Multipartite Entanglement of a Two-Separable State

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We consider a six-partite, continuous-variable quantum state that we have effectively generated by the parametric down-conversion of a femtosecond frequency comb. We show that, though this state is two-separable, i.e., it does not exhibit "genuine entanglement," it is undoubtedly multipartite entangled. The consideration of not only the entanglement of individual mode decompositions, but also of combinations of those, solves the puzzle and exemplifies the importance of studying different categories of multipartite entanglement.

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Introduction.—Entanglement is, nowadays, a major subject of research in quantum physics, long after the pioneering contributions of Einstein, Podolsky, Rosen [1], and Schrödinger [2]. It is the main quantum resource in a vast number of applications in quantum information [3]. Entanglement witnesses uncover such quantum correlations [4,5] in either discrete variables, using measurements with photon counters, or in continuous variables by employing homodyne detection, for characterizing the quantum states of light.

Pure entangled states have been first considered in bipartite systems. The case of mixed correlated states turns out to be more involved. An intermediate situation between factorized and entangled states has been introduced, the separable states, which are statistical mixtures of factorized pure states [6]. A number of entanglement probes for continuous-variable systems have been studied [4,5], partial transpose being one of the most popular inseparability tests. These criteria are in most cases only sufficient to detect the different levels of correlation. The problem simplifies for bipartite Gaussian states, for which the partial transposition of the covariance matrix is a necessary and sufficient entanglement identifier [7–9].

The complexity of the separability problem increases substantially when one studies multipartite systems. In these situations, one has a rapidly increasing number of choices in the bunching of parties on which one searches for a possible factorization. Hence, the inseparability between the individual degrees of freedom exhibits a much richer and complex structure which begins to be studied [10–12]. For example, the difference between bipartite and multipartite systems is highlighted by the existence of multimode Gaussian states whose entanglement cannot be uncovered by the partial transposition [13,14].

As a special case, combinations of bipartitions of the total system are a subject of many studies. A state which is not a statistical mixture of bipartite factorized density matrices (i.e., not two-separable) is also called "genuinely" multipartite entangled [15]. The detection of genuine entanglement is at the focus of attention [16–24]. This interest can be explained by the fact that genuine entanglement implies multipartite entanglement for every other separation of the modes. However, if a state does not exhibit this specific kind of entanglement (i.e., is two-separable), no conclusions on other forms of multipartite quantum correlations can be drawn. Thus, it is indispensable to study what happens beyond genuine entanglement. This is the subject of the present Letter.

In experiments, continuous-variable quantum correlated states have been produced by mixing, in an appropriate way, different squeezed states on beam splitters [25]. More recently, multimode Gaussian states (either spatial or frequency modes) have been directly generated by a multimode optical nonlinear device [26,27]. In the multifrequency case, the experimental determination of the full covariance matrix of a ten-mode "quantum frequency comb" has allowed us to uncover the complex structure of its quantum properties, in particular, the entanglement of all its possible partitions. [28–30].

In this Letter, we characterize states which are two-separable and yet exhibit a rich multipartite entanglement structure. For achieving this, we formulate different notions of separability and entanglement, then, we provide a method for qualifying them in a general case. Using this technique, we uncover the structure of multimode entanglement in an experimentally produced six-mode Gaussian state, using multimode parametric down-conversion of a femtosecond frequency-comb light source. Even though this state is two-separable, it includes all other forms of higher-order entanglement.

Combinations of modal partitions.—We consider multimode states which are based on an N-fold Hilbert space

$\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$, where $\mathcal{H}_j$ is the local Hilbert space of the jth mode. A particular K-partition $\mathcal{I}_1 : \ldots : \mathcal{I}_K$ decomposes the set of modes, $\{1, \ldots, N\}$, into K nonempty, disjoint subsets $\mathcal{I}_k$ (for $k = 1, \ldots, K$). We will call such a partition an individual K-partition.

