

From Cylindrical to Stretching Ridges and Wrinkles in Twisted Ribbons

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Twisted ribbons under tension exhibit a remarkably rich morphology, from smooth and wrinkled helicoids, to cylindrical or faceted patterns. This complexity emanates from the instability of the natural, helicoidal symmetry of the system, which generates both longitudinal and transverse stresses, thereby leading to buckling of the ribbon. Here, we focus on the tessellation patterns made of triangular facets. Our experimental observations are described within an “asymptotic isometry” approach that brings together geometry and elasticity. The geometry consists of parametrized families of surfaces, isometric to the undeformed ribbon in the singular limit of vanishing thickness and tensile load. The energy, whose minimization selects the favored structure among those families, is governed by the tensile work and bending cost of the pattern. This framework describes the coexistence lines in a morphological phase diagram, and determines the domain of existence of faceted structures.

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Sheets subjected to external forces store the exerted work in elastic deformations that underlie wrinkled and crumpled states. Under tension, the exerted work is typically stored as stretching energy. When the forces are compressive, strain is negligible and the exerted work is instead stored as bending energy. For instance, a sheet resting on a soft substrate and compressed uniaxially deforms isometrically into wrinkles or folds [1,2]. However, when compression results from geometrical constraints, the final shape may involve a complex combination of bending and stretching energies. For instance, confining a thin sheet in 3D (i.e., a crumpled paper ball) is often described as an assembly of flat polygonal facets delimited by ridges where stretching and bending predominate [3]. Such a faceted morphology is an efficient minimizer of stretching since it is isometric to the undeformed sheet (i.e., strainless) everywhere except at those narrow ridges. Two such types of ridges have been reported: (i) isometric (cylindrical) ridges, which involve only bending, and (ii) stretching ridges, in which the bending and stretching energies are comparable, leading to a width $w_r \sim L^{2/3}t^{1/3}$, where $L \gg w_r$ is the ridge length and $t \ll w_r$ is the sheet thickness [3,4]. In addition, transitions from isometric to stretching ridges were recently reported for simple geometries. Witten showed that a single stretching ridge becomes isometric when both ends are truncated [5], and Fuentealba *et al.* have demonstrated a similar phenomenon for a tearing flap, when the pulling force exceeds a threshold $F_c \sim B/(t^{2/3}L^{1/3})$ (where $B \sim Et^3$ is the bending modulus and E is the Young’s modulus of the sheet) [6]. Both studies suggest that the curvature at the ridge’s end completely determines its shape.

In this Letter, we investigate the transition between isometric and stretching ridges in a twisted ribbon, and

characterize its impact on the mechanics of ribbons. Indeed, twisting ribbons was suggested as an original method to geometrically constrain thin sheets. In this setup, first proposed by Green [7,8], the two short edges of a flat ribbon are clamped and held apart by a tensile force while a twist is applied. In contrast to crumpling experiments, the existence of two degrees of freedom, the twist angle η and the tensile force T , enriches the variety of observed morphologies. The twist and tension determine (i) the stretching of the edges, and (ii) the contraction of the midline, $\chi = \eta^2/24 - T$ (where $\chi = 1 - L_{ee}/L$, L_{ee} being the end-to-end distance). The helicoid is the basic shape appearing for moderate twist angles $\eta < \sqrt{24T}$ [Fig. 1(d)]. Upon increasing the twist below a critical tension T^* , the helicoid midline is contracted and undergoes a buckling instability, whereby longitudinal wrinkles (also described as a “zigzag” fold) form around its midline, $\eta > \sqrt{24T}$ [9] [see Fig. 1(c)]. Further increasing the twist leads to a faceted morphology, also called a “creased helicoid” [10] or “ribbon crystal” [11] [Figs. 1(a) and 1(b)]. Additionally, the helicoid buckles in the transverse direction upon increasing the twist for a tension larger than T^* [12], and reaches a cylindrical shape [Fig. 1(e)]. Parts of these morphologies were recently organized in a tension-twist phase diagram by Chopin and Kudrolli [10]. We must clarify that we refer to tensile loads as “large,” “moderate,” or “small” according to their ratio with certain powers of the thickness t , but even a “large tension” corresponds to characteristic strains $< 10^{-2}$, deeply in the Hookean regime of the material response. In this sense, the morphological transitions reported in Refs. [7,8,10] and here are universal and not material specific.

