Necessary Condition for Emergent Symmetry from the Conformal Bootstrap

Yu Nakayama and Tomoki Ohtsuki
Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan
(Received 16 March 2016; revised manuscript received 29 August 2016; published 21 September 2016)

We use the conformal bootstrap program to derive the necessary conditions for emergent symmetry enhancement from discrete symmetry (e.g., $\mathbb{Z}_n$) to continuous symmetry [e.g., $U(1)$] under the renormalization group flow. In three dimensions, in order for $\mathbb{Z}_2$ symmetry to be enhanced to $U(1)$ symmetry, the conformal bootstrap program predicts that the scaling dimension of the order parameter field at the infrared conformal fixed point must satisfy $\Delta_1 > 1.08$. We also obtain the similar necessary conditions for $\mathbb{Z}_3$ symmetry with $\Delta_1 > 0.580$ and $\mathbb{Z}_4$ symmetry with $\Delta_1 > 0.504$ from the simultaneous conformal bootstrap analysis of multiple four-point functions. As applications, we show that our necessary conditions impose severe constraints on the nature of the chiral phase transition in QCD, the deconfinement criticality in Néel valence bond solid transitions, and anisotropic deformations in critical $O(n)$ models. We prove that some fixed points proposed in the literature are unstable under the perturbation that cannot be forbidden by the discrete symmetry. In these situations, the second-order phase transition with enhanced symmetry cannot happen.

DOI: 10.1103/PhysRevLett.117.131601

Introduction.—Symmetry in physics is the most helpful guideline to understand nature. The philosophy of the renormalization group (RG), in particular, dictates that the symmetry of the physical system governs its universality class. We further ensure its ubiquity from the observation that the symmetry does not have to originate microscopically, but it may appear as an emergent phenomenon. Such emergent symmetry plays a significant role in theoretical physics.

Take a lattice system, for example. Suppose the defining Hamiltonian possesses certain discrete or continuous symmetry. This does not mean that the infrared (IR) physics has the same symmetry. Rather, it often shows enhanced symmetry, especially when the system is at criticality. Indeed, the emergence of global continuous symmetry out of discrete lattice symmetry is ubiquitous in strongly interacting systems, and it has played a key role in understanding the nature of quantum criticality that is outside the scope of the traditional Wilson-Landau-Ginzburg (WLG) paradigm of phase transitions [1,2].

In this Letter, we derive the universal necessary conditions for such an emergent symmetry enhancement from discrete symmetry to continuous symmetry under the RG flow by using the recently developed technique of the numerical conformal bootstrap program in three dimensions [3–11]. We will show that the conformal symmetry imposes a strong constraint on when the emergent symmetry enhancement can or cannot occur.

Let us rephrase the question in terms of conformal field theories (CFTs). Suppose we have a system with emergent $U(1)$ symmetry in the IR. Can we realize the same system with smaller discrete symmetry [e.g., $\mathbb{Z}_2 \in U(1)$] without fine-tuning? The $\mathbb{Z}_2$ symmetry forbids the perturbation of the $U(1)$ symmetric fixed point under the smallest charged operators that are $\mathbb{Z}_2$ odd. However, with only $\mathbb{Z}_2$ symmetry, one cannot forbid a perturbation by twice $U(1)$ charged operators that are $\mathbb{Z}_2$ even. In order to obtain the emergent $U(1)$ symmetry, all the $\mathbb{Z}_2$ even but $U(1)$ charged operators must be irrelevant. The conformal bootstrap program tells exactly when this can happen. In this case, we find that the scaling dimension of the $\mathbb{Z}_2$ odd order parameter field must satisfy $\Delta_1 > 1.08$ in three dimensions. Otherwise, we always have $\mathbb{Z}_2$ even but $U(1)$ charged relevant deformations that we cannot forbid without fine-tuning.

