Nonequilibrium Dynamics and Phase Transitions in Holographic Models

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We study the poles of the retarded Green's functions of strongly coupled field theories exhibiting a variety of phase structures from a crossover up to a first order phase transition. These theories are modeled by a dual gravitational description. The poles of the holographic Green's functions appear at the frequencies of the quasinormal modes of the dual black hole background. We establish that near the transition, in all cases considered, the applicability of a hydrodynamic description breaks down already at lower momenta than in the conformal case. We establish the appearance of the spinodal region in the case of the first order phase transition at temperatures for which the speed of sound squared is negative. An estimate of the preferential scale attained by the unstable modes is also given. We additionally observe a novel diffusive regime for sound modes for a range of wavelengths.

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Introduction.—One of the most surprising discoveries in contemporary theoretical physics, the anti–de Sitter/conformal field theory (AdS/CFT) correspondence [1], provides for us a way to investigate the dynamics of strongly coupled quantum field theories by means of general relativity methods. An important field of research exploiting this new relation between geometry and physics aims at exploring the real-time dynamics of strongly interacting hot matter [2]. In particular, real-time response of a thermal equilibrium state has been quantified in the case of $\mathcal{N}=4$ super-Yang-Mills theory by means of the poles of the retarded Green's function [3]. The locations of these poles correspond to quasinormal mode (QNM) frequencies in the dual gravitational theory. Initial steps towards the extension of this case to nonconformal field theories that still admit a gravitational dual description were taken in Ref. [4,5] where QNM frequencies of an external scalar field were studied.

In this Letter we take another step in quantifying real-time response of a strongly coupled nonconformal field theory. First, we analyze all allowed channels of energy-momentum tensor perturbations and corresponding two-point correlation functions. Secondly, we concentrate on the phenomena appearing in the vicinity of a nontrivial phase structure of various type: a crossover (motivated by QCD), a second order phase transition, and a first order phase transition. These cases are modeled by choosing appropriate scalar field self-interaction potentials in a holographic gravity-scalar theory used in [6]. We focus here on phase transitions postponing more details of the other cases to [7]. Thirdly, we concentrate on the lowest nonhydrodynamic QNM in relation to the hydrodynamic ones as this provides insight into the range of applicability of the hydrodynamic description.

Because of the ubiquity of employing a hydrodynamic description for quark gluon plasma it is important to understand its limitations and the influence of other nonhydrodynamic degrees of freedom. Let us emphasize, however, that in the present investigation we are targeting quite different physics from "early thermalization" that has so far been predominantly studied within the AdS/CFT correspondence. Here we expect that the plasma has already thermalized, which could occur at higher temperatures, e.g., in the almost conformal regime. Indeed, the investigations of [4,5] indicate that nonconformality should not influence this physics much. Subsequently the plasma cools and approaches a phase transition of appropriate type or a crossover. It is this final stage of plasma evolution that we are considering in this Letter and, in particular, the pattern of excitations of such a plasma system and the range of applicability of a hydrodynamic description in the vicinity of a phase transition.

The background.—The black hole background (BH) solutions for the QNM calculations follow from the action

$$S=\frac{1}{2\kappa_5^2}\int d^5x\sqrt{g}\left[R-\frac{1}{2}(\partial\phi)^2-V(\phi)\right], \qquad (1)$$

where $\kappa_5=\sqrt{8\pi G}$ and $V(\phi)$ is thus far arbitrary. These solutions are similar to those studied in Ref. [6], but since our goal is to determine the quasinormal mode frequencies, it is convenient to express them in Eddington-Finkelstein coordinates, which have been proven useful in the case of the scalar field modes [4]. The line element reads

$$ds^2=e^{2A(r)}[-h(r)dt^2+d\vec{x}^2]-2e^{A(r)+B(r)}dtdr, \quad (2)$$

and for the scalar field $\phi(r)=r$. Since we require that asymptotically the geometry is that of AdS space the potential needs to have the following small ϕ expansion:

TABLE I. Potentials chosen to study different equations of state (EoS) exhibiting different phase structure and corresponding conformal dimension of the scalar field.

