the form of Eq. (3) stronger for a mixed ensemble. This new relation is experimentally tested here for the first time. Our experiments study whether the uncertainty vanishes or remains for mixed states, illuminating its residual character.

The considerations so far are all valid for a general, arbitrary pair of noncommuting observables. As the simplest case, spin-½ observables, represented by a set of Pauli operators, have been a major focus of investigations of EDURs. Branciard [7] showed that for binary measurements with $A^2 = B^2 = \mathbb{1}$ and $\langle A\rangle = \langle B\rangle = 0$, where $\langle\cdots\rangle$ stands for the expectation value in the system state, Eq. (3) can be strengthened to a stronger EDUR. Ozawa demonstrated that replacement of the bound C'_{AB} by D_{AB} [28] improves the inequality in the binary case as well.

We investigate projective spin measurements where Eq. (2) can be simplified to operator biases

$$\epsilon(A)^2 = \langle (O_A - A)^2\rangle,$$
$$\eta(B)^2 = \langle (O_B - B)^2\rangle, \tag{4}$$

between the observables actually measured O_A [O_B] and the observables intended to be measured A [B]. The output observables are linked to the decomposed meter observable $M_m = |A = m\rangle\langle O_A = m|$ of the measurement apparatus, where m can have the outcome ± 1, by $O_A = \sum_m m M_m^\dagger M_m$ and $O_B^{(k)} = \sum_m M_m^\dagger B^k M_m$. In our experiments we choose $A = \sigma_z$, $B = \sigma_y$ and consider a mixed ensemble ρ satisfying $\langle A\rangle = \langle B\rangle = 0$; then, ρ is generally parametrized as $\rho_x(\alpha) = \frac{1}{2}(\mathbb{1} + \alpha\sigma_x)$. In this case, the bound $D_{AB} = 1$ is constant and yields the tight relation [28]

$$(\epsilon(A)^2 - 2)^2 + (\eta(B)^2 - 2)^2 \leq 4, \tag{5}$$

for any $\rho_x(\alpha)$ independent of the mixture of the state, while the bound C'_{AB} does depend on the parameter α.

We carried out a neutron polarimeter experiment at the 250 kW TRIGA Mark-II research reactor at TU Wien, Austria, as depicted in Fig. 1. The incident neutrons with a wavelength $\lambda \cong 2.02$ Å, are polarized by a spin-dependent reflection on a multilayer structure, referred to as supermirror. A guide field between polarizer and analyzer 1 and between analyzer 1 and analyzer 2 in the +z direction is applied and determines the quantization axis. Based on the previous performance of the studies of the EDUR for pure states [8–10], we extend here the investigation by applying two procedures, i.e., the generation of mixed states and modification of the first measurement process in apparatus ($M1$) by unitary transforming the output states. The former allows the study of the EDUR for mixed states and the latter enables us to tune the disturbance.

The polarimeter setup consists of three stages: (i) state preparation, (ii) apparatus $M1$ performing a projective O_A measurement plus the correction procedure and (iii) apparatus $M2$ performing the B measurement. Larmor precession induced by magnetic fields B_x in the dc coils allows us to orient all required directions of the spin measurements. The mixing of the state can be tuned by a noise magnetic field $B_{\rm noise}$ [29]. In practice, we realize $\pi/2$ rotations with noisy fields by one dc coil (DC-1), where the required mixture can be adjusted by the amplitude of the noise signal.

In the first stage, the input states are chosen to be $\rho_x(\alpha) = \frac{1}{2}(\mathbb{1} + \alpha\sigma_x)$. The case $\alpha = 1$ corresponds to the pure input state $|x\rangle$, whereas $\alpha = 0$ yields a completely mixed ensemble. For the entire experiment five different mixtures $\alpha = \{1, 0.75, 0.5, 0.25, 0\}$ were realized by tuning the strength of the noisy magnetic field. The orientation and degree of mixture of the input states were verified by measuring the expectation values of the Pauli-spin operators $\mathrm{Tr}(\sigma_i\rho_x)$ for $i = x,\ y,\ z$ each. Typical state fidelity $F = \mathrm{Tr}(\sqrt{\sqrt{\rho_x}\rho_x^{\rm exp}\sqrt{\rho_x}})$ of the pure input state ρ_x was 0.982(5). We applied the so-called "three-state-method" [6]. Two additional states to the original input state $\rho_x(\alpha)$ are generated and sent to the apparatus: in practice, ancilla states are set as $\rho_x(-\alpha)$ $[= A\rho_x(\alpha)A]$ and A's eigenstate $|+z\rangle$ $(= A|+z\rangle)$ $\{\rho_x(-\alpha)[= B\rho_x(\alpha)B]$ and B's eigenstate $|+y\rangle$ $(= B|+y\rangle)\}$ to determine the error (disturbance) (see the Supplemental Material for more details on this).

