Lieb-Robinson Bound and the Butterfly Effect in Quantum Field Theories

Daniel A. Roberts*
Center for Theoretical Physics and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

Brian Swingle†
Stanford Institute for Theoretical Physics and Department of Physics, Stanford University, Stanford, California 94305, USA

(Received 25 May 2016; published 23 August 2016)

As experiments are increasingly able to probe the quantum dynamics of systems with many degrees of freedom, it is interesting to probe fundamental bounds on the dynamics of quantum information. We elaborate on the relationship between one such bound—the Lieb-Robinson bound—and the butterfly effect in strongly coupled quantum systems. The butterfly effect implies the ballistic growth of local operators in time, which can be quantified with the "butterfly" velocity v_B. Similarly, the Lieb-Robinson velocity places a state-independent ballistic upper bound on the size of time evolved operators in nonrelativistic lattice models. Here, we argue that v_B is a state-dependent effective Lieb-Robinson velocity. We study the butterfly velocity in a wide variety of quantum field theories using holography and compare with free-particle computations to understand the role of strong coupling. We find that v_B remains constant or decreases with decreasing temperature. We also comment on experimental prospects and on the relationship between the butterfly velocity and signaling.

DOI: 10.1103/PhysRevLett.117.091602

In relativistic systems with exact Lorentz symmetry, causality requires that spacelike-separated operators commute. In nonrelativistic systems, there is no analogous notion: a local operator $V(0)$ at the origin need not commute with another local operator $W(x,t)$ at position x at a later time t, even if the separation is much larger than the elapsed time $|x| \gg t$. This can be understood by considering the Baker-Campbell-Hausdorff formula for the expansion of $W(x,t) = e^{iHt}W(x)e^{-iHt}$,

$$W(x,t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \underbrace{[H,\ldots[H}_{k}, W(x)]\ldots], \qquad (1)$$

where H is the Hamiltonian which is assumed to consist of bounded local terms. As long as there is some sequence of terms in H that connect the origin and point x (and absent any special cancellations), the operator $W(x,t)$ will generically fail to commute with $V(0)$.

This does not necessarily imply that the magnitude commutator $[W(x,t), V(0)]$ between distance operators must be large. A bound of Lieb and Robinson [1], along with many subsequent improvements [2–4], limits the size of commutators of local operators separated in space and time, even in nonrelativistic systems. In terms of the Heisenberg operator $W(x,t)$ at position x and time t and an operator V at the origin of space and time, the bound reads

$$\|[W(x,t), V(0)]\| \leq K_0 \|W\| \|V\| e^{-(|x|-v_{\mathrm{LR}}t)/\xi_0}, \qquad (2)$$

where K_0 and ξ_0 are constants, $\|\cdot\|$ indicates the operator norm, and v_{LR} is the Lieb-Robinson velocity. The growth of the commutator is controlled by v_{LR}, which is a function of the parameters of the Hamiltonian. Hence, although operators separated by a distance x may cease to exactly commute for any $t > 0$, the Lieb-Robinson bound implies that their commutator cannot be $O(1)$ until $t \gtrsim x/v_{\mathrm{LR}}$.

Thus, the Lieb-Robinson velocity provides a natural notion of a "light" cone for nonrelativistic systems. Even for relativistic systems, where causality implies that the commutator of local operators must be exactly zero for $t < x$ (in this Letter we have set the speed of light to unity, $c = 1$), the Lieb-Robinson cone, if more restrictive, determines where in space-time acting with an early operator $V(0)$ can nontrivially effect a later operator $W(x,t)$.

Commutators $[W(x,t), V(0)]$ of local Hermitian operators separated by time and space can also be used to characterize the butterfly effect in many-body quantum systems [5,6]. The butterfly effect can be defined in terms of such a commutator, which expresses the dependence of later measurements of distant operators $W(x,t)$ on an earlier perturbation $V(0)$. For strongly chaotic systems, such a commutator can exhibit exponential growth in time, in analogy to the classical butterfly effect. This connection was recently made sharp within the AdS/CFT correspondence, where quantum chaos in strongly coupled large-N gauge theories was shown to be connected to universal properties of high-energy scattering in the vicinity of the horizons of black holes [7]. This has inspired a large body of additional work [8–30].