Potential	a	γ	b_2	b_4	b_6	Δ
V_{QCD}	0	0.606	1.4	−0.1	0.0034	3.55
V_{2nd}	0	$1/\sqrt{2}$	1.958	0	0	3.38
V_{1st}	0	$\sqrt{7/12}$	2.5	0	0	3.41
V_{IHQCD}	1	$\sqrt{2/3}$	6.25	0	0	3.58

$$V(\phi) = -\frac{12}{L^2} + \frac{1}{2}m^2\phi^2 + O(\phi^4). \tag{3}$$

Here L is the AdS radius, which we set $L = 1$ by the freedom of the choice of units. The relation of the scalar mass and corresponding operator conformal dimension is $\Delta(\Delta - 4) = m^2$. In general we consider a family of potentials [6,8],

$$V(\phi) = -12(1 + a\phi^2)^{1/4}\cosh(\gamma\phi) + \sum_{n=1}^{3} b_{2n}\phi^{2n}, \tag{4}$$

with parameter values shown in Table I. The $a = 1$ case is called the improved holographic QCD (IHQCD) [8]. We are interested in solutions possessing a horizon, which requires that the function h should have a 0 at some $\phi = \phi_H$, i.e., $h(\phi_H) = 0$. We solve the coupled equations of motion using spectral discretization and the Newton-Raphson iterative algorithm. The corresponding equations of state are obtained from standard procedures in holography. The entropy $s(\phi_H)$ is obtained from $1/4$ of the area of the horizon, while the temperature $T(\phi_H)$ is obtained from the nonsingularity of the Euclidean horizon. Consequently one gets the EoS in the form of a temperature dependence of the entropy $s(T)$.

In the case of the V_{1st} potential (cf., left panel of Fig. 1) there are two stable black hole branches at low (small BH) and high (large BH) temperatures. They are separated by a branch of black holes that exhibit spinodal instability. A first order phase transition appears at the temperature $T_c \simeq 1.05T_m$ between large and small black hole solutions [7]. For the V_{2nd} potential the system exhibits a second order phase transition at $T = T_c$ that is defined by $c_s(T_c) = 0$ [6] (cf., left panel of Fig. 1). The specific heat critical exponent is $\alpha \simeq 0.658$ [7].

Quasinormal modes.—We consider perturbations of the background in the following form: $g_{ab}(r,z) = g^{(0)}_{ab}(r) + h_{ab}(r)e^{-i\omega t+ikz}$, $\phi(r,z) = r + \psi(r)e^{-i\omega t+ikz}$. Following [3,4] we consider infinitesimal diffeomorphism transformations, $x^a \mapsto x^a + \xi^a$, of the form $\xi_a = \xi_a(r)e^{-i\omega t+ikz}$, and look for linear combinations of metric and scalar perturbations that are invariant under those transformations. There are four families of such modes, two of which are decoupled and two of which are coupled. Written explicitly, the coupled modes read

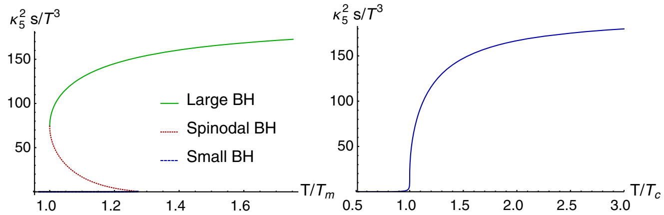

FIG. 1. Left panel: equation of state for V_{1st} potential with different regions marked. The meeting point of red and green lines defines the corresponding minimal temperature T_m. Right panel: equation of state for V_{2nd}.

$$Z_1(r) = H_{aa}(r)\left[\frac{k^2h'(r)}{2A'(r)} + k^2h(r) - \omega^2\right] + k^2h(r)H_{tt}(r) + \omega[2kH_{tz}(r) + \omega H_{zz}(r)], \tag{5}$$

$$Z_2(r) = \psi(r) - \frac{H_{aa}(r)}{2A'(r)}. \tag{6}$$

In the above $h_{aa}(r) = h_{xx}(r) = h_{yy}(r)$ are transverse metric components and we have factorized the background from the metric perturbations in the following way: $h_{tt}(r) = h(r)e^{2A(r)}H_{tt}(r)$, $h_{tz}(r) = e^{2A(r)}H_{tz}(r)$, $h_{aa}(r) = e^{2A(r)}H_{aa}(r)$, $h_{zz}(r) = e^{2A(r)}H_{zz}(r)$. Comparing with Ref. [3] we can see that the $Z_1(r)$ mode corresponds to the sound mode, while the $Z_2(r)$ might be called a nonconformal mode, since it is intimately related to the scalar field. The third decoupled mode is the shear one and is expressed as $Z_3(r) = H_{xz}(r) + (\omega/k)H_{tx}(r)$. The dynamics of the fourth mode is governed by an equation of motion that is similar to the external massless scalar equation, which was studied in [4]. As usual at the horizon we take the ingoing boundary conditions, which in our coordinates means a regular solution. The conformal boundary ($r \sim 0$) asymptotic is

$$Z_1(r) \sim A_1 + B_1 r^{(4/4-\Delta)}, \qquad Z_2(r) \sim A_2 r + B_2 r^{(\Delta/4-\Delta)}. \tag{7}$$