The faceted morphology [Figs. 1(a) and 1(b)] has been described as an isometric shape by solving effective

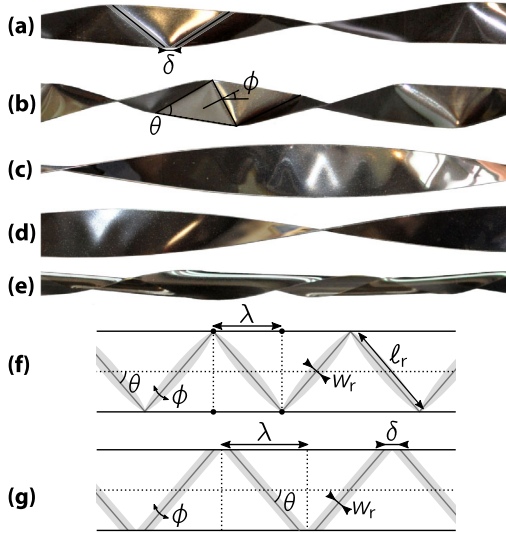


FIG. 1. (a)–(e) Pictures of the different morphologies. (a) Isometric ridges (FIRs). (b) Stretching ridges (FSRs). (c) Wrinkled helicoid (WH). (d) Helicoid. (e) Cylinder. (f) Parametrization of the stretching ridges illustrated in (b); the gray areas denote the ridges. (g) Parametrization of the isometric ridges illustrated in (a).

equations for the ribbon’s midline [13], or by assuming triangular facets separated by isometric ridges [11]. However, facets are observed over a whole region of the twist-tension phase diagram, where the tension is small but nonzero [10], suggesting that this morphology accommodates a finite amount of stretching. Furthermore, upon increasing the tension at a moderate twist, the facets turn into longitudinal wrinkles, which are clearly stretched [10] [see Fig. 1(c)]. Motivated by these observations, we focus here on the faceted morphology: we determine experimentally its domain of existence and propose a theoretical framework to explain how facets separated by isometric ridges (FIRs) turn into facets separated by stretching ridges (FSRs), and then to longitudinal wrinkles (WH), by increasing the tension.

We use ribbons of length L , width W , and thickness t under an external tension T and clamped at their short edges, which are twisted relative to each other by a prescribed angle Θ . Our ribbons are composed of polyethylene terephthalate (PET) (Young’s modulus $E \approx 3$ GPa). To simplify the discussion, we assume the Poisson ratio $\nu = 0$ (the numerical coefficients may depend on the Poisson ratio, but the qualitative results, including the transverse buckling instability, do not [12]). We use W as a unit of length, and the stretching modulus $Y = tE$ as a unit of in-plane stress (i.e., $W = Y = 1$), and introduce the twist per unit ribbon length $\eta = \Theta W/L$, and the energy per unit length U . The different observed shapes are shown in Fig. 1 and are organized in a phase diagram (Fig. 2), which focuses on the longitudinally buckled morphologies and complements the one reported in Ref. [10].

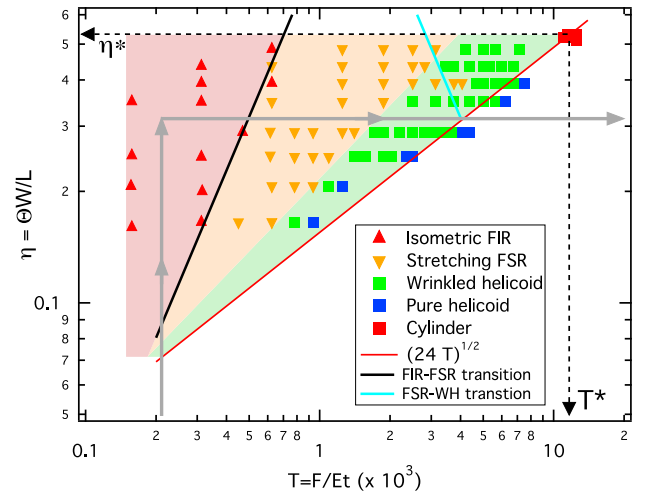


FIG. 2. Phase diagram of a PET ribbon with $t \approx 0.012$ and $L \approx 18$. The solid lines correspond to the coexistence lines (see text). The gray lines with arrows represent the experimental trajectory. T^* and η^* correspond to the helicoid-cylinder transition at a fixed length [10].