Prior to our work, our expectations for the emergent symmetry have been based on an explicit ultraviolet Lagrangian or Hamiltonian with naive dimensional counting or, at best, with the perturbative computations, e.g., large N expansions or ϵ expansions (see, e.g., [12–17] for the examples we will study). We show that the conformal bootstrap program gives the more stringent and precise necessary conditions for the emergent symmetry. Our results are nonperturbative, rigorous, and universal, so they should be applied to any critical phenomena in nature as long as the conformal symmetry is realized at the fixed point.

In this Letter, among many possibilities, we offer applications to two widely discussed controversies in the theoretical physics community. The one is the finite-temperature chiral phase transition in quantum chromodynamics (QCD), and the other is the deconfinement criticality in Néel valence bond solid (VBS) transitions. We also test our necessary conditions against anisotropic deformations of $O(n)$ critical vector models. The conformal bootstrap program predicts that the $O(4) \times U(1)_A$ symmetric fixed point proposed in the QCD chiral phase transition and noncompact CP^{N-1} fixed points in Néel-VBS transitions are unstable under the perturbation that cannot be forbidden by the discrete symmetry. In these systems, therefore, the second-order phase transition with the enhanced continuous symmetry cannot happen.

Necessary conditions for emergent symmetry enhancement from the conformal bootstrap.—The foremost basis of our discussion is the conformal hypothesis: Under the RG flow, the system reaches a critical point described by a unitary CFT. In particular, not only scale symmetry but also Lorentz and special conformal symmetry should emerge. The hypothesis seems to be valid in many classical as well as quantum critical systems as long as we trust the effective field theory description with emergent Lorentz symmetry. In particular, in the examples we will study in this Letter, there are no perturbative candidates for the virial current in the effective action, which is the obstruction for conformal invariance in the scale-invariant field theory, so the scale invariance most likely implies conformal invariance. See, e.g., [18] for a review on this argument.

Once conformal invariance is assumed, we may study the consistency of four-point functions that results in the conformal bootstrap equations. In our case, we are interested in the consistency of four-point functions $\langle O_q O_q^\dagger O_{q'} O_{q'}^\dagger \rangle$ of $U(1)$ charge q local scalar operators O_q, whose scaling dimension is denoted by Δ_q, demanding the crossing equations in $U(1)$ symmetric unitary CFTs. By mapping the crossing equations to a semidefinite problem [9], numerical optimization yields a bound on the scaling dimension of the operators that appear in the operator product expansion (OPE), e.g., $O_q \times O_{q'} \sim O_{q+q'}$. The idea of studying the bootstrap equation with $U(1)$ symmetry was first developed in four-dimensional CFTs in Ref. [19]. See Supplemental Material [20], Sec. I, which includes Refs. [21–24], for the details of our implementation.

Let us begin with emergent $U(1)$ symmetry from $\mathbb{Z}_2$. The upper bound on Δ_2 as a function of Δ_1 in $U(1)$ symmetric CFTs is straightforwardly obtained as in Ref. [6] by studying $\langle O_1 O_1^\dagger O_1 O_1^\dagger \rangle$. The plot in Fig. 1 shows that, when the scaling dimension Δ_1 of the charge-one operator O_1 is smaller than 1.08, there always exists a charge-two operator O_2 whose scaling dimension is less than 3, which means that we have a relevant operator that cannot be forbidden by the lattice $\mathbb{Z}_2$ symmetry. Therefore, we conclude that the necessary condition for the symmetry enhancement from $\mathbb{Z}_2$ to $U(1)$ is $\Delta_1 > 1.08$.