Transformation to the usual Fefferman-Graham coordinates close to the boundary, $r \mapsto \rho^{4-\Delta}$, reveals that $Z_1(\rho)$ has the asymptotics of a massless scalar field like the perturbations considered in [3]. This perturbation corresponds to the sound mode of the theory. On the other hand $Z_2(\rho)$ has the asymptotics of the background scalar field ϕ and is similar to the case studied in [9]. According to the AdS/CFT dictionary the boundary conditions are the requirement of vanishing sources, i.e., $A_1 = A_2 = 0$. The shear mode perturbation $Z_3(r)$ has the same asymptotics as $Z_1(r)$ and requires a standard Dirichlet boundary condition at $r = 0$.

The problem of determining the quasinormal frequencies is a form of a generalized eigenvalue equation, which for given k results in a well-defined set of frequencies $\omega(k)$ [7]. Note that all modes, for which $\mathrm{Re}\omega(k) \neq 0$, come in pairs, i.e., $\omega(k) = \pm|\mathrm{Re}\omega(k)| + i\mathrm{Im}\omega(k)$.

Results.—For all the potentials we have made natural consistency checks. For high temperatures, i.e., small horizon radius in the sound and the shear channels, we have an agreement with the conformal results of Ref. [3]. An important thing to note here is that due to the coupled nature of the modes $Z_1(r)$ and $Z_2(r)$ all frequencies, except for the hydrodynamical one, come in pairs. This effect is present even at high temperatures, where the system is expected to be conformal. The second most damped nonhydrodynamic mode turns out to be the most damped one found in Ref. [3].

The hydrodynamical QNMs are defined by the condition $\lim_{k\to 0}\omega_H(k) = 0$, and are related to transport coefficients in the following way, $\omega \simeq -i(\eta/sT)k^2$, $\omega \simeq \pm c_s k - i\Gamma_s k^2$, respectively, in the shear and sound channels. Those formulas are approximate in the sense that in general higher order transport coefficients should be considered [10]. However, for appropriately small momenta, second order expansion is enough, and we use it to read off the lowest transport coefficients of the model. The sound attenuation constant, Γ_s, is related to shear η and bulk ζ viscosities by $\Gamma_s = [(\zeta + 4\eta/3)/2sT]$. Also these formulas were used to make a second check of the results: compute the speed of sound c_s from the hydrodynamic mode and values of the shear viscosity and compare it respectively to the ones obtained from the gravitational background calculations and predictions known in the literature [11,12]. Both of them are always satisfied, and $\eta/s = 1/4\pi$ in all cases considered in this Letter.

In the analysis below we measure the momentum and the frequency in the units of temperature by setting $q = k/2\pi T$, $\varpi = \omega/2\pi T$. There are a few novel observations that we make from the pattern of QNM frequencies. The first is an estimate of the momentum, or equivalently the length scale, at which the hydrodynamic description of the plasma system breaks down. For the CFT case this was estimated to be $q = 1.3$ where in the shear channel first nonhydrodynamic QNM dominated the system dynamics [13], being less damped than the hydrodynamic shear mode. The new effect we find is that away from conformality, in the vicinity of a phase transition, we see this crossing not only in the shear channel but also in the sound channel, as illustrated in Fig. 2. This shows that the influence of a nontrivial phase structure of the background affects the applicability of hydrodynamics in a qualitative way. Moreover, the momentum at which this crossover happens in the sound channel is smaller, which means that the applicability of hydrodynamics near the phase transition is more restricted than in the high temperature case. In Table II we summarize the critical values of momenta