The corresponding pure factorized states are product states, $|s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\rangle = |a_{\mathcal{I}_1}\rangle \otimes \ldots \otimes |a_{\mathcal{I}_K}\rangle$, consisting of states $|a_{\mathcal{I}_k}\rangle \in \bigotimes_{j\in\mathcal{I}_k} \mathcal{H}_j$. Subsequently, a mixed $\mathcal{I}_1 : \ldots : \mathcal{I}_K$-separable state is defined as

$$\hat{\sigma}_{\mathcal{I}_1:\ldots:\mathcal{I}_K} = \int dP(s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}) |s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\rangle\langle s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}|, \quad (1)$$

where P is a classical probability distribution over the set of pure (continuous-variable) separable states.

A state is called K-separable if it can be written as a statistical mixture of separable states with respect to the different K-partitions $\mathcal{I}_1 : \ldots : \mathcal{I}_K$,

$$\hat{\sigma}_K = \sum_{\mathcal{I}_1:\ldots:\mathcal{I}_K} p_{\mathcal{I}_1:\ldots:\mathcal{I}_K} \hat{\sigma}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}, \quad (2)$$

where $p_{\mathcal{I}_1:\ldots:\mathcal{I}_K}$ are probabilities and $\hat{\sigma}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}$ are the corresponding $\mathcal{I}_1 : \ldots : \mathcal{I}_K$-separable states in Eq. (1). We will refer to this combination of individual K-partitions as a convex combination of K-partitions. A state is called K-entangled if it cannot be written in the manner specified in Eq. (2). In particular, a state which is not "biseparable" ($K = 2$) is precisely the genuinely multipartite entangled state studied in the literature.

Figure 1 shows the different kinds of separability in the tripartite scenario, $N = 3$, in a schematic Venn diagram. Needless to say, it is impossible to illustrate the full structure of the partitions in infinite dimensional Hilbert spaces. The circles represent pure states that are factorizable with respect to a defined partitioning. These states have to be extremal points of the convex sets, since they are not combinations of any other states. The highlighted areas in between these points represent the considered convex hull of mixed separable states. States lying outside of these sets are entangled in that particular notion for arbitrary, compound Hilbert spaces.

For instance for $K = 3$, we have the statistical mixture of pure, fully separable states $|s_{\{1\}:\{2\}:\{3\}}\rangle$ (red area; left pattern in top row of Fig. 1). In order to get an area, we selected three pure state representatives (D, E, and F) from the equivalence class of all three-separable pure states. Any state that is outside this convex (red) area symbolizes a three-entangled state.

The $K = 2$-separable states (green area; center pattern in top row of Fig. 1) lie in the convex hull of three individual bipartitions, which are depicted in the middle row of Fig. 1. We select the point H to be one pure state representative $|s_{\{1,2\}:\{3\}}\rangle$, which is not of the form $|s_{\{1\}:\{2\}:\{3\}}\rangle$, $|s_{\{1\}:\{2,3\}}\rangle$, or $|s_{\{1,3\}:\{2\}}\rangle$. Similarly, the states represented by point B or points C and G are exclusively separable with

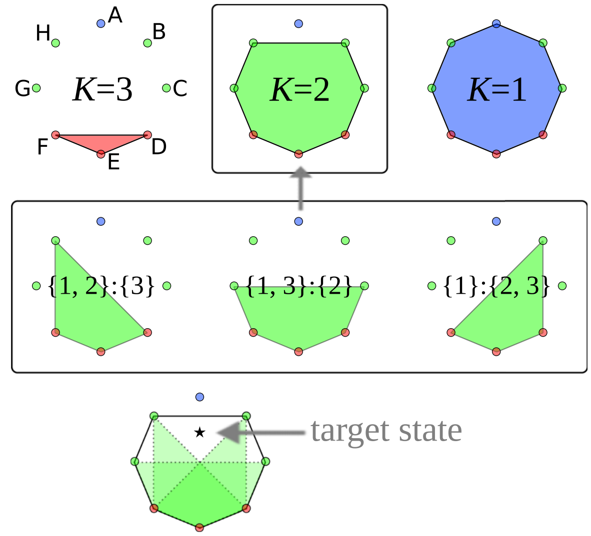

FIG. 1. Partitioning of a tripartite system. The circles indicate pure state representatives for different notions of separability or entanglement (distinguished by the lightness [color]). The top row depicts the convex sets of K-partitions. The $K = 2$ case is a convex combination of the individual bipartitions $\mathcal{I}_1 : \mathcal{I}_2$, which are given in the pattern in the middle row. The bottom shows the overlay of the individual bipartitions (dashed bordered sets) as an inset into the convex set of all bipartitions (solid borders). The target state (filled star) is a convex combination of two-partite separable states. It is, therefore, two-separable, but three-entangled.