For the $\mathbb{Z}_3$ enhancement, we study the simultaneous consistency of three four-point functions $\langle O_1 O_1^\dagger O_1 O_1^\dagger \rangle$, $\langle O_1 O_1^\dagger O_2 O_2^\dagger \rangle$, and $\langle O_2 O_2^\dagger O_2 O_2^\dagger \rangle$ from the mixed correlator conformal bootstrap analysis [10,25]. In order to make the bound relevant for us, we make two additional assumptions: (i) All the charge-four operators are irrelevant, and (ii) all the charge-neutral operators (above the identity) have a scaling dimension larger than 1.044. The latter

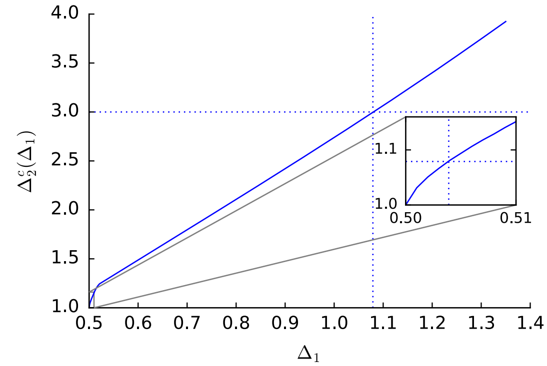

FIG. 1. The upper bound on the scaling dimension Δ_2^c of the lowest-dimensional charge-two scalar operator appearing in the $O_1 \times O_1$ OPE as a function of Δ_1. The same bound applies to $O_2 \times O_2 \sim O_4$.

assumption is motivated from our setup, because by using the conformal bootstrap analysis we can numerically prove that if there exists a neutral scalar operator with a scaling dimension less than 1.044, there also exists another neutral scalar operator whose scaling dimension is less than 3 as shown in Supplemental Material [20], Sec. II, which includes Refs. [26,27]. However, in all of our applications, there is only one neutral scalar operator that must be tuned, so the assumption is justifiable.

Figure 2 shows the upper bound on Δ_3 as a function of Δ_1 and Δ_2. When $\Delta_1 \geq 0.585$, there exists an allowed region of Δ_2 where Δ_3 can be irrelevant. As soon as the bound on Δ_3 touches 3, it shows a conspicuous jump that is similar to the one observed in the fermionic conformal bootstrap analysis [11]. Without knowing the value of Δ_2, the plot shows that the necessary condition is

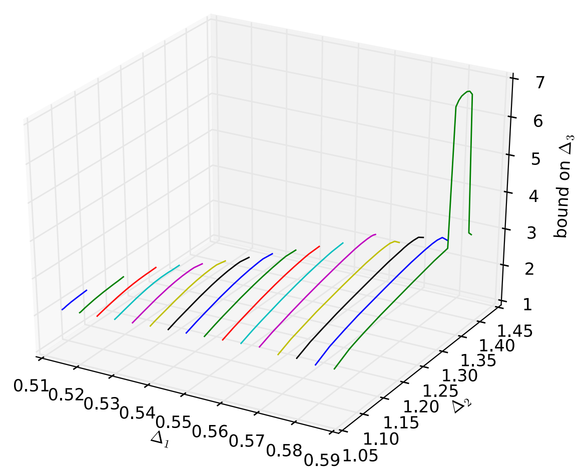

FIG. 2. Upper bounds on the scaling dimension of the lowest-dimensional charge-three scalar operator appearing in the $O_1 \times O_2$ OPE as a function of Δ_1 and Δ_2. The jump in the bounds appears as soon as they touch the value 3. Note that $1.08 < \Delta_2 < \Delta_2^c(\Delta_1)$ must hold from the assumption that all the charge-four operators are irrelevant. See Fig. 1.

$\Delta_1 > 0.580$. See Supplemental Material [20], Sec. IV, for two-dimensional projections of the plot.

In a similar manner, we can study the bound on Δ_4 for the $\mathbb{Z}_4$ enhancement. We obtain the simplest bound by studying $\langle O_1 O_1^\dagger O_1 O_1^\dagger \rangle$ and $\langle O_2 O_2^\dagger O_2 O_2^\dagger \rangle$ independently, which immediately gives $\Delta_1 > 0.504$ (see Fig. 1). The study of the simultaneous consistency of three four-point functions $\langle O_1 O_1^\dagger O_1 O_1^\dagger \rangle$, $\langle O_1 O_1^\dagger O_2 O_2^\dagger \rangle$, and $\langle O_2 O_2^\dagger O_2 O_2^\dagger \rangle$ gives a stronger bound in principle, but in practice, without introducing further assumptions, it does not improve much.