To study the typical matrix elements of $[W(x,t), V(0)]$, it is useful to consider the average of its square,

$$C(x,t) \equiv -\langle [W(x,t), V(0)]^2 \rangle_\beta, \tag{3}$$

where $\langle \cdot \rangle_\beta$ indicates thermal average at inverse temperature $\beta = 1/T$. $C(x,t)$ characterizes the strength of the butterfly effect at x at time t after an earlier perturbation of $V(0)$ at the origin. The statement of many-body chaos is that such commutators should grow to be large for almost all choices of operators W, V [11,12] and should remain large for a long time thereafter. The time when the commutator grows to be $O(1)$ (for suitably normalized operators) is known as the "scrambling" time [7,31,32] and is usually denoted t_*. For large-N gauge theories with $O(N^2)$ degrees of freedom per site, the early-time approach to scrambling is governed by an exponential growth with time,

$$C(x,t) = \frac{K}{N^2} e^{\lambda_L (t - x/v_B)} + O(N^{-4}), \tag{4}$$

for some constants K, λ_L, and v_B that can depend on the choice of operators in the commutator W, V. (It can be that $K = 0$, in which case the early-time growth is governed by the $O(N^{-4})$ term.) The onset of chaos is characterized by the two quantities: λ_L and v_B.

λ_L has been called a "Lyapunov" exponent [11]—in analogy with classical chaos—and characterizes the growth of chaos in time. (In Ref. [19] it is argued that the classical limit of λ_L does not always map onto the classical definition of the Lyapunov exponent; hence, our use of "quotes.") Maldacena, Shenker, and Stanford [14] showed that this exponent is bounded by the temperature $\lambda_L \leq 2\pi/\beta$, with conjectured saturation for systems with thermal states that have a large-N holographic black holes description whose near-horizon region is well described by Einstein gravity.

v_B is a velocity—the "butterfly" velocity—and characterizes the speed at which the small perturbation grows [7]. Considering the commutator Eq. (3) as quantifying the effect of the perturbation $V(0)$ on $W(x,t)$, one may understand the butterfly effect as the growth of the operator $V(0)$ under time evolution. The speed of the growth is characterized by v_B, and the commutator begins to increase when $t \approx x/v_B + \lambda_L^{-1} \log N^2$ [10]. This defines an effective light cone for chaos, a butterfly cone, outside of which the system is not affected by the perturbation.

In this Letter, we explore the relationship between the Lieb-Robinson bound and the butterfly effect. A similarity was first noticed in Ref. [10], where it was pointed out that v_{LR} can be used to bound the rate of growth of operators. Here, we would more directly like to contrast v_{LR} with v_B. We will argue that the butterfly velocity can play the role of a low-energy Lieb-Robinson velocity.

To elaborate, the Lieb-Robinson bound holds for any local lattice model of spins with bounded norm interactions. The constants K_0, ξ_0, and v_{LR} depend on the model, but the general conclusion that there exists an effective light cone does not. However, as a bound on the operator norm of the commutator, it has some important limitations. It requires that W and V have finite operator norm. Also, the constants appearing in the bound are microscopic; from the point of view of a low-energy description of the physics in terms of an emergent quantum field theory, they are UV sensitive. The reason for these limitations is that the Lieb-Robinson bound is state independent. One could hope that given further information about the state of the system a tighter bound might hold. Such a bound would constitute a bound on matrix elements of the commutator between states of interest—e.g., the butterfly commutator Eq. (3)—instead of a bound on the operator norm.