respect to the individual partition $\{1\}:\{2,3\}$ or $\{1,3\}:\{2\}$, respectively, forming nonidentical sets of mixed separable states of the individual partitions (middle row). For symmetry reasons, we consider two points (G and C) for the partition $\{1,3\}:\{2\}$. All of them are two-separable in the convex combination of all bipartitions (top row, center pattern).

For $K = 1$, we get all states (blue area, right pattern in top row of Fig. 1). The point A serves as an element of the equivalence class of all pure, two-entangled states. Those are nonfactorizable. As every mixed state is trivially in the full partition $\{1,2,3\}$, this (blue) set includes all quantum states.

In the bottom row of Fig. 1, we also show the three individual bipartitions $\{1,2\}:\{3\}$, $\{1,3\}:\{2\}$, and $\{1\}:\{2,3\}$ (dashed borders) included into the convex combination of biseparable states (solid border). Our target state (indicated by a star) is entangled with respect to all individual bipartitions, but it is separable with respect to the convex combination of bipartitions. These states are particularly interesting as they are not genuinely multipartite entangled (two-separable). We will show explicitly for a six-partite system, $N = 6$, that such states still exhibit rich multipartite entanglement properties.

Entanglement criteria for convex combinations.—A criterion for detecting entanglement of states is based on the entanglement witnesses [31]. It can be formulated as follows: A state is entangled if a Hermitian operator $\hat{L}$

exists, whose expectation value is smaller than the minimal attainable value for all separable states $\hat{\sigma}$ [32],

$$\langle\hat{L}\rangle < \inf_{\hat{\sigma}}\{\mathrm{tr}(\hat{L}\,\hat{\sigma})\}. \tag{3}$$

This criterion is general and covers any kind of inseparability. It applies, therefore, to either individual partitions or convex combinations.

Let us show that the bound in (3) can be achieved by pure state representatives. For this reason, we apply the following property of convex (statistical) mixtures:

$$\inf\left\{\int dP(x)x|\int dP=1 \quad \text{and} \quad P\geq 0\right\} = \inf_x\{x\}. \tag{4}$$

This allows us to derive the lower bound in (3) for K-separable states,

$$\begin{aligned}\inf_{\hat{\sigma}_K}\{\mathrm{tr}(\hat{L}\hat{\sigma}_K)\} &\overset{(2),(4)}{=} \min_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\inf_{\hat{\sigma}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}}\{\mathrm{tr}(\hat{L}\hat{\sigma}_{\mathcal{I}_1:\ldots:\mathcal{I}_K})\}\\ &\overset{(1),(4)}{=} \min_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\inf_{|s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\rangle}\\ &\quad\times\{\langle s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}|\hat{L}|s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\rangle\}.\end{aligned}$$

Equation labels over the equal signs indicate that those equations have been used for rewriting. Thus, the minimal expectation value of $\hat{L}$ for separable states in convex combinations of K-partitions is identical to the least expectation value ($\min_{\mathcal{I}_1:\ldots:\mathcal{I}_K}$) achievable by pure states of the individual partitions $\mathcal{I}_1:\ldots:\mathcal{I}_K$ ($\inf_{|s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\rangle}$).