The second stage represents the apparatus $M1$ in which the coils dc-2/3 plus analyzer 1 carry out projective measurement of the observable $O_A = \cos(\theta_{OA})\sigma_z + \sin(\theta_{OA})\sigma_y$ for all input states. The parameter θ_{OA} is the detuning angle of this measurement. For $\theta_{OA} = 0$ the observable O_A is identical to A for which the error is expected to vanish, while for $\theta_{OA} = \pi/2$, the observable O_A equals B where the disturbance is supposed to be the smallest. After the measurement the neutron leaves the apparatus in the eigenstates $|O_A = \pm 1\rangle$ of the first measurement.

In the correction stage, we investigate the influence of an unitary transformation $U^{\rm corr}$ on the output state $|O_A = \pm 1\rangle$, which rotates the state just after measurement in $M1$ and before the second one. Thereby, one can realize an optimal (and anti-optimal) correction by adjusting $U^{\rm corr}$. Note that in our previous study [8–10] the unitary operation $U^{\rm corr}$ is not applied and fixed as $U^{\rm corr} = \mathbb{1}$ in practice. The last stage consists of the measurement of $B = \sigma_y$ in the state $U^{\rm corr}|O_A = \pm 1\rangle$ in apparatus $M2$ which is accomplished again by a dc coil (DC-4) plus analyzer (analyzer 2) combination. Results of the second measurement are expected to be affected both by the disturbance at the first measurement and the intermediate unitary transformation $U^{\rm corr}$. Note that a spin rotation after the apparatus $M2$ is not applied, since it has no influence on the detected count rates. The entire sequence of the successive projective measurements is given by

$$\begin{cases} \rho_x(\alpha) \\ \rho_x(-\alpha) \overset{M1}{\to} |O_A = \pm 1\rangle \overset{\rm Corr}{\to} U^{\rm corr}|O_A = \pm 1\rangle \\ |z\rangle, |y\rangle \end{cases}$$
$$\overset{M2}{\to} |O_A = \pm 1, O_B = \pm 1\rangle.$$

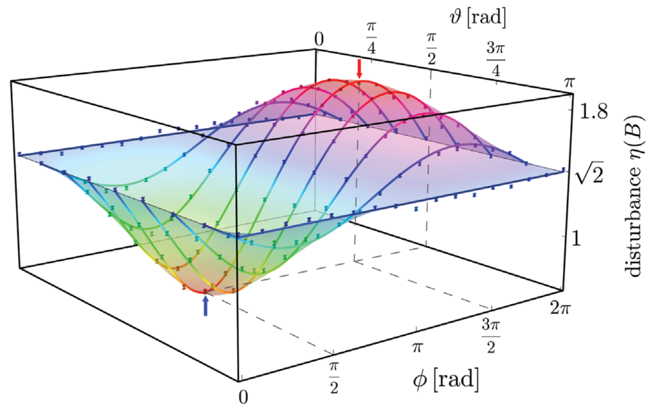

FIG. 2. Influence of the correction procedure on the disturbance. After the projective measurement of $O_A(\theta_{OA} = 5\pi/18)$ plus unitary rotations $U^{\rm corr}$ [with angle parameters (ϑ, ϕ) for the output state of the apparatus $M1$], the measurement of $B = \sigma_y$ is performed in apparatus $M2$. The angles identify the output states $|\psi(\vartheta,\phi)\rangle = (\cos(\vartheta/2), e^{i\phi}\sin(\vartheta/2))^T$ and $|{-\psi}(\vartheta,\phi)\rangle = |\psi(\pi-\vartheta, \phi+\pi)\rangle$ of the unitary operation. Blue and red arrow indicate the position of the minimal $(\pi/2, 3\pi/2)$ and maximal $(\pi/2, \pi/2)$ disturbance.