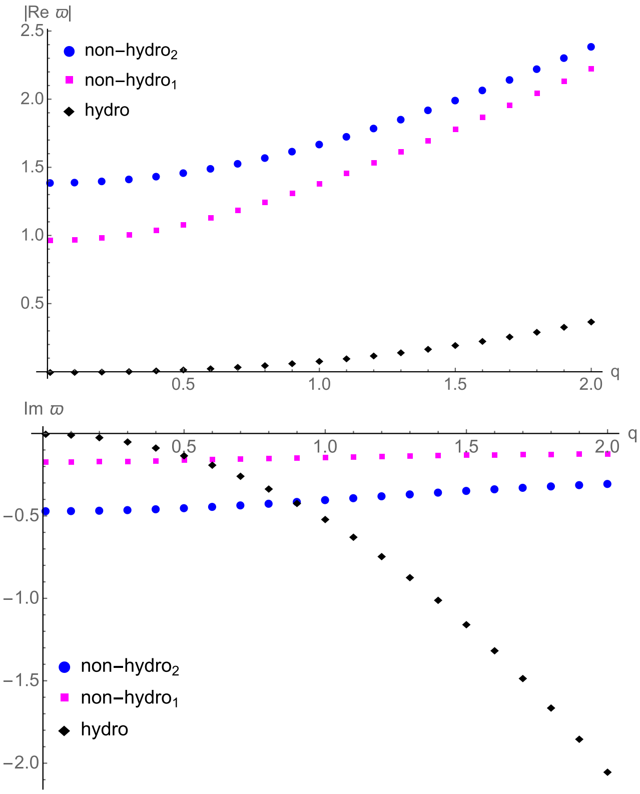

FIG. 2. Quasinormal modes for the potential $V_{2\mathrm{nd}}$ at T_c: real part (upper panel) and imaginary part (lower panel). The speed of sound and the ratio of bulk viscosity to the entropy density are calculated, namely, $c_s \simeq 0$, $\zeta/s \simeq 0.061$ (cf. [14]).

where the hydrodynamic description of the system breaks in sound and shear channels for the potentials we considered in this Letter.

The second observation is the bubble formation in the spinodal region in the case of the first order phase transition [15]. This happens when $c_s^2 < 0$, which means that the hydrodynamic mode is purely imaginary, $\omega_H = \pm i|c_s|k - i\Gamma_s k^2$. For small k, the mode with the plus sign is in the unstable region, i.e., $\mathrm{Im}\omega_H > 0$. For larger momenta the other term starts to dominate, so that there is $k_{\mathrm{max}} = |c_s|/\Gamma_s$ for which the hydromode becomes again stable. The scale of the bubble is the momentum for which positive imaginary part of the hydrodynamic mode attains

TABLE II. The momenta for which the crossing phenomena between the hydrodynamic and first nonhydrodynamic QNM happen. Values are given at corresponding critical temperatures (T_m for $V_{1\mathrm{st}}$ and V_{IHQCD}).

Potential	Sound channel q_c	Shear channel q_c
V_{QCD}	0.8	1.1
$V_{2\mathrm{nd}}$	0.55	0.9
$V_{1\mathrm{st}}$	0.8	1.15
V_{IHQCD}	0.14	1.25

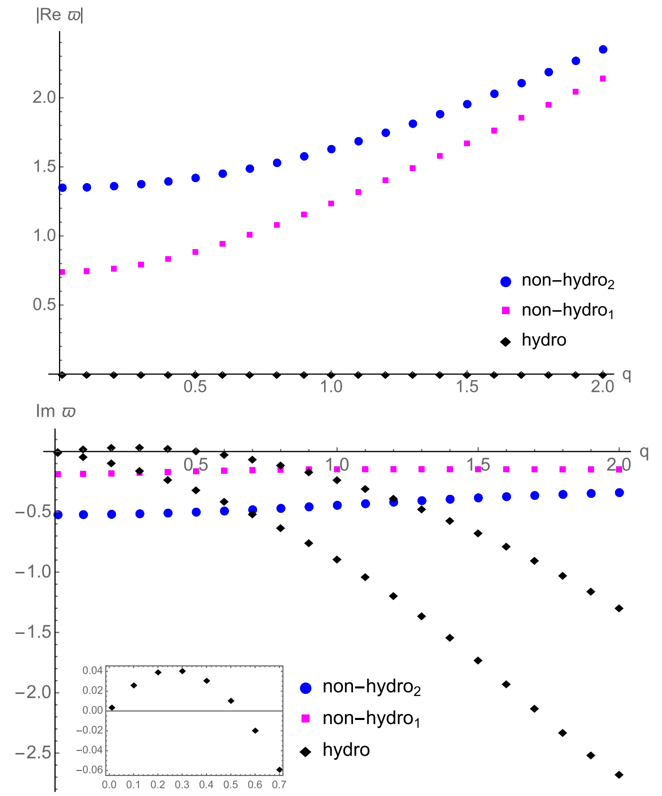

FIG. 3. Sound channel quasinormal modes for the potential $V_{1\mathrm{st}}$ at $T \simeq 1.067 T_m \gtrsim T_c$ in the unstable region of the EoS. An instability of the spinodal region is shown in the inset.

