Floquet Time Crystals

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We define what it means for time translation symmetry to be spontaneously broken in a quantum system and show with analytical arguments and numerical simulations that this occurs in a large class of many-body-localized driven systems with discrete time-translation symmetry.

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Introduction.—Spontaneous symmetry-breaking (SSB) is a pivotal concept in physics, with implications for condensed matter and high-energy physics. It occurs when the ground state or low-temperature states of a system fail to be invariant under symmetries of the Hamiltonian. The Ising model is a prototypical example for this behavior: Here, the symmetry is a simultaneous flip of all the spins, which leaves the energy of a state unchanged. In the ferromagnetic phase, low-energy states are formed with a nonzero magnetization. For almost every symmetry imaginable, there is a model whose ground state breaks it: crystals break the continuous translational and rotational symmetries of Coulomb interactions; magnetically ordered materials break time-reversal symmetry and spin symmetry, and superfluids break global gauge symmetry. The lone holdout, thus far, has been time-translation symmetry. In this Letter, we give a definition of time-translation symmetry breaking and construct an example of this behavior in a driven many-body localized system.

Definition of time translation symmetry-breaking.—Systems that spontaneously break time-translation symmetry (TTS) have been dubbed "time crystals," in analogy with ordinary crystals, which break spatial translational symmetries [1,2]. Even defining this notion correctly requires considerable care, and putative models have proven inconsistent [3–9]. The most obvious definition of time-translation symmetry breaking (TTSB) would be that the expectation values of observables are time-dependent in thermal equilibrium. However, this is clearly impossible, since a thermal equilibrium state $\rho = (1/Z)e^{-\beta H}$ is time independent by construction (because $[\rho, H] = 0$). A more sophisticated definition of TTSB in terms of correlation functions in the state ρ has been proposed—and ruled out by a no-go theorem—in Ref. [10].

Therefore, we must look beyond strict thermal equilibrium. This should not be too surprising, as the state ρ preserves *all* the symmetries of H, which would suggest that *no* symmetry can be spontaneously broken. For symmetries other than time translation, the resolution to this paradox is well known: in a system with a spontaneously broken symmetry, there is ergodicity breaking, and the lifetime of a symmetry-breaking state diverges as the system size grows. Thus, in the thermodynamic limit, the state ρ is unphysical and is never reached. This suggests that an analogous phenomenon should be possible for time translation symmetry, where the time taken to reach a time-independent steady state (such as the thermal state ρ) diverges exponentially with system size.

To turn these considerations into a more useful definition, we observe that, in a quantum system, the ergodicity breaking in a phase with a spontaneously broken symmetry can be seen at the level of eigenstates. For example, the symmetry-respecting ground states of an Ising ferromagnet are $|\pm\rangle = (1/\sqrt{2})(|\uparrow\cdots\uparrow\rangle \pm |\downarrow\cdots\downarrow\rangle$. Such long-range correlated "cat states" are unphysical, will immediately decohere given any coupling to the environment, and can never be reached in finite time by any unitary time evolution starting from a short-range correlated starting state. On the other hand, the "physical" combinations $|\uparrow\cdots\uparrow\rangle$ and $|\downarrow\cdots\downarrow\rangle$ break the Ising symmetry.

In the TTSB case, we also need to invoke the intuition that oscillation under time evolution requires the superposition of states whose phases wind at different rates. That is, whereas in the Ising ferromagnet, the two cat states $|\pm\rangle$ are degenerate in the thermodynamic limit, in a time crystal they would need to have *different* eigenvalues under the time-evolution operator. Indeed, consider, for simplicity, a discrete time evolution operator U_f (which describes periodically driven "Floquet" systems as we discuss further below.) Suppose that the states $|\pm\rangle$ have eigenvalues $e^{i\omega_\pm}$ under U_f. Then, although the unphysical cat states $|\pm\rangle$ are time invariant (up to a phase), a physical state such as $|\uparrow\cdots\uparrow\rangle$ will evolve according to $(U_f)^n|\uparrow\rangle \propto \cos(\omega n)|\uparrow\cdots\uparrow\rangle + i\sin(\omega n)|\downarrow\cdots\downarrow\rangle$, where $\omega = (\omega_+ - \omega_-)/2$.

The above considerations motivate two equivalent definitions of TTSB, using the following terminology or notation. We will say that a state $|\psi\rangle$ has "short-ranged correlations" if, for any local operator $\Phi(x)$, $\langle\psi|\Phi(x)\Phi(x')|\psi\rangle - \langle\psi|\Phi(x)|\psi\rangle\langle\psi|\Phi(x')|\psi\rangle \to 0$ as $|x - x'| \to \infty$, i.e., if cluster decomposition holds. Note that the superpositions defined above are not short-range correlated under this definition, while a state such as $|\uparrow\uparrow\ldots\uparrow\rangle$ is.