Our first study examines the influence of the unitary transformation in apparatus ($M1$). First, pure input states are generated and the detuning angle θ_{OA} is fixed at $5\pi/18$. Then, the eigenstate of O_A after apparatus $M1$ is unitarily transformed to the state $|\psi(\vartheta,\phi)\rangle = (\cos(\vartheta/2), e^{i\phi}\sin(\vartheta/2))^T = U^{\rm corr}|O_A(5\pi/18)\rangle$. The measured disturbance as a function of the polar and azimuthal angle (ϑ, ϕ) is plotted in Fig. 2. This plot clearly exhibits the decrease and increase of disturbance by the choice of ϑ and ϕ. It is apparent that the minimal and maximal disturbances, illustrated by blue and red arrows in Fig. 2, are achieved when the output state after measurement in ($M1$) is unitarily transformed by $U^{\rm corr}$ into eigenstates of the observable $B = \sigma_y$. (see Supplemental Material [30] for theoretical details of the correction or anticorrection procedure).

After determination of the disturbance-minimizing or -maximizing unitary transformations, the EDUR given by Eq. (5) is analyzed. The experimentally determined error versus maximum and minimum disturbances are plotted in Fig. 3 for pure states together with the theoretically predicted bound. The red shaded area marks the forbidden region. The lower and upper bound was measured for angle

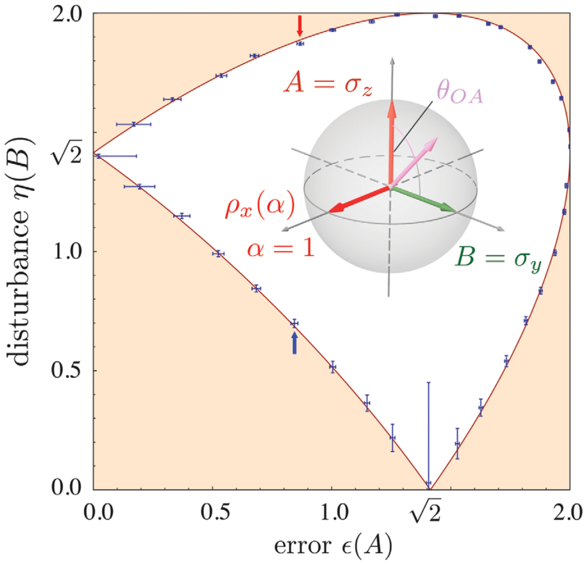

FIG. 3. Error-disturbance uncertainty relation as indicated by inequality Eq. (5) measured with pure states: not only the lower but also upper bounds of the disturbance are found. For a detuning angle of $\theta_{OA} = 0$ the output observable O_A coincides with $A = \sigma_z$ at which point $(\epsilon(A), \eta(B)) = (0, \sqrt{2})$. For increasing angles θ_{OA} the error increases as well and disturbance spreads between the maximum and minimum values. The extremal points are reached at $\theta_{OA} = \pi/2$, at which O_A equals B. For angles from $\pi/2$ to π the EDUR evolves back. Blue and red arrows indicate the points denoted to the maximal and minimal disturbance in Fig. 2.

$\theta_{OA} = [0, \pi]$ with a step of $\pi/18$. For $\theta_{OA} = 0$ we have $\epsilon(A) = 0$ at which point the disturbance is unique. When $\theta_{OA} = \pi/2$ ($O_A = B$), the disturbance reaches its (maximum) minimum value, depending on the unitary (anti-) optimal correction transformation. When $\theta_{OA} = \pi$, $O_A = -A$ the error is maximal and disturbance is independent of the transformation once again. Note that larger statistical errors are seen in the plot for small values than for large $\epsilon(A)$, $\eta(B)$ values: this is due to the nonlinearity, i.e., larger slope of the square-root function for smaller values [see definitions of error and disturbance in Eq. (4)] and due to the propagation of statistical uncertainty. All error intervals represent 1σ of confidence level.