The numerical accuracy of our necessary condition is set by the dimensions of the search space Λ in the numerical conformal bootstrap program. The larger the dimensionality, the more severe the bound becomes. For the $\mathbb{Z}_2$, we obtained $\Delta_1 > 0.565$ with $\Lambda = 15$, while $\Delta_1 > 0.580$ with $\Lambda = 19$. Similarly, for the $\mathbb{Z}_4$, we obtained $\Delta_1 > 1.07$ with $\Lambda = 19$, while $\Delta_1 > 1.08$ with $\Lambda = 23$. Note that increasing Λ makes our necessary condition only stronger, so our results are, albeit not necessarily the strongest, still rigorous.

Applications.—Chiral phase transition in QCD. The order of the chiral phase transition in finite-temperature QCD has been controversial over many years without reaching a consensus. In the WLG paradigm, we may translate the problem into the (non)existence of a RG fixed point in a certain three-dimensional WLG model whose order parameter is given by the quark bilinear scalar field $\Phi_{\bar{i}j} = \bar{\psi}_{\bar{i}} \psi_j$ (where $\bar{i}$, $j = 1, 2$ runs the number of approximately massless quarks in nature) with the manifest $O(4) \sim SU(2)_R \times SU(2)_L$ symmetry. To reveal the nature of the RG flow, it is crucial to discuss whether the anomalous $U(1)_A$ symmetry is restored in the IR limit of the effective WLG model. If the $U(1)_A$ symmetry is restored, we expect that the chiral phase transition is described by a RG fixed point with the symmetry of $O(4) \times U(1)_A$ [8,15]. Otherwise, it is described by a RG flow only with the symmetry of $O(4)$ [28].

In Refs. [12,29,30], it was shown that, under mild assumptions, the $\mathbb{Z}_2$ subgroup of the anomalous $U(1)_A$ is microscopically restored, which raises the second question if the $\mathbb{Z}_2$ can be further enhanced to the full $U(1)_A$ under the RG flow of the effective WLG model in three dimensions. This is exactly the problem we have discussed, and the conformal bootstrap program gives a definite answer.

A study of the RG properties of this effective WLG model is notoriously hard, but the conformal bootstrap analysis of Ref. [8] tells that the scaling dimension of the $\mathbb{Z}_2$ odd operator at the $O(4) \times U(1)_A$ symmetric fixed point is $\Delta_1 = 0.82(2)$ (see also [15–17] for earlier computations). It turns out that this value does not satisfy the necessary condition for the $U(1)$ symmetry enhancement that we have obtained. We therefore conclude that the microscopic $O(4) \times \mathbb{Z}_2$ symmetry cannot be enhanced to $O(4) \times U(1)_A$ without fine-tuning. It means that the chiral phase transition in QCD does not accompany the full restoration of the $U(1)_A$ symmetry and does not show the second-order phase transition described by the fixed point studied in Refs. [8,15] unless further symmetry enhancement is assumed.

Deconfinement criticality in Néel-VBS transitions. The deconfinement criticality in Néel-VBS transitions in $2+1$ dimensions is proposed to be an example of critical phenomena whose description is beyond the traditional framework of the WLG effective field theory. In Refs. [1,2], they argued that the effective field theory description with N component spin near the critical point is given by the noncompact CP^{N-1} model [31–33] or a $U(1)$ gauge theory coupled with N charged scalars with $SU(N)$ flavor symmetry. While we have no rigorous proof, it was argued that the system shows a conformal behavior once we tune one parameter, the $SU(N)$ singlet mass of the charged scalars.