To that end, we compute the rate of growth of commutators of generic local operators Eq. (3) for a wide variety of holographic states of matter. We show that butterfly commutators Eq. (3) grow ballistically with a butterfly velocity v_B, which is UV insensitive and only depends on IR quantities such as temperature and certain thermodynamic exponents. We use these results to argue that the butterfly velocity is a state-dependent effective Lieb-Robinson velocity, which can be used to bound the growth of commutators. For comparison, we provide a direct calculation of these butterfly commutators in a free-fermion system. To further support our argument, we show that, if we are allowed only low-energy operators, the butterfly velocity places an upper bound on the speed with which signals can be sent between distant parties. Finally, we briefly discuss the prospect that our results can be probed experimentally using a recently proposed framework for measuring the scrambling time and the butterfly velocity [25].

In the Supplemental Material [33] we provide additional technical details: our holographic calculations, our free fermion calculations, and the specifics of our signaling argument.

Chaos in holographic models has so far been studied mostly in the special case of conformal field theories. Here, we will compute the rate of growth of commutators of local operators for a much wider class of holographic theories, specifically, those with a finite density of charge for some conserved $U(1)$ symmetry.

The holographic backgrounds we study are solutions of the Einstein-Maxwell-dilaton theory and describe the dynamics of a metric coupled to a gauge field and an uncharged scalar. With an electric flux for the gauge field turned on, such holographic models are dual to quantum field theories at finite density, that is, field theories with a conserved $U(1)$ charge perturbed by a chemical potential. (Strictly speaking, only some subset of these models have known string theory embeddings; see, e.g., Ref. [43]. The remaining models appear to be consistent gravitational theories at low energies, but their UV status is not clear.) These models were originally studied in an attempt to describe non-Fermi-liquid states of electrons within the so-called AdS/condensed matter theory program [43–46].

The models are characterized by the bulk action of the Einstein-Maxwell-dilaton theory; the important information

therein is the potential energy of the dilaton and the coupling of the dilaton to the field strength of the gauge field. In terms of observable parameters, the backgrounds are characterized by two "critical exponents," z and θ. The dynamical exponent z relates momentum to energy via $\omega \sim k^z$. Correlations obey power laws at zero temperature, but at finite temperature the field theory develops a correlation length given by $\xi \sim T^{-1/z}$. The exponent θ enters via the thermal entropy density,

$$s(T) \sim T^{(d-\theta)/z} \sim \left(\frac{1}{\xi(T)}\right)^{d-\theta}, \tag{5}$$

where d is the spatial dimension of the field theory. The physics is this: at a conventional quantum critical point the entropy would scale as the inverse thermal length ξ^{-1} to the power d, but when $\theta > 0$, the entropy density scales like ξ^{-1} to the power $d - \theta$. "Hyperscaling" is violated, so θ is called the hyperscaling violation exponent. For additional details about these geometries, see the Supplemental Material [33].

The simplest example of such a hyperscaling violating theory is a Fermi gas at finite density. This system has $z = 1$ and $\theta = d - 1$. Since the fermions fill up their single particle energy levels up to a Fermi energy equal to the chemical potential, the locus of zero energy states in momentum space is generically $d - 1$ dimensional instead of zero dimensional. The extent to which the holographic backgrounds we consider can describe conventional electronic states remains a topic of research. (For instance, note that imposing the null energy condition forbids $z = 1$ and $\theta = d - 1$ in holographic models.) However, since the results we find in the holographic model depend only on z and θ in a rather simple way, we conjecture that they are more broadly applicable, as we elaborate on below.

To calculate the growth of the commutator $C(x, t)$, we will study black holes geometries perturbed by a localized operator $W(x, t)$ [7,10]. For large t such that $G_N e^{2\pi t/\beta} \sim 1$ (where G_N is Newton's constant), backreaction will become important, and the perturbation will create a shock wave with a profile $h(x, t)$. Considering $C(x, t)$ in the $t = 0$ frame, the difference between the state created by $W(x, t)V(0)$ and the state created by $V(0)W(x, t)$ is a null shift of the $V(0)$ quanta by $h(x, t)$ due to the shock wave [8,10] (see also Refs. [47,48]). Taking into account that the commutator is determined by the real part of $\langle W(x, t)V(0)W(x, t)V(0)\rangle_\beta$ [10], $C(x, t)$ behaves at early times such that $\beta < t < x/v_B + t_*$ as

$$C(x, t) \sim h(x, t)^2 + \cdots, \tag{6}$$

where the scrambling time is given by $t_* = (\beta/2\pi) \times \log(\ell^d/G_N)$. We emphasize that this calculation is a statement about commutators of generic operators. This is due to the universal coupling of energy to gravity: any operator that adds energy can backreact and generate a shock wave (and operators that do not add energy must commute with H and do not scramble).