Finally, the influence of the mixture of the input states is studied, by applying the optimal correction procedure for minimal disturbances and tuning the mixture of $\rho_x(\alpha) = \frac{1}{2}(\mathbb{1} + \alpha\sigma_x)$. The results are plotted in Fig. 4. Each plot exhibits optimal EDUR for a particular mixture with theoretical predictions by D_{AB} and C'_{AB}. It can immediately be seen that the error-disturbance uncertainty is insensitive to dephasing or amplitude damping of the input states caused by the fluctuating magnetic field and that the bound is preserved. The measured values always saturate inequality Eq. (5); for mixed spin states no dependence on the mixture appears. Only the bound given by D_{AB} leads to saturation of the error-disturbance uncertainty relation. This statement is also true for different configurations of the observables A and B. In the Supplemental Material [30] results for other choices of observable B are depicted.

The successive nature of the measurement made it obvious how the correction procedure, i.e., a unitary transformation, can be incorporated to the whole measurement. Disturbance is strongly affected by this correction and we have observed the maximum and the minimum disturbance by optimal and anti-optimal corrections. Our experiment successfully demonstrates the tightness of the bound D_{AB} and the nontightness of the simply extended Robertson bound C'_{AB}. We confirmed the independence of the EDUR on the mixture of the states for the case of dichotomic observables A, B with $\langle A\rangle = \langle B\rangle = 0$. This is considered to be due to the fact that the observed uncertainty for Pauli operators is originated more in observables than in input states: this reminds us of another state

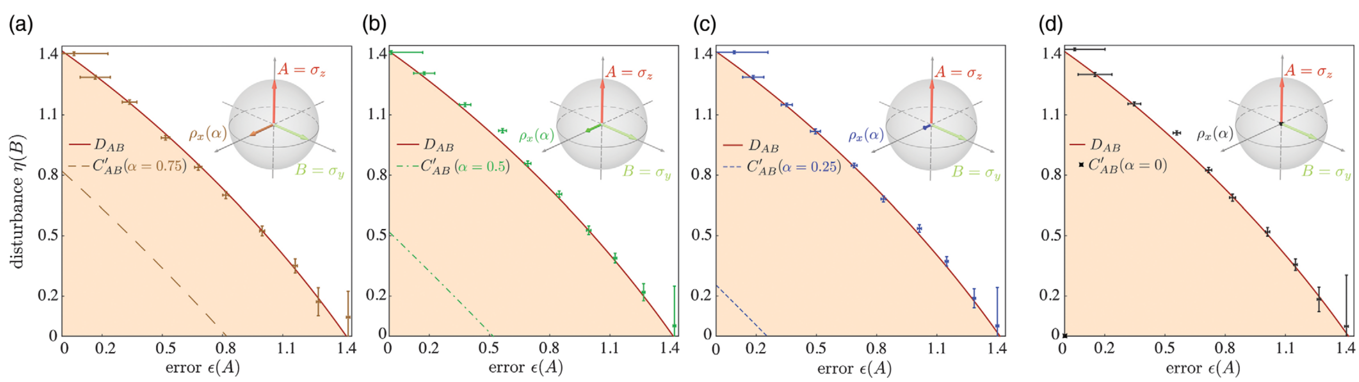

FIG. 4. Error $\epsilon(A)$ versus disturbance $\eta(B)$ for the standard configuration ($A = \sigma_z$, $B = \sigma_y$) with four different mixtures of the state $\rho_x(\alpha) = \frac{1}{2}(\mathbb{1} + \alpha\sigma_x)$: (a) $\alpha = 0.75$, (b) $\alpha = 0.5$, (c) $\alpha = 0.25$, and (d) $\alpha = 0$. The red shaded areas are forbidden according to Eq. (5). The border indicates the theoretical prediction of the lower bound $D_{AB} = 1$ which is saturated by our data points. The behavior of the bound C'_{AB} is indicated by the colored dashed lines. A change of the mixture parameter α has no effect on the final error-disturbance relation in the standard configuration as initially predicted by the expectation value C'_{AB}.

independence appearing in quantum contextuality, which was confirmed in an ion experiment [31]. Since quantum states, practically used in application such as quantum communication and computation, are more mixed ensemble due to (unavoidable) dephasing and decoherence than in a laboratory, our study shed a light on the new aspects of quantum measurements available for practical applications.