However, it turns out that the actual realization of this critical behavior in the lattice simulation has been controversial for years. From the effective field theory viewpoint, a difficulty comes from the existence of monopole operators. One can argue that the lattice symmetry forbids the smallest charged monopole operator, but not necessarily so for the higher charged monopole operators [34,35]. For instance, if we use the rectangular lattice, one can only preserve the $\mathbb{Z}_2$ subgroup of the $U(1)$ monopole charge, if we use the honeycomb lattice, it is $\mathbb{Z}_3$, and if we use the square lattice, it is $\mathbb{Z}_4$. If the higher charged monopole operators that are not forbidden by the lattice symmetry are relevant, then we cannot reach the noncompact CP^{N-1} model in the IR limit without further fine-tuning, and we typically expect the first-order phase transition in lattice simulations.

Therefore, the central question we should address is under which condition the higher charged monopole operators can become irrelevant once we know the scaling dimension of the lowest monopole charged operator. Again, this is precisely the question we have studied. To reiterate our results, in order to obtain $U(1)$ symmetry enhancement, we need $\Delta_1 > 1.08$ from $\mathbb{Z}_2$, $\Delta_1 > 0.580$ from $\mathbb{Z}_3$, and $\Delta_1 > 0.504$ from $\mathbb{Z}_4$.

In the following, we critically review various predictions about the nature of Néel-VBS phase transitions in the literature for different N. We may find a convenient summary of the scaling dimensions of operators proposed in the literature in Supplemental Material [20], Sec. III, which includes Refs. [36–47].

Let us begin with the $N = 2$ case, which has the most experimental significance. The predictions of $\Delta_1 = (1 + \eta_{\text{VBS}})/2$ in the literature ranges between 0.57 and 0.68. For whichever values, our necessary condition for $\mathbb{Z}_4$ tells us that the charge-four operators can be irrelevant, and it is consistent with the observation of the second-order phase transition on the square lattice. On the other hand, our necessary condition $\Delta_1 > 1.08$ for $\mathbb{Z}_2$ tells us that the charge-two monopole operator must be relevant, so we predict the first-order phase transition on the rectangular lattice as observed indeed in Ref. [34].

The most controversial question is if the charge-three monopole operator must be relevant or not. Our necessary condition $\Delta_1 > 0.580$ is consistent with it being either relevant or irrelevant, depending on the value of Δ_1. We note that the scaling dimensions obtained in Ref. [48] [i.e., $\Delta_1 = 0.579(8)$, $\Delta_2 = 1.42(7)$, and $\Delta_3 = 2.80(3)$] are very close to the bound. In particular, our results show that, given their values of Δ_1 and Δ_2, the charge-three monopole operator must be relevant, supporting their claim that they observed the first-order phase transition on the honeycomb lattice.

Let us next consider the $N = 3$ case. In Ref. [35], they obtained $\Delta_1 = 0.785$, so our necessary condition implies that it can show the second-order phase transition on the square lattice but it cannot on the rectangular lattice, in agreement with what is observed. The direct measurement of $\Delta_2 = 2.0$ there turns out to be close to but slightly below our bound with $\Delta_1 = 0.785$. However, we also note that the earlier estimate of $\Delta_1 = 0.71(2)$ in Ref. [49] may be inconsistent with $\Delta_2 = 2.0$.

For $N = 4$, our result reveals inconsistency among the literature. In Ref. [35], they obtained $\Delta_1 = 0.865$, and our necessary condition predicts that the charge-two monopole operator is relevant. On the other hand, in Ref. [34], they claim that the charge-two monopole operator is irrelevant and the phase transition is second order on the rectangular lattice. These statements cannot be mutually consistent, and one of them or our conformal hypotheses must be wrong.