In the Supplemental Material [33], we present the technical details of our calculation of $h(x, t)$ using the shock wave techniques of Refs. [7,10]. From this, we extract the butterfly velocity for hyperscaling violating geometries,

$$v_B = \left(\frac{\beta_0}{\beta}\right)^{1-1/z} \sqrt{\frac{d + z - \theta}{2(d - \theta)}}, \tag{7}$$

with $1/\beta_0$ a cutoff temperature above which the holographic solution breaks down. This is the main result of our Letter. As promised, the butterfly velocity is a function of IR quantities: thermodynamic exponents z, θ and the inverse temperature β. The only dependence on high-energy physics is the scale β_0, which simply sets the units of the temperature [49]; the β_0 dependence can be canceled by taking an appropriate ratio, so the prefactor is meaningful even when $z \neq 1$. Various comments are in order.

First, we note that in the limit of AdS $z = 1$, $\theta = 0$, we recover the previously reported velocity for Einstein gravity [7,10]:

$$v_B(z = 1, \theta = 0) = \sqrt{\frac{d + 1}{2d}}. \tag{8}$$

On the other hand, for the z and θ appropriate to a Fermi gas at finite density, our result predicts

$$v_B(z = 1, \theta = d - 1) = 1, \tag{9}$$

as appropriate for a theory that lives in effectively $1 + 1$ dimensions.

Second, we note that for $z < 1$ the butterfly velocity diverges at $T = 0$, but microscopic causality requires bounded v_B. Thus, if the hyperscaling violation geometry is to describe the deep IR, we require $z \geq 1$. This means that v_B has a temperature dependence of

$$v_B \sim T^{1-1/z}, \qquad 1 - 1/z > 0, \tag{10}$$

and increases with temperature. v_B behaves as an effectively "renormalized" Lieb-Robinson velocity, with a magnitude that depends on a negative power of the thermal scale β.

Third, the allowed values of z and θ are constrained by the null energy conditions. For example, when $z = 1$ the null energy condition implies that $\theta < 0$ or $\theta > d$. Furthermore, if $z \geq 1$, then $d - \theta + z \geq 0$. In fact, the stronger condition $d - \theta + z > 1$ follows from finiteness of energy fluctuations in the ground state [50].

Finally, our computation also shows that for hyperscaling violating theories

$$\lambda_L = \frac{2\pi}{\beta}, \tag{11}$$

which, unlike v_B, is unchanged from its value in Einstein gravity [7]. We also point out that λ_L behaves like a state-dependent effective Lieb-Robinson growth rate [equivalent to v_{LR}/ξ_0 in Eq. (2)], a quantity for which a stronger bound was successfully derived [14].

The main result of this Letter is a computation of the butterfly velocity in a class of strongly coupled quantum field theories dual, via AdS/CFT, to Einstein-Maxwell-Dilaton theories of gravity. In fact, the computation relied only on the form of the metric, so any set of matter fields coupled to Einstein gravity which produces such a solution is sufficient. We framed the calculation in terms of the possibility of a bound on the growth of commutators analogous to the Lieb-Robinson bound in lattice many-body systems, so we now briefly discuss the thesis that the butterfly velocity functions as a low-energy Lieb-Robinson velocity.