We acknowledge support by the Austrian Science Fund, FWF (No. P27666-N20 and No. P24973- N20), the European Research Council, ERC (No. MP1006), the John Templeton Foundations (ID 35771), and Japan Society for the Promotion of Science, JSPS KAKENHI (No. 26247016). We thank H. Rauch, (Vienna), M. J. W. Hall (Brisbane), F. Busemi (Nagoya), and A. Hosoya (Tokyo) for their helpful comments.

*Hasegawa@ati.ac.at

[1] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, 1983).
[2] W. Heisenberg, Z. Phys. **43**, 172 (1927).
[3] E. Kennard, Z. Phys. **44**, 326 (1927).
[4] H. P. Robertson, Phys. Rev. **34**, 163 (1929).
[5] M. Ozawa, Phys. Rev. A **67**, 042105 (2003).
[6] M. Ozawa, Ann. Phys. (Amsterdam) **311**, 350 (2004).
[7] C. Branciard, Proc. Natl. Acad. Sci. U.S.A. **110**, 6742 (2013).
[8] J. Erhart, S. Sponar, G. Sulyok, G. Badurek, M. Ozawa, and Y. Hasegawa, Nat. Phys. **8**, 185 (2012).
[9] G. Sulyok, S. Sponar, J. Erhart, G. Badurek, M. Ozawa, and Y. Hasegawa, Phys. Rev. A **88**, 022110 (2013).
[10] S. Sponar, G. Sulyok, J. Erhart, and Y. Hasegawa, Adv. High Energy Phys. **44**, 36 (2015).
[11] L. A. Rozema, A. Darabi, D. H. Mahler, A. Hayat, Y. Soudagar, and A. M. Steinberg, Phys. Rev. Lett. **109**, 100404 (2012).
[12] S.-Y. Baek, F. Kaneda, M. Ozawa, and K. Edamatsu, Sci. Rep. **3**, 2221 (2013).
[13] M. Ringbauer, D. N. Biggerstaff, M. A. Broome, A. Fedrizzi, C. Branciard, and A. G. White, Phys. Rev. Lett. **112**, 020401 (2014).
[14] F. Kaneda, S.-Y. Baek, M. Ozawa, and K. Edamatsu, Phys. Rev. Lett. **112**, 020402 (2014).
[15] M. M. Weston, M. J. W. Hall, M. S. Palsson, H. M. Wiseman, and G. J. Pryde, Phys. Rev. Lett. **110**, 220402 (2013).
[16] P. Busch, P. Lahti, and R. F. Werner, Phys. Rev. Lett. **111**, 160405 (2013).
[17] P. Busch, P. Lahti, and R. F. Werner, Rev. Mod. Phys. **86**, 1261 (2014).
[18] F. Buscemi, M. J. W. Hall, M. Ozawa, and M. M. Wilde, Phys. Rev. Lett. **112**, 050401 (2014).
[19] X.-M. Lu, S. Yu, K. Fujikawa, and C. H. Oh, Phys. Rev. A **90**, 042113 (2014).
[20] J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
[21] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, New York, 1994).
[22] L. E. Ballentine, *Quantum Mechanics: A Modern Development* (World Science, New York, 1998).
[23] G. Süssmann, *Über den Messvorgang*, (Abh. Bayer. Akad., Munich, 1958), Vol. 88.
[24] B.-G. Englert, Eur. Phys. J. D **67**, 238 (2013).
[25] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).
[26] E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, Berlin, 2003).
[27] C. Branciard, Phys. Rev. A **89**, 022124 (2014).
[28] M. Ozawa, arXiv:1404.3388v1.
[29] J. Klepp, S. Sponar, S. Filipp, M. Lettner, G. Badurek, and Y. Hasegawa, Phys. Rev. Lett. **101**, 150404 (2008).
[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.117.140402 for a description of the experiment's theoretical framework, data treatment and mixed state generation.
[31] G. Kirchmair, F. Zähringer, R. Gerritsma, M. Kleinmann, O. Guhne, A. Cabello, R. Blatt, and C. F. Roos, Nature (London) **460**, 494 (2009).