The minimization of $\langle s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}|\hat{L}|s_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\rangle$ for pure, $\mathcal{I}_1:\ldots:\mathcal{I}_K$-separable states has been treated in Ref. [32]. There, so-called "separability eigenvalue equations" have been derived. The solution of those equations for a given observable $\hat{L}$ yields the minimal separability eigenvalue $g^{\min}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}$, which is also the desired infimum for separable states of the individual K-partition $\mathcal{I}_1:\ldots:\mathcal{I}_K$. We can conclude: A state is inseparable with respect to the convex combination of all K-partitions (K-entangled), if and only if there exists a Hermitian operator $\hat{L}$, such that

$$\langle\hat{L}\rangle < g^{\min}_K = \min_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\{g^{\min}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\}. \tag{5}$$

Although this condition clearly differs from the approach for individual partitions [30], it is remarkable that we can use a similar calculus. The method of separability eigenvalues was introduced to uncover entanglement of individual K-partitions $\mathcal{I}_1:\ldots:\mathcal{I}_K$, via $\langle\hat{L}\rangle < g^{\min}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}$ [32]. Now, it serves for detecting entanglement among convex combinations of all K-partitions [inequality (5)].

Witnessing multimode Gaussian states.—A Gaussian state is fully described by its covariance matrix. In the following, we will use the vector

$$\hat{\boldsymbol{\xi}} = (\hat{x}_1,\ldots,\hat{x}_N,\hat{p}_1,\ldots,\hat{p}_N)^T, \tag{6}$$

including the amplitude ($\hat{x}_j$) and phase ($\hat{p}_j$) quadratures of all possible modes ($j=1,\ldots,N$). The covariance matrix C of a Gaussian state can be written in terms of the symmetrically ordered elements $C^{ij} = \langle\hat{\xi}_i\hat{\xi}_j + \hat{\xi}_j\hat{\xi}_i\rangle/2 - \langle\hat{\xi}_i\rangle\langle\hat{\xi}_j\rangle$. As local displacements do not affect the entanglement, it is sufficient to analyze the covariance matrix of a Gaussian state, assuming $\langle\hat{\xi}_j\rangle = 0$. Thus, the most general form of a Gaussian test operator $\hat{L}$ is the quadratic combination

$$\hat{L} = \sum_{i,j=1}^{2N} M_{ij}\hat{\xi}_i\hat{\xi}_j, \tag{7}$$

with a symmetric, positive definite $2N\times 2N$ matrix $M=(M_{ij})_{i,j=1}^{2N}$. Note that Williamson's theorem allows us to diagonalize such a matrix M into a form $\mathrm{diag}(\lambda_1,\ldots,\lambda_N,\lambda_1,\ldots,\lambda_N)$ in terms of symplectic operations, see, e.g., [33].

The minimal separability eigenvalue of $\hat{L}$ in Eq. (7) for an individual partition $\mathcal{I}_1:\ldots:\mathcal{I}_K$ is given by [30]

$$g^{\min}_{\mathcal{I}_1:\ldots:\mathcal{I}_K} = \sum_{j=1}^{K}\sum_{k=1}^{|\mathcal{I}_j|}\lambda_k^{\mathcal{I}_j}, \tag{8}$$

where $|\mathcal{I}_j|$ is the cardinality of $\mathcal{I}_j$, and $\lambda_k^{\mathcal{I}_j}$ are the diagonal values of the Williamson decomposition of the submatrix which solely consists of the rows and columns of M that are in the index set $\mathcal{I}_j$ (see, also, the Supplemental Material of Ref. [30]). Finally, the entanglement condition (5) is given by the bound

$$g^{\min}_K = \min_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\{g^{\min}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}\text{in Eq.}(8)\}. \tag{9}$$

Hence, we have formulated an infinite number (for any positive, symmetric matrix M) of multipartite K-entanglement probes in an analytical form. This includes, as a subclass, Gaussian tests for two-entanglement, $g^{\min}_{K=2} > \langle\hat{L}\rangle$, which have been recently studied [24]. The analytical minima $g^{\min}_K$ in Eq. (9) are needed in order to correctly apply our entanglement condition (5).