To begin, it is interesting to compare the strongly coupled holographic results to results in the opposite limit of zero coupling. In the Supplemental Material [33], we record calculations of commutators for free-particle lattice models. In these free-particle models Wick's theorem implies that all commutators of composite operators are controlled by the basic commutator (or anticommutator) between the elementary bosons (or fermions). For example, a one-dimensional free-fermion hopping model with hopping matrix element w is known to flow to a free CFT at low energies, and the anticommutator of two fermions at integer positions x and y is

$$\{c(x,t), c^\dagger(y,0)\} = e^{-i\pi(x-y)/2} J_{x-y}(2wt), \tag{12}$$

where $J_\nu(x)$ is a Bessel function of index ν. This anticommutator grows outward like a shell,

$$\{c(x,t), c^\dagger(y,0)\} \sim \frac{t^{|x-y|}}{|x-y|!} \quad \text{(early times)}, \tag{13}$$

$$\sim t^{-1/2} \quad \text{(late times)}, \tag{14}$$

showing an initial rise followed by a slow decay to zero. This is to be contrasted to a strongly coupled system, where commutators should grow like a ball [10].

The anticommutator Eq. (12) is state independent and sensitive to the UV details of the lattice model. (While naively this violates the bound of Ref. [14], we note that the assumptions of the proof fail to hold: the relevant time-ordered four point functions fail to factorize after a "dissipation time.") However, by considering fermion operators corresponding to low-energy wave packets, it is possible to exhibit an anticommutator that grows instead at some group velocity given by the momentum derivative of the energy evaluated at the average momentum of the wave packet. If the dispersion relation near zero energy is $\epsilon \sim k^z$ with k the momentum, then in a thermal state where the typical energy is T, the typical thermal group velocity will be $T^{1-1/z}$, as in the holographic results. With the additional reasonable assumption that weak interactions cause high-energy particles to decay towards low energy, it is plausible that our holographic result applies at both weak and strong coupling, at least in terms of its temperature dependence. Note also that this result is not a trivial consequence of dimensional analysis when $\theta \neq 0$, since there are additional scales in the problem.

In addition to these statements about commutators, there is an alternative interpretation of v_B in terms of quantum information flow. As we show in the Supplemental Material [33], within a certain model of communication using only low-energy observables, if the relevant commutators between operators controlled by two distant parties are small, then these parties cannot communicate much information. The butterfly velocity thus defines an information theoretic light cone in which quantum information cannot be spread in space faster than v_B. Relatedly, in Ref. [18] it was argued that v_B controls the flow of information through time, and in Ref. [51] it will be shown that the butterfly velocity v_B controls the rate of growth of the entanglement wedge in holographic theories. Thus, v_B is both a measure of the growth of operators or commutators and also a measure of the spreading of information under time evolution. These results all support our identification of the butterfly velocity as a low-energy Lieb-Robinson velocity.

It is also interesting to compare v_B with the speed of hydrodynamic sound. Conformal field theories support a hydrodynamic sound mode with velocity $v_s = (1/\sqrt{d})$. Consistent with v_B being an upper bound on operator growth, we find $v_B(z=1,\theta=0) \geq v_s$ for a CFT. In the case where the shock wave calculation completely determines the butterfly velocity, our results provide an interesting constraint on the speed of the sound: if a hydrodynamic sound mode exists, its velocity must go to zero with temperature at least as fast as v_B [52,53]. The constraint can be avoided by adding extra ingredients into the bulk, e.g., probe D-branes [54]. Presumably a proper calculation of the butterfly velocity in the presence of the D-brane would be sensitive to any sound mode on the D-brane world volume.

To summarize, we have seen that the butterfly velocity v_B characterizes the spread of quantum information in quantum many-body systems and can provide a low-energy analog of the microscopic Lieb-Robinson velocity v_{LR}. We have also computed the dependence of the butterfly velocity on temperature and on the thermodynamic exponents z and θ at weak and strong coupling. Given the interest in experimental measures of the spread of quantum information [55–58], it is natural to ask if our results can be tested experimentally.

Recently, an experimental protocol was devised to measure out-of-time-order correlation functions [25]. Such correlators are simply related to the squared

commutator $C(x,t)$, and therefore these measurements are necessary to gain access to the butterfly velocity. Furthermore, although the simpler measurement of an average commutator $\langle[W(x,t),V(0)]\rangle$ would give information about the butterfly velocity (at least a lower bound), a measurement of $C(x,t)$ is more desirable because it gives information about typical off-diagonal matrix elements.