Our experimental “trajectory” is depicted by the gray lines in Fig. 2. To avoid hysteresis, the tension at each segment is either constant or increases. We start at a very small tension and zero twist, and progressively increase the twist. Once the chosen twist angle is attained, the tension is increased progressively. First, the ribbon takes a helicoidal shape [Fig. 1(d)]. As η is increased further, the centerline of the helicoid is under compression ($\eta > \sqrt{24T}$). A linear stability analysis shows that buckling occurs when $\eta_c \approx \sqrt{24T} + 10t$, forming first wrinkled helicoids [7–10,12]. The prediction for the critical twist is compared to our experimental results in Fig. 3, and shown in the phase diagram (Fig. 2, red line).

Facets separated by isometric ridges.—Increasing the twist beyond the buckling threshold, at a fixed small tension, we observe a shape resembling facets separated by rounded cylindrical ridges [see Fig. 1(a)]. We follow Ref. [11] and model it with flat triangular facets separated by cylindrical ridges. Such a shape is parametrized by the two angles θ and ϕ , formed, respectively, between the ridges and the ribbon’s midline, and between adjacent facets [Figs. 1(b) and 1(g)], and by the radius of curvature of the ridges R_c . In contrast to Ref. [11], we find θ and R_c by energy minimization.

We evaluate the energy using the general framework of asymptotic isometries (AI) [12], which is valid for physically admissible states in the doubly asymptotic limit of vanishing tension T and thickness t . The energy of such states can be approximated by the sum of a tensile work and an elastic energy U^{el} , both of which vanish in the limit t , $T \rightarrow 0$:

$$U = U^{\text{el}} + \chi T, \quad (1)$$

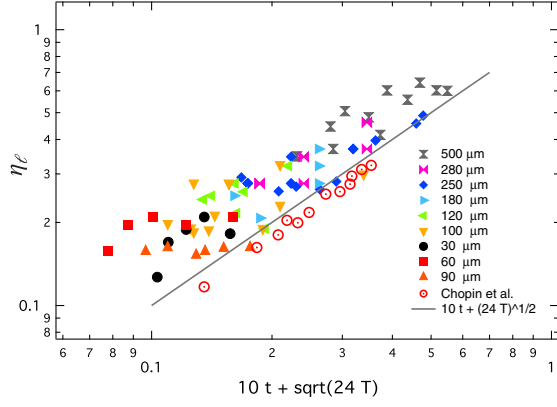


FIG. 3. Critical twist angle for the helicoid-facets transition, η_c , as a function of $\sqrt{24T} + 10t$ for different ribbons of PET (thicknesses are as indicated).

This formalism allows us to compute both U^{el} and χ from geometrical and mechanical considerations, yielding expressions that have no explicit dependence on T and which can be formally evaluated at $T = 0$. We can thus make a tension-independent construction by using parameters with geometrical meaning (θ , ϕ , R_c), whose actual dependence on T is found when minimizing the whole energy (1). Notice that the (unwrinkled) helicoid, for instance, is not an asymptotic isometry of the ribbon, since its elastic energy is proportional to η^4 and does not vanish as t , $T \rightarrow 0$ [12]. More generally, in the asymptotic limit $t \rightarrow 0$, the AI framework is valid only if the ratio η^2/T , between the twisted-induced stress η^2 and the exerted tension T , is sufficiently large. Since here $\eta^2/T \lesssim 100$, we cannot expect a perfect quantitative agreement of the results obtained in this framework with the experiments. We will use this framework not only for the FIRs but also for the other shapes obtained upon increasing the tension. A similar approach has been used in other studies of sheets (or shells) on which a Gaussian curvature is imposed in the presence of a small tension [14–16].