Let us relate our method with other covariance based entanglement probes. For example, in [34,35], entanglement tests are constructed based on the partial transposition. As each mode can be either transposed or not, those criteria are only sensitive to individual bipartitions and, in a convex combination, two-entanglement. As our operator $\hat{L}$ [Eq. (7)] is defined in the most general second-order-moment form, it is well suited for uncovering entanglement in Gaussian states of any partitioning. Even if a test for two-entanglement fails, we can still probe for $K\geq 3$-entanglement. Other methods for inferring multipartite entanglement were introduced, e.g., in [36], based on semi-definite problems. Those approaches are limited to certain states, e.g., Gaussian ones, or they are computationally demanding. Extending the operator in Eq. (7) beyond quadratic terms and solving its separability eigenvalue equations generalizes our method, so it can be used to verify entanglement, also, in non-Gaussian states by inequality (3).

In order to get the best entanglement signature of all test operators $\hat{L}$ in terms of matrices M [Eq. (7)], we take the analytical solutions in Eqs. (8) and (9) and numerically minimize the signed significances

$$\Sigma_{\mathcal{I}_1:\ldots:\mathcal{I}_K} = \frac{\langle \hat{L} \rangle - g^{\min}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}}{\Delta \langle \hat{L} \rangle} \quad \text{and} \quad \Sigma_K = \frac{\langle \hat{L} \rangle - g^{\min}_K}{\Delta \langle \hat{L} \rangle}, \tag{10}$$

where $\Delta\langle \hat{L} \rangle$ denotes the experimental error of $\langle \hat{L} \rangle$, by finding the optimal matrix M for each of those significances. The signed significance is negative, $\Sigma_\chi < 0$, if the state is entangled with respect to the given notion of separability, $\chi = K$ or $\chi = \mathcal{I}_1:\ldots:\mathcal{I}_K$, which is certified with a significance of $|\Sigma_\chi|$ standard deviations. The numerical minimization was performed with a genetic algorithm [30] which can, in principle, not only find local minima, but also global ones [37]. Hence, one could claim that a positive value Σ_χ corresponds to a χ-separable covariance matrix. However, we will more carefully state, in such a case, that no χ-entanglement can be detected. As the resulting minima Σ from the genetic algorithm are upper bounds for Σ_χ, $\Sigma_\chi < 0$ is a reliable bound to the full entanglement, and it cannot overestimate the entanglement in our system.

Characterization of the SPOPO multimode quantum state.—The highly multimode light state that we consider in the following is a femtosecond frequency comb of zero mean value spanning over roughly $\sim 10^5$ individual equally spaced frequency components, generated by parametric down-conversion of a pump frequency comb in a synchronously pumped optical parametric oscillator (SPOPO). Details on its experimental generation and characterization can be found in Refs. [28,29] and [38]. The 12×12 covariance matrix C, containing the quadrature noise variances in six different frequency bands covering the whole spectrum of the SPOPO state, as well as the correlations between them, has been experimentally determined. The SPOPO state, being generated by an intense pump laser in a weakly nonlinear medium, is Gaussian to a very good approximation. Thus, the covariance matrix contains the whole information about the generated quantum state, at least within the frequency resolution given by the width of the frequency bands used in the measurements. The generated state is clearly mixed, as its purity, $\mathrm{tr}\hat{\rho}^2 = (\det C)^{-1/2} = 86.4\%$, is below one.

Entanglement structure of a six-mode SPOPO state.—For an $N = 6$-mode state, 203 possible individual partitions exist. That is one trivial partition $\mathcal{I}_1 = \{1, \ldots, 6\}$, 31 bipartitions $\mathcal{I}_1:\mathcal{I}_2$, 90 tripartitions, 65 four-partitions, 15 five-partitions, and one six-partition $\{1\}:\ldots:\{6\}$. Hence, we have six convex combinations of K-partitions.

The results of our analysis are shown in Fig. 2 in terms of the minimized signed significances in Eq. (10). The trivial partition $K = 1$ yields $\Sigma_{K=1} > 0$, which means that the measured covariance is a physical one. The value $\Sigma_{K=2} > 0$ shows that no detectable two-entanglement exists in the SPOPO quantum frequency comb. Yet, for all $K \geq 3$, K-entanglement is verified with a significance of at least seven standard deviations, $\Sigma_{K>2} < -7$. Such types of multipartite entanglement are not accessible with entanglement probes that are only sensitive to two-entanglement.