For $N = 5$, the situation is again subtle. Reference [34] claims that it shows the second-order phase transition on the rectangular lattice. Our result then demands that $\Delta_1 > 1.08$ to make the charge-two monopole operator irrelevant. The values they obtained on square and honeycomb lattices $\Delta_1 = 1.0(1)$ are quite marginal. In contrast, the value of $\Delta_1 = 0.85(1)$ they obtained on a rectangular lattice is clearly inconsistent, so it is likely that the charge-two monopole operator is actually relevant and the phase transition on the rectangular lattice is first order.

Finally, for $N \geq 6$, the results in Ref. [34] as well as $1/N$ expansions (see, e.g., [50]) suggest $\Delta_1 > 1.08$. Then our necessary condition implies that charge-two monopole operators can be irrelevant and the phase transition on the rectangular lattice can be second order, in agreement with the claims in the literature.

Anisotropic deformations in critical $O(n)$ models. Critical $O(n)$ vector models are canonical examples of conformal fixed points naturally realized in the WLG effective field theory. Under the RG flow, the $O(n)$ critical point is achieved by adjusting one parameter that is $O(n)$ singlet (e.g., the temperature), but the fixed point may be unstable under anisotropic deformations. The stability under anisotropic deformations is an example of emergent symmetry and is subject to our general discussions.

Let us focus on $n = 2$ (i.e., the XY model). It is believed that the $O(2)$ invariant conformal fixed point is unstable under anisotropic deformations with charge $q = 2$ and $q = 3$ but stable under $q \geq 4$ deformations. Our current best estimate based on the Monte Carlo (MC) simulation is $\Delta_1 = 0.51905(10)$ and $\Delta_2 = 1.2361(11)$ [51]. These values are in agreement with the conformal bootstrap analysis of $O(2)$ invariant CFTs in Refs. [4,10]. The MC simulations on scaling dimensions of $q = 3$ and $q = 4$ deformations are also available in Ref. [52] as $\Delta_3 = 2.103(15)$ and $\Delta_4 = 3.108(6)$. See also [14,53,54] for perturbative computations.

Now given Δ_1 and Δ_2, our conformal bootstrap program gives a rigorous upper bound on Δ_3. With more accurate data to be compared, we have tried to derive the more accurate bound by increasing the approximation in our conformal bootstrap program with $\Lambda = 23$. The resulting bound is $\Delta_3 < 2.118$ for $\Delta_1 = 0.51905$ and $\Delta_2 = 1.234$. It turns out that the number quoted above is very close to (or almost saturating) the bound we have obtained.

On the other hand, the estimate of $\Delta_4 = 3.108(6)$ seems slightly below the conformal bootstrap bound $\Delta_4 < 3.52$. It is not obvious if our bound is saturated, but it is at least consistent that our conformal bootstrap bound does not predict that $q = 4$ anisotropy must be relevant.

For $n > 2$, it is a challenging problem to determine exactly when the cubic anisotropy becomes irrelevant. In the large n limit the cubic anisotropy is believed to be relevant, but for smaller n it is believed to become irrelevant, showing the enhanced $O(n)$ symmetry. The current estimate of the critical n is around $n = 3$ (see, e.g., [14]). However, it is still an open question if it is strictly smaller than 3.

Our conformal bootstrap program might shed some light on this problem. Repeating our analysis now with $O(n)$ symmetry rather than $U(1)$ on the four-point functions $\langle O_{[ij]} O_{[kl]} O_{[mn]} O_{[pq]} \rangle$, we can rigorously show that the cubic anisotropy must be relevant for $n = 10$ with no assumptions and for $n = 6$ with mild assumptions (i.e., nonconserved vector operators have a scaling dimension larger than 3). For $n = 3$ and $n = 4$, the conformal bootstrap bound we have obtained is not conclusive yet.

Discussions.—In this Letter, we have numerically proved the necessary conditions for emergent symmetry enhancement from discrete symmetry to continuous symmetry under the RG flow. Our necessary conditions are universally valid, but, given a concrete model with concrete predictions on critical exponents, we may be able to offer more stringent bounds. Even modest partial data such as $\Delta_0 = 3 - 1/\nu$ will make the constraint more nontrivial. We are delighted to test the consistency of future predictions obtained from other methods upon request.