Cold bosonic atoms moving in an optical lattice constitute one experimental setting where such measurements could be performed. Because the sign of the Hamiltonian can be effectively reversed by combining lattice modulation with Feshbach resonance, the echolike measurement necessary to measure $C(x,t)$ is conceivable.

We thank Alexei Kitaev, Mark Mezei, and Douglas Stanford for discussions and Adam Brown, Leonard Susskind, and Ying Zhao for collaboration on related work. D. A. R. is supported by the Fannie and John Hertz Foundation and also acknowledges the U.S. Department of Energy under cooperative research agreement Contract No. DE-SC0012567. B. S. is supported by the Simons Foundation and the Stanford Institute for Theoretical Physics.

Note added.—Recently, Blake [28] also computed the butterfly velocity in hyperscaling violating geometries.

*drob@mit.edu
†bswingle@stanford.edu

[1] E. H. Lieb and D. W. Robinson, Commun. Math. Phys. **28**, 251 (1972).
[2] B. Nachtergaele, Y. Ogata, and R. Sims, J. Stat. Phys. **124**, 1 (2006).
[3] M. B. Hastings and T. Koma, Commun. Math. Phys. **265**, 781 (2006).
[4] M. B. Hastings, arXiv:1008.5137.
[5] A. Larkin and Y. Ovchinnikov, JETP **28**, 1200 (1969).
[6] A. Almheiri, D. Marolf, J. Polchinski, D. Stanford, and J. Sully, J. High Energy Phys. 09 (2013) 018.
[7] S. H. Shenker and D. Stanford, J. High Energy Phys. 03 (2014) 067.
[8] S. H. Shenker and D. Stanford, J. High Energy Phys. 12 (2014) 046.
[9] S. Leichenauer, Phys. Rev. D **90**, 046009 (2014).
[10] D. A. Roberts, D. Stanford, and L. Susskind, J. High Energy Phys. 03 (2015) 051.
[11] A. Kitaev, *presented at the Fundamental Physics Prize Symposium, Stanford University, Stanford*, 2014 (unpublished).
[12] D. A. Roberts and D. Stanford, Phys. Rev. Lett. **115**, 131603 (2015).
[13] S. H. Shenker and D. Stanford, J. High Energy Phys. 05 (2015) 132.
[14] J. Maldacena, S. H. Shenker, and D. Stanford, arXiv:1503.01409.
[15] A. Kitaev, presented at Kavli Institute for Theoretical Physics, Santa Barbara, 2015 (unpublished).
[16] J. Polchinski, arXiv:1505.08108.
[17] D. Marolf, Classical Quantum Gravity **32**, 245003 (2015).
[18] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, J. High Energy Phys. 02 (2016) 004.
[19] D. Stanford, arXiv:1512.07687.
[20] A. L. Fitzpatrick and J. Kaplan, J. High Energy Phys. 05 (2016) 070.
[21] J. Polchinski and V. Rosenhaus, J. High Energy Phys. 04 (2016) 001.
[22] B. Michel, J. Polchinski, V. Rosenhaus, and S. J. Suh, J. High Energy Phys. 05 (2016) 048.
[23] Y. Gu and X.-L. Qi, arXiv:1602.06543.
[24] P. Caputa, T. Numasawa, and A. Veliz-Osorio, arXiv:1602.06542.
[25] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, arXiv:1602.06271.
[26] E. Perlmutter, arXiv:1602.08272.
[27] N. Sircar, J. Sonnenschein, and W. Tangarife, J. High Energy Phys. 05 (2016) 091.
[28] M. Blake, preceding Letter, Phys. Rev. Lett. **117**, 091601 (2016).
[29] J. Maldacena and D. Stanford (to be published).
[30] A. Kitaev (to be published).
[31] P. Hayden and J. Preskill, J. High Energy Phys. 09 (2007) 120.
[32] Y. Sekino and L. Susskind, J. High Energy Phys. 10 (2008) 065.
[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.117.091602, which includes Refs. [34–42], for additional details on hyperscaling violating black holes, for the technical details of the holographic calculation and the free-fermion calculations, and for our argument that the butterfly velocity puts a bound on speed of information propagation.
[34] K. Sfetsos, Nucl. Phys. **B436**, 721 (1995).
[35] J. M. Maldacena, J. High Energy Phys. 04 (2003) 021.
[36] M. Fannes, Commun. Math. Phys. **31**, 291 (1973).
[37] H. Liu and S. J. Suh, Phys. Rev. Lett. **112**, 011601 (2014).
[38] T. Hartman and J. Maldacena, J. High Energy Phys. 05 (2013) 014.
[39] M. Alishahiha, A. Faraji Astaneh, and M. R. Mohammadi Mozaffar, Phys. Rev. D **90**, 046004 (2014).
[40] P. Fonda, L. Franti, V. Keränen, E. Keski-Vakkuri, L. Thorlacius, and E. Tonni, J. High Energy Phys. 08 (2016) 051.
[41] H. Casini, H. Liu, and M. Mezei, J. High Energy Phys. 07 (2016) 077.
[42] T. Hartman and N. Afkhami-Jeddi, arXiv:1512.02695.
[43] X. Dong, S. Harrison, S. Kachru, G. Torroba, and H. Wang, J. High Energy Phys. 06 (2012) 041.
[44] N. Iizuka, N. Kundu, P. Narayan, and S. P. Trivedi, J. High Energy Phys. 01 (2012) 094.
[45] N. Ogawa, T. Takayanagi, and T. Ugajin, J. High Energy Phys. 01 (2012) 125.
[46] B. Swingle and T. Senthil, Phys. Rev. B **87**, 045123 (2013).
[47] G. 't Hooft, Nucl. Phys. **B335**, 138 (1990).
[48] Y. Kiem, H. L. Verlinde, and E. P. Verlinde, Phys. Rev. D **52**, 7053 (1995).
[49] Suppose a hyperscaling violation metric of the form $ds^2 = r^{2\theta/d}(-Ar^{2-2z}dT^2 + BdX^2 + Cdr^2)/r^2$ (with constants A,