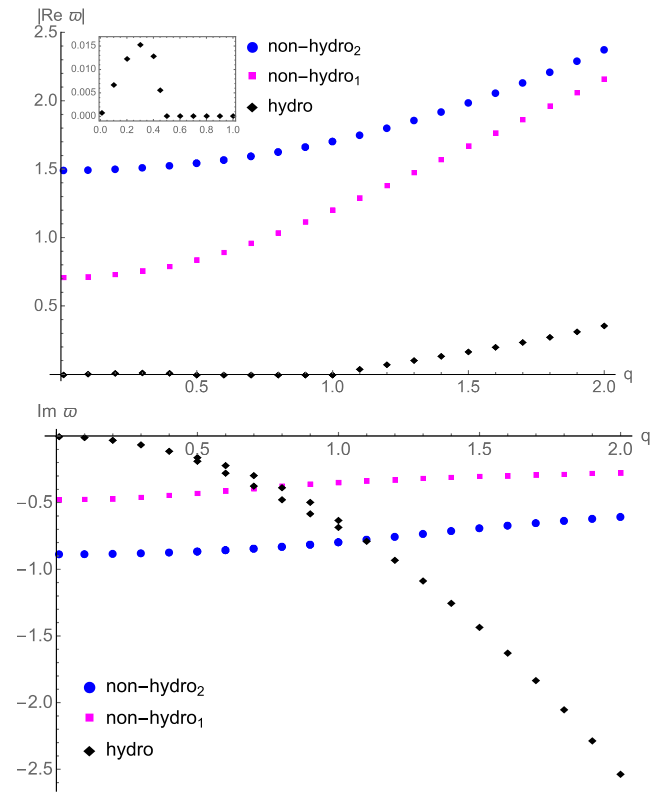

FIG. 4. Quasinormal modes for the potential $V_{1\mathrm{st}}$ at $T \simeq 1.00004 T_m$ in the stable region of the EoS, below the transition $T < T_c \simeq 1.05 T_m$: real part (upper panel) and imaginary part (lower panel).

the maximal value. The imaginary part of the unstable hydrodynamic mode is called the growth rate [15], and is illustrated in the inset in Fig. 3. This phenomenon is very similar to a Gregory-Laflamme instability [16]. When ω_H is purely imaginary, one can express it as $\omega_H = \pm i O(k) - i E(k)$, with $O(-k) = -O(k)$ and $E(-k) = E(k)$. Then there are two separated branches of the hydrodynamic modes, as seen on Figs. 3 and 4. In summary, in all cases considered, whenever there was an indication of thermodynamic instability in equations of state, the lowest QNM displayed a dynamical instability. Also, nonhydrodynamic modes were always stable, even for large k. This supports the expectation spelled out in Ref. [17].

The third observation is that near the minimal temperature in the case of first order phase transition the hydrodynamic mode (cf. Fig. 4), and in the IHQCD case also the lowest nonhydrodynamic modes, become purely imaginary for a range of momenta [7]. The interpretation of this fact is that the corresponding wavelengths cannot propagate at a linearized level, and there is a diffusionlike mechanism for those modes. The onset of the appearance of a nonpropagating sound mode in the deeply overcooled phase has been observed earlier in a related model [18]. Our analysis shows that the range of momenta for which the nonpropagating mode appears is finite. It is important to note that generically the ultralocality [4] of the nonhydrodynamic mode is still present in the critical region of the phase diagram. The only exception observed is the IHQCD potential, where the modes exhibit a nontrivial behavior [7]. Most of the interesting dynamics and effects observed are due to the different behavior of the hydrodynamic modes. This includes the instability and the bubble formation in the case of the first order phase transition. However, no interesting structure in QNMs appears at T_c (in contrast to the vicinity of T_m), which suggests that at the linearized level one cannot detect the transition.

Conclusions.—In the present Letter we performed an extensive study of the linearized dynamics of excitations in strongly coupled field theories in the vicinity of a nontrivial phase structure of various kinds. We observed a number of novel features that were not present in the conformal case. First, for relatively small k, the propagating hydrodynamical sound modes become more damped than the lowest nonhydrodynamic degrees of freedom. This provides a more stringent restriction on the applicability of hydrodynamics and indicates the necessity of incorporating these other degrees of freedom on appropriate length scales. This is in contrast to the conformal case where a similar phenomenon only occurred in the shear channel and only at a higher value of k. Secondly, we explicitly determined the instability in the spinoidal branch of a first order phase structure and estimated the length scale for bubble formation. Thirdly, close to the point $T = T_m$ on the first order equation of state, the sound mode frequencies become purely imaginary for a range of momenta, thus indicating that these modes effectively do not propagate at these length scales. The richness of phenomena appearing in the linearized regime strongly suggests that it is important to study the corresponding real-time dynamics also at the nonlinear level.

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