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We assume that time evolution is described by a time-dependent Hamiltonian $H(t)$, with a discrete time translation symmetry such that $H(t) = H(t + T)$ for some T. Note that we have not assumed a *continuous* time translation symmetry, which will allow us to consider Floquet systems driven at a frequency $\Omega = 2\pi/T$. Let $U(t_1, t_2)$ be the corresponding time evolution operator from time t_1 to t_2. We now define (in the thermodynamic limit)

TTSB-1: TTSB occurs if for each t_1, and for every state $|\psi(t_1)\rangle$ with short-ranged correlations, there exists an operator Φ such that $\langle\psi(t_1 + T)|\Phi|\psi(t_1 + T)\rangle \neq \langle\psi(t_1)|\Phi|\psi(t_1)\rangle$, where $|\psi(t_1 + T)\rangle = U(t_1 + T, t_1)|\psi(t_1)\rangle$.

TTSB-2: TTSB occurs if the eigenstates of the Floquet operator $U_f \equiv U(T, 0)$ cannot be short-range correlated.

In what follows, we will show how to construct a time-dependent Hamiltonian $H(t)$ which satisfies the conditions for TTSB given above. In such a system, even though the time evolution is invariant under the discrete TTS generated by time translation by T, the expectation value of some observables is only invariant under translations by nT for some $n > 1$. In other words, the system responds at a fraction Ω/n of the original driving frequency.

The first definition puts the time dependence front and center and is directly connected to how TTSB would be observed experimentally: prepare a system in a short-range correlated state and observe its subsequent time evolution, which will not be invariant under the TTS of the time evolution operator. But since, in a Floquet eigenstate, observables would necessarily be invariant under the discrete TTS generated by time translation by T, definition TTSB-1 implies that Floquet eigenstates cannot be short-range correlated, thereby implying TTSB-2. Conversely, if it is impossible to find Floquet eigenstates that are short-range correlated (which is TTSB-2), then it means that short-range correlated states can only be formed by taking superpositions of Floquet eigenstates with different eigenvalues. In such states, observables will not be invariant under the discrete TTS generated by time translation by T, thereby implying TTSB-1. Hence, the two definitions are equivalent. The second definition will prove to be particularly useful for analyzing the results of numerical exact diagonalization of the Floquet operator. When discrete TTS by T is broken down to TTS by nT, the eigenstates of U_f must be superpositions of n different short-range-ordered states. Then, in any Floquet eigenstate, the mutual information $I(A, B) \equiv S_A + S_B - S_{AB}$, where A and B are spatially separated regions of the system and S_X is the von Neumann entropy of the reduced density matrix for region X, satisfies $I(A, B) \to \ln n$ as the system size as well as the sizes of the regions A and B and their separation is taken to infinity [11,12].

Floquet-many-body-localization.—Generic translationally invariant many-body Floquet systems likely cannot have TTSB, as their eigenstates resemble infinite temperature states and, hence, are short-range correlated [13–15]. (Nevertheless, an initial state that is not an eigenstate could potentially heat very slowly, leading to nontrivial intermediate-time dynamics [16–20].) This is analogous to the fact (which follows from the results of Ref. [10]) that, for continuous time-translation symmetry, TTSB is impossible so long as the eigenstate thermalization hypothesis [21–24] is satisfied. However, we can build upon recent developments in the study of Floquet-many-body-localized (Floquet-MBL) systems [25–35], for which the eigenstates do not resemble infinite temperature states. Instead, the Floquet states of such systems exhibit the characteristics of the energy eigenstates of static MBL [36–45] systems: the eigenstates are local product states, up to finite-depth unitary quantum circuits [46].

In MBL systems, all eigenstates (of the Hamiltonian in the static case or of the Floquet operator in the driven case) behave as ground states and, therefore, SSB or topological order can occur in all eigenstates [46–48]. In the SSB case, simultaneous eigenstates of the Floquet operator and of the Cartan subalgebra of the symmetry generators cannot be short range correlated. TTSB-2 can then be viewed as a special case of this in which there are no other symmetry generators besides U_f.

In the next paragraph, we construct a Floquet operator and show that it exhibits discrete TTSB. In subsequent paragraphs, we show that this soluble Floquet operator sits in a finite window in parameter space over which TTSB occurs—i.e., that there is a TTSB phase. Models which exhibit TTSB (though not identified as such) have previously been considered in Refs. [30,35]. These models also break another symmetry spontaneously, but this is not essential to achieve TTSB. Our model will be a generalization of that of Refs. [30,35], with the extra symmetry explicitly broken. By contrast, the models of Refs. [49,50] rely crucially on an additional symmetry.