B, and C) is the IR limit of a solution which is asymptotically AdS [with metric $ds^2 = D(-dT^2 + dX^2 + dr^2)/r^2$ and constant D]. Then, the change of variables $X = (C/B)^{1/2}x$ and $T = (C/B)^{1/2}t$ preserves the speed of light $c = 1$ and puts the metric into the standard hyperscaling violating form with $r_0^{2z-2} = A/B$.

[50] B. Swingle and J. McGreevy, Phys. Rev. B **93**, 205120 (2016).

[51] M. Mezei and D. Stanford (to be published).

[52] For example, a weakly interacting thermal gas of particles with dispersion $\omega \sim k^2$, i.e., nonrelativistic particles, may be approximately described as an ideal gas with sound speed $\propto T^{1/2}$.

[53] Note that in the ground state, scrambling can still occur but might be slower [12]. For example, consider a $2d$ large-c CFT on an infinite line. In this case, the commutator will vanish outside the butterfly cone due to causality (since $v_B = 1$), but then it will begin to grow only as a power law $(t - x)^4$.

[54] M. Ammon, M. Kaminski, and A. Karch, J. High Energy Phys. 11 (2012) 028.

[55] M. Cheneau, P. Barmettler, D. Poletti, M. Endres, P. Schauß, T. Fukuhara, C. Gross, I. Bloch, C. Kollath, and S. Kuhr, Nature (London) **481**, 484 (2012).

[56] P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, and C. Monroe, Nature (London) **511**, 198 (2014).

[57] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, Nature (London) **511**, 202 (2014).

[58] J. Eisert, M. Friesdorf, and C. Gogolin, Nat. Phys. **11**, 124 (2015).