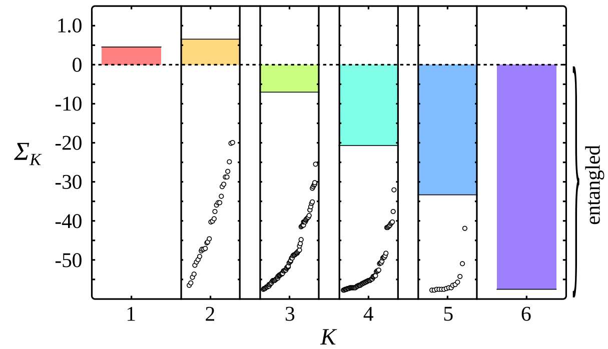

FIG. 2. Signed significance Σ_K (bars), for $1 \leq K \leq 6$, calculated from the data of the SPOPO state. The insets for $2 \leq K \leq 5$ give the values for the individual partitions, $\Sigma_{\mathcal{I}_1:\ldots:\mathcal{I}_K}$ (circles), sorted in increasing order. For better visibility, the positive part of the ordinate has a different scaling than the negative (entangled) part. Despite no signature of two-entanglement, $\Sigma_2 > 0$, the state shows highly significant other forms of multipartite entanglement.

Considering the circles in the insets for $1 \leq K \leq 5$ in Fig. 2, it can be seen that the same six-mode state is entangled with respect to all nontrivial, individual partitions—even for $K = 2$. Therefore, the SPOPO state is entangled with respect to any individual bipartition, even though it cannot be identified as a two-entangled state: The subtle structures of multipartite entanglement are invisible for genuine entanglement probes.

Here, we clearly see that entanglement of some or even all individual partitions $\mathcal{I}_1:\ldots:\mathcal{I}_K$ of the length K does not necessarily imply K-entanglement. Rather, it is the convex combination of the individual partitions that is responsible for the separability or inseparability. The inverse, however, is true: K-entanglement implies entanglement with respect to all individual K-partitions. This follows from the condition (5) by taking a proper test operator $\hat{L}$ for the convex combination and the same $\hat{L}$ for every individual K-partition, as $g^{\min}_K \leq g^{\min}_{\mathcal{I}_1:\ldots:\mathcal{I}_K}$. Finally, let us stress that this approach can be extended to study other convex combinations of some individual partitions which are not limited by a fixed K value.

Summary and conclusion.—We have studied different forms of K-party entanglement in multimode states. An analytical approach to construct the corresponding entanglement tests was derived and further elaborated for covariance based entanglement probes. To optimize over the resulting infinite set of all analytical Gaussian witnesses, a numerical optimization was performed. This approach allows us to classify entanglement in Gaussian states with an arbitrary number of modes.

We applied this approach to a parametrically generated multimode frequency comb. It was shown, for a six-mode example, that our system shows an interesting form of entanglement. That is, the SPOPO state turns out to be a biseparable state which is K-entangled for any $K = 3, \ldots, 6$. Moreover, we detected entanglement with respect to all individual partitions, even all the individual bipartitions. Thus, the absence of so-called genuine entanglement does not give any insight into the entanglement structure.

This work proves of great interest for investigating entanglement beyond two-entanglement in highly multipartite systems. A lot of questions remain to be investigated concerning other possible types of multipartite entanglement and, in particular, their relation to quantum computation and communication protocols between multiple parties. Our construction of general entanglement criteria, likely to access multipartite quantum correlations beyond bipartitions, provides a good starting tool for tackling such problems.

As a comment to the Einstein-Podolsky-Rosen paradox [1], Schrödinger emphasized that a compound quantum system includes more information than provided by the individual subsystems [2]. Considering our scenario at hand, we may extend such a statement. Namely, multipartite entanglement is much richer than the entanglement one can infer from bipartitions only.

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