The twist $\eta(\theta, \phi, R_c)$ and contraction $\chi(\theta, \phi, R_c)$ of a FIRs can be obtained from geometrical arguments (see Ref. [11] and the Supplemental Material [17]). Upon expansion for small η and R_c , we obtain

$$\chi_{\text{FIR}} = \frac{\eta^2}{8} + \frac{\eta^3 R_c}{\sin(2\theta)} + \left(\frac{1}{48 \sin^2(\theta)} - \frac{5}{384} \right) \eta^4 + \mathcal{O}(\eta^5 R_c). \quad (2)$$

The bending energy of a single ridge is given by $u_r \sim t^2 \eta / [R_c \sin^2(\theta)]$ [using $\phi \approx \eta / \sin(\theta) + \mathcal{O}(\eta^3)$ [17]]. Assuming a small width of the ridge compared to the wavelength, $w_r \ll \lambda$, there are $N/L \approx 1/\lambda = \tan(\theta)$ ridges per unit ribbon length, and the elastic energy per unit ribbon length becomes

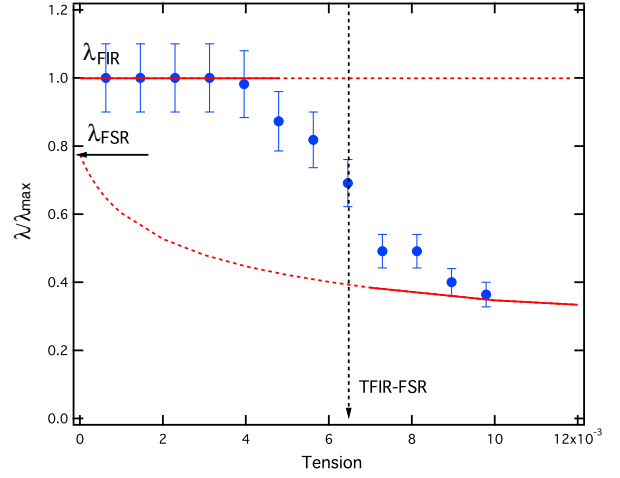


FIG. 4. Evolution of the measured size of the facets $\lambda/\lambda_{\text{max}}$ (filled symbols) for different PET ribbons ($\lambda_{\text{max}} \approx 1.3$, $t = 0.012$, $\eta = 0.37$). The red lines corresponds to the theoretical $\lambda(T)$ curves for FIRs and FSRs.

$$U_{\text{FIR}}^{\text{el}} \sim \frac{t^2 \eta}{R_c \sin(2\theta)}. \quad (3)$$

Minimizing the global energy $U_{\text{FIR}}^{\text{el}} + \chi_{\text{FIR}} T$ (keeping terms up to order η^3 in the contraction) yields

$$\theta = \pi/4, \quad \lambda = 1, \quad R_c \sim t/(\eta\sqrt{T}), \quad (4)$$

$$U_{\text{FIR}} \sim t\eta^2\sqrt{T} + \eta^2 T + \mathcal{O}(\eta^4). \quad (5)$$

Notably, the independence of the wavelength λ on tension indicates the robust, geometrical nature of the FIRs shape, despite the nontrivial dependence of its energy on the tensile load T . In hindsight, this robustness explains the validity of the purely geometric approach of Refs. [11,13] for describing the general structure of the FIRs. In Fig. 4, we test the prediction $\lambda = 1$ by varying the tension T at constant η . Only the low- T regime, where λ is constant, exhibits the FIRs. The independence of λ on T is in agreement with the theory, despite a slight discrepancy between the observed and predicted values of the wavelength that may be attributed to the finite size of the ribbon ($L \approx 15$), and to the global rearrangement needed to adjust the wavelength. We also note that a similar scaling of R_c with t and T was obtained in Ref. [6] for a completely different system, namely, the width of a pulling flap in an isometric configuration. Finally, inspection of Eq. (5) shows that the tensile work and elastic energy are balanced only for $T \sim t^2$. Therefore, for FIRs at a very small tension ($T \ll t^2$), the work done by the torque upon twisting the ribbon is stored efficiently as bending energy in the ridges, whereas the observation of FIRs for a larger tension ($T \gg t^2$) implies that the twister transmits its work to

the puller and the ribbon becomes a “bad capacitor” of energy.

Facets separated by stretching ridges.—Upon increasing the tension, the radius of curvature of the ridges decreases, until the ridges pinch along the ribbon’s long edges [Fig. 1(b)]. From visual inspection and the study of pulling flaps [6], we hypothesize that the ribbon’s shape consists of facets separated by stretching ridges (FSRs).