The symmetry enhancement we have discussed is mainly $U(1)$, but it is possible to discuss non-Abelian enhancement as well from the conformal bootstrap program. For example, there is an interesting conjecture [55,56] that the noncompact CP^1 model shows further symmetry enhancement to $SO(5)$ by combining the Néel order parameter and VBS order parameter. However, the conformal bootstrap analysis tells us that the currently observed value of Δ_1 (i.e., $\eta_{\rm VBS} \simeq \eta_{\rm Néel}$) is too small for the conjecture to hold that it has only one singlet relevant deformation under $SO(3) \times SO(2)$ within $SO(5)$.

Finally, we should stress that our discussions are entirely based on the emergent conformal symmetry. In quantum critical systems, this is more nontrivial than in classical critical systems. While the Lorentz-invariant conformal fixed points are typically stable under Lorentz-breaking deformations allowed on the lattice, it would be important to understand precisely under which condition the conformal symmetry itself emerges.

We thank S. Aoki, N. Kawashima, A. Nahum, and T. Okubo for discussions. This work is supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT. T. O. is supported by JSPS Research Fellowships for Young Scientists and the Program for Leading Graduate Schools, MEXT.

[1] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science **303**, 1490 (2004).
[2] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, Phys. Rev. B **70**, 144407 (2004).
[3] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86**, 025022 (2012).
[4] F. Kos, D. Poland, and D. Simmons-Duffin, J. High Energy Phys. 06 (2014) 091.
[5] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, J. Stat. Phys. **157**, 869 (2014).
[6] F. Kos, D. Poland, and D. Simmons-Duffin, J. High Energy Phys. 11 (2014) 109.
[7] Y. Nakayama and T. Ohtsuki, Phys. Rev. D **89**, 126009 (2014).
[8] Y. Nakayama and T. Ohtsuki, Phys. Rev. D **91**, 021901 (2015).
[9] D. Simmons-Duffin, J. High Energy Phys. 06 (2015) 174.
[10] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, J. High Energy Phys. 11 (2015) 106.
[11] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin, and R. Yacoby, J. High Energy Phys. 03 (2016) 120.
[12] S. Aoki, H. Fukaya, and Y. Taniguchi, *Proc. Sci.*, LATTICE2013 (2014) 139 [arXiv:1312.1417].
[13] E. Dyer, M. Mezei, S. S. Pufu, and S. Sachdev, J. High Energy Phys. 06 (2015) 037.
[14] J. M. Carmona, A. Pelissetto, and E. Vicari, Phys. Rev. B **61**, 15136 (2000).
[15] A. Pelissetto and E. Vicari, Phys. Rev. D **88**, 105018 (2013).
[16] P. Calabrese, P. Parruccini, A. Pelissetto, and E. Vicari, Phys. Rev. B **70**, 174439 (2004).
[17] P. Calabrese, A. Pelissetto, and E. Vicari, Nucl. Phys. **B709**, 550 (2005).
[18] Y. Nakayama, Phys. Rep. **569**, 1 (2015).
[19] R. Rattazzi, S. Rychkov, and A. Vichi, J. Phys. A **44**, 035402 (2011).
[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.117.131601 for technical details about our implementation of the conformal bootstrap program, review on the literature and additional plots.