For the FSRs, the contraction χ_{FSR} can be directly evaluated by considering $R_c \ll \eta$ [18], retaining only the η^2 and η^4 terms of Eq. (2). The elastic energy of a single stretching ridge is given by $u_r \sim t^{5/3} \ell_r^{1/3} \phi^{7/3}$, where $\ell_r = 1/\sin(\theta)$ is the length of a ridge [19,20]. Using the above geometric relationships between ϕ , λ , θ , and η , we deduce that the elastic energy per unit ribbon length is

$$U_{\text{FSR}}^{\text{el}} \sim \frac{t^{5/3} \eta^{7/3}}{\sin(\theta)^{5/3} \cos(\theta)}. \quad (6)$$

Again, the total energy $U_{\text{FSR}}^{\text{el}} + \chi_{\text{FSR}} T$ should be minimized. The angle θ is the solution of $8[3\tan(\theta)^2 - 5]\sin(\theta)^{1/3}/\cos\theta = (\eta/t)^{5/3} T$. Interestingly, a physical solution only exists for $\theta \geq \theta_c = \arctan(\sqrt{5/3})$, which determines the size of the facets at vanishing tension, $\lambda = \sqrt{3/5} \approx 0.78$. For a small tension, the wavelength slightly decreases with tension as $\lambda \approx 0.78 - 6.4 \times 10^{-3} (\eta/t)^{5/3} T$. The energy of the FSRs ribbon at small tension becomes

$$U_{\text{FSR}} \sim t^{5/3} \eta^{7/3} + \eta^2 T + \mathcal{O}(\eta^4). \quad (7)$$

For isometric ridges (FIRs), increasing the tension decreases the radius of curvature R_c of the ridges [Eq. (4)], thus increasing the elastic energy of the ribbon. At some critical value of the tension, it becomes energetically favorable to switch to stretching ridges (FSRs), which enable saving some bending energy. Comparing the total energy of both faceted shapes [Eqs. (5) and (7)], we find that the FSRs appears for tensions above

$$T_{\text{FIR-FSR}} \sim t^{4/3} \eta^{2/3}. \quad (8)$$

This prediction is shown in the phase diagram (Fig. 2), in good agreement with our experimental observations. This scaling can also be obtained through a direct comparison of the widths of the isometric, $\eta R_c \sim t/\sqrt{T}$, and stretching, $(t/\eta)^{1/3}$, ridges. We note that the predicted dependence of the tension in t is similar to the transition force found in Ref. [6] between the isometric and stretching ridges for pulling flaps. Figure 4 provides significant support for the theoretical prediction of a sharp transition of the wavelength λ from a tension-independent plateau in a low- T regime (FIRs) to a tension-dependent branch (FSRs).

Another recent work [21] also found a sharp transition of the facet’s size upon increasing the tension.

In the FSRs regime, the evolution of the wavelength as a function of tension can be obtained numerically up to an unknown numerical factor multiplying T (see Fig. 4). To determine the asymptotic behavior of the wavelength λ at large tension, we expand the two terms in the energy in $\lambda = (\pi/2) - \theta$:

$$U_{\text{FSR}}^{\text{el}} + \chi_{\text{FSR}} T \sim \frac{t^{5/3} \eta^{7/3}}{\lambda} + T \eta^2 [1 + \eta^2 (1 + \lambda^2)], \quad (9)$$

whose minimization leads to $\lambda \sim (t/\eta)^{5/9} T^{-1/3}$. This relation is consistent with the general trend for the wavelength. Inserting the wavelength expression in Eq. (9) yields the total energy of the FSRs ribbon for large tensions, i.e., when the angle θ is close to $\pi/2$,

$$U_{\text{FSR}}^T \sim t^{10/9} \eta^{26/9} T^{1/3} + T \eta^2. \quad (10)$$

The FSRs description assumes that the width of the ridges, which is given by $w_r \sim (t/\eta)^{1/3}$, remains small compared to the size of the facets. This assumption holds as long as $T < (t/\eta)^{2/3}$.