[21] M. Hogervorst, H. Osborn, and S. Rychkov, J. High Energy Phys. 08 (2013) 014.
[22] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, J. High Energy Phys. 12 (2008) 031.
[23] Sage Developers, Sage Mathematics Software version 6.8.0, http://www.sagemath.org.
[24] An example of T. Ohtsuki's implementation is available at https://github.com/tohtsky/cboot.
[25] M. Lemos and P. Liendo, J. High Energy Phys. 01 (2016) 025.
[26] J. Zinn-Justin, Phys. Rep. **344**, 159 (2001).
[27] J. A. Lipa, J. A. Nissen, D. A. Stricker, D. R. Swanson, and T. C. P. Chui, Phys. Rev. B **68**, 174518 (2003).
[28] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).
[29] S. Aoki, H. Fukaya, and Y. Taniguchi, Phys. Rev. D **86**, 114512 (2012).
[30] T. Kanazawa and N. Yamamoto, J. High Energy Phys. 01 (2016) 141.
[31] B. I. Halperin, T. C. Lubensky, and S.-k. Ma, Phys. Rev. Lett. **32**, 292 (1974).
[32] M. Kamal and G. Murthy, Phys. Rev. Lett. **71**, 1911 (1993).
[33] O. I. Motrunich and A. Vishwanath, Phys. Rev. B **70**, 075104 (2004).
[34] M. S. Block, R. G. Melko, and R. K. Kaul, Phys. Rev. Lett. **111**, 137202 (2013).
[35] K. Harada, T. Suzuki, T. Okubo, H. Matsuo, J. Lou, H. Watanabe, S. Todo, and N. Kawashima, Phys. Rev. B **88**, 220408 (2013).
[36] F.-J. Jiang, M. Nyfeler, S. Chandrasekharan, and U.-J. Wiese, J. Stat. Mech. (2008) P02009.
[37] A. B. Kuklov, M. Matsumoto, N. V. Prokof'ev, B. V. Svistunov, and M. Troyer, Phys. Rev. Lett. **101**, 050405 (2008).
[38] S. Pujari, K. Damle, and F. Alet, Phys. Rev. Lett. **111**, 087203 (2013).
[39] S. Pujari, F. Alet, and K. Damle, Phys. Rev. B **91**, 104411 (2015).
[40] R. K. Kaul and A. W. Sandvik, Phys. Rev. Lett. **108**, 137201 (2012).
[41] G. J. Sreejith and S. Powell, Phys. Rev. B **89**, 014404 (2014).
[42] G. Murthy and S. Sachdev, Nucl. Phys. **B344**, 557 (1990).
[43] M. Hermele, T. Senthil, and M. P. A. Fisher, Phys. Rev. B **72**, 104404 (2005).
[44] M. Hermele, Y. Ran, P. A. Lee, and X.-G. Wen, Phys. Rev. B **77**, 224413 (2008).
[45] S. S. Pufu, Phys. Rev. D **89**, 065016 (2014).
[46] S. M. Chester, M. Mezei, S. S. Pufu, and I. Yaakov, arXiv:1511.07108.
[47] S. M. Chester and S. S. Pufu, J. High Energy Phys. 08 (2016) 019.

[48] G. J. Sreejith and S. Powell, Phys. Rev. B **92**, 184413 (2015).
[49] J. Lou, A. W. Sandvik, and N. Kawashima, Phys. Rev. B **80**, 180414 (2009).
[50] M. A. Metlitski, M. Hermele, T. Senthil, and M. P. A. Fisher, Phys. Rev. B **78**, 214418 (2008).
[51] M. Campostrini, M. Hasenbusch, A. Pelissetto, and E. Vicari, Phys. Rev. B **74**, 144506 (2006).
[52] M. Hasenbusch and E. Vicari, Phys. Rev. B **84**, 125136 (2011).
[53] P. Calabrese, A. Pelissetto, and E. Vicari, Phys. Rev. B **67**, 054505 (2003).
[54] M. De Prato, A. Pelissetto, and E. Vicari, Phys. Rev. B **68**, 092403 (2003).
[55] C. Xu and S. Sachdev, Phys. Rev. Lett. **100**, 137201 (2008).
[56] A. Nahum, P. Serna, J. T. Chalker, M. Ortuno, and A. M. Somoza, Phys. Rev. Lett. **115**, 267203 (2015).