Wrinkled helicoid.—Turning now to higher tension values, it is natural to ask how the FSRs state [Fig. 1(b)] transforms into the wrinkled helicoid [Fig. 1(c)], observed for tensions slightly smaller than the fixed ribbon length limit $T = \eta^2/24$ (see Fig. 2) [22]. For $T < \eta^2/24$, the stress field within a twisted ribbon can be divided into three parts. The central part of the helicoid $|r| < r_{\text{wr}}$ is under compression, while the two outer parts ($r_{\text{wr}} < |r| < 1/2$) remain stretched. A basic description of the wrinkled helicoid state has been developed in Ref. [12], using a far-from-threshold theory, whereby the inner zone around the helicoid’s midline is decorated with longitudinal wrinkles that fully relax the compression while the two outer strips are stretched. The width $2r_{\text{wr}}$ of the wrinkled zone is determined by the ratio T/η^2 , vanishing for $T/\eta^2 \rightarrow 1/24$ and close to 1 at asymptotic isometry, where $T/\eta^2 \rightarrow 0$. In this limit, the wrinkled helicoid provides a remarkable example of a state that is arbitrarily close to isometric deformation of the ribbon, although the Gaussian curvature K of the envelope (helicoidal) shape is finite ($-\eta^2$). This type of “nondevelopable isometry” [23] is strictly different from a piecewise-developable shape (e.g., the faceted shapes discussed above), for which $K = 0$ almost everywhere, echoing ideas from the mathematical literature [24]. While a full characterization of the wrinkled helicoid is beyond the scope of the current work, we use below a scaling analysis to explore the possibility of a transition from FSRs to a wrinkled helicoid at the asymptotic isometry limit $t, T \rightarrow 0$. Note that this is obviously a crude approximation of the shape observed in Fig. 1(b), where the wrinkles do not reach the edges.

Assuming that the ribbon does approach a fully wrinkled helicoid shape at $T \ll \eta^2$, the contraction χ_{WH} can be computed by assuming that the edges, $|r| = r_{\text{wr}} \approx 1/2$, are neither wrinkled nor stretched. A simple calculation yields

$$\chi_{\text{WH}} = 1 - \sqrt{1 - \frac{\eta^2}{4}} = \frac{\eta^2}{8} + \frac{\eta^4}{128} + \mathcal{O}(\eta^6). \quad (11)$$

Note that this expression corresponds to the contraction of the facets given by Eq. (2), in the limit $R_c \rightarrow 0$, $\theta \rightarrow \pi/2$. In order to evaluate the elastic energy, we must determine the amplitude A and wavelength λ of the wrinkles. Since the wrinkles completely relax the compression, we obtain the ‘‘slaving condition’’: $A/\lambda \sim \eta$ [12]. The elastic energy is governed by stretching in the transverse direction, $U_{\text{str}} \sim A^4$, and bending in the longitudinal direction, $U_{\text{bend}} \sim t^2 A^2/\lambda^4$. Minimizing the total energy with respect to λ yields

$$\lambda \sim (t/\eta)^{1/3}, \quad U_{\text{WH}} \sim t^{4/3} \eta^{8/3} + T\eta^2. \quad (12)$$

Unsurprisingly, the wavelength scales with the thickness similarly to the width of the stretching ridge [19], reflecting the same type of energy balance used in both cases. Comparing the energy estimates for both morphologies [Eqs. (10) and (12)] we see that the WH is energetically favorable for tensions above

$$T_{\text{FSR-WH}} \sim (t/\eta)^{2/3}. \quad (13)$$

Remarkably, this occurs when the size of the facets becomes comparable to the width of a ridge in the FSRs. Note that this value is much larger than the tension where the FSRs appears [see Eq. (8)], which guarantees a large domain of existence for the FSRs. However, the predicted value of the transition $T_{\text{FSR-WH}}$ is larger than the critical tension $T^* \sim t$ (see Fig. 2) where transverse buckling instability occurs [12], meaning that the WH cannot exist in the asymptotic isometry limit. The experimental data show a transition at $T/\eta^2 \approx 1/40$ (Fig. 2), rather close to the onset of longitudinal buckling, $T/\eta^2 < 1/24$. We are thus far from the asymptotic isometry regime ($T/\eta^2 \rightarrow 0$).

In conclusion, we employed here the framework of asymptotically isometric shapes, together with an analysis of the energy of elastic ridges, to classify the various types of longitudinally buckled morphologies attained by a twisted ribbon at a very small thickness and tensile load: facets separated by isometric or stretching ridges, and a wrinkled helicoid. We hope that the rich plethora of distinct patterns and transitions will lead to further studies of this system as a model for the spontaneous emergence of morphological complexity in elastic sheets under geometric constraints.

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