Model and soluble point.—We consider one-dimensional spin-1/2 systems with Floquet unitaries of the form

$$U_f = \exp(-it_0 H_{\rm MBL}) \exp\left(it_1 \sum_i \sigma_i^x\right). \qquad (1)$$

We choose $t_1 \approx \pi/2$, such that the application of $\sum_i \sigma_i^x$ in this stroboscopic time evolution has the effect of approximately flipping all of the spins since $\exp(i(\pi/2)\sum_i \sigma_i^x) = \prod_i i\sigma_i^x$. This is followed by time evolution for an interval t_0 under the Hamiltonian

$$H_{\rm MBL} = \sum_i (J_i \sigma_i^z \sigma_{i+1}^z + h_i^z \sigma_i^z + h_i^x \sigma_i^x), \qquad (2)$$

where J_i, h_i^z, and h_i^x are uniformly chosen from $J_i \in [(J/2), (3J/2)]$, $h_i^z \in [0, h^z]$, $h_i^x \in [0, h]$ where $h \ll J$ is the regime of interest. The period of the drive is $T = t_0 + t_1$. For $h = 0$ and $t_1 = \pi/2$, the eigenstates of $H_{\rm MBL}$ are eigenstates of the individual σ_i^z. Call such an eigenstate $|\{s_i\}\rangle$ with $s_i = \pm 1$ so that $\sigma_k^z|\{s_i\}\rangle = s_k|\{s_i\}\rangle$. Then $H|\{s_i\}\rangle = [E^+(\{s_i\}) + E^-(\{s_i\})]|\{s_i\}\rangle$ where

$E^+(\{s_i\}) = \sum_i (J_i s_i s_{i+1})$ and $E^-(\{s_i\}) = \sum_i (h_i^z s_i)$. The Floquet eigenstates are $e^{it_0 E^-(\{s_i\})/2}|\{s_i\}\rangle \pm e^{-it_0 E^-(\{s_i\})/2}|\{-s_i\}\rangle$, and the corresponding Floquet eigenvalues are $\pm\exp[it_0 E^+(\{s_i\})])$. Hence, TTSB-2 is satisfied for $h = 0$ and $t_1 = \pi/2$.

Stability of TTSB.—We now argue that the preceding conclusions are no fluke: arbitrary weak local T-periodic perturbations of the Floquet operator, such as nonzero h or deviations of the length of the second time interval from $\pi/2$, do not destroy TTSB, so long as a reasonable but nontrivial assumption about resonances holds. Ordinarily, there would be little doubt that SSB of a discrete symmetry is stable to weak perturbations at zero temperature in 1D. But since the symmetry in question is TTS, more care seems necessary.

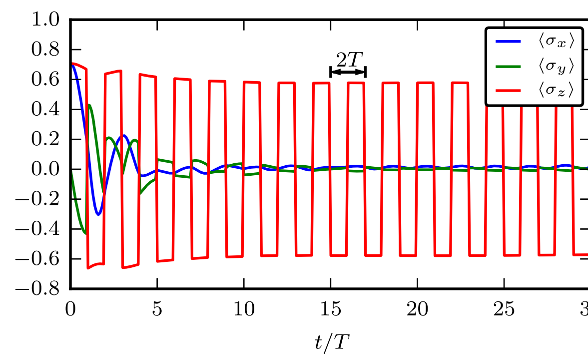
To build confidence in the stability of TTSB, we can exploit the discrete local connectivity of fully MBL systems: that is, for any eigenstate $|i\rangle$, and point x, there is only a finite number of eigenstates $|j\rangle$ such that the matrix elements $\langle i|\Phi(x)|j\rangle \neq 0$ for some operator $\Phi(x)$ acting locally at x. In particular, generically, the (quasi-) energy difference $\omega_j - \omega_i$ for eigenstates connected in this way will not be close to zero. In systems with such a local spectral gap, one expects that "local perturbations perturb locally" [51–54], or more precisely, that there exists a single local unitary $\mathcal{U}$ (that is, a unitary which can be expressed as the time evolution of a local Hamiltonian S) which relates perturbed eigenstates to unperturbed eigenstates [46]. Such a local unitary $\mathcal{U}$ cannot possibly connect short-range correlated states with the long-range correlated eigenstates found above. Therefore, the eigenstates of the perturbed Floquet operator still satisfy TTSB-2.

We make these ideas more precise in the Supplemental Material [55]. There, we construct the unitary $\mathcal{U}$ order by order in perturbation theory and show that it remains local at all orders, provided that the local spectral gap condition holds. The skeptic might argue, however, that there will always be rare regions (known as "resonances") in which the local spectral gap is arbitrarily small, and that this will spoil the convergence of the perturbation theory. A rigorous treatment of resonances is a difficult problem; however, the principle of "local perturbations perturb locally" has, in fact, been proven (given certain reasonable assumptions), at least for a particular model of stationary MBL [56].

On the other hand, for sufficiently large perturbations, resonances will proliferate and TTSB (and possibly MBL) will be destroyed. The TTSB and MBL will be particularly fragile for low frequencies $T^{-1} \ll J$ (see the Supplemental Material [55] for details.)

Numerical analysis of U_f.—In order to confirm the stability of TTSB, we will simulate the time evolution for one class of perturbations, namely nonzero h in Eq. (1). In the Supplemental Material [55], we also numerically demonstrate stability with respect to variations of t_1 (see, also, Ref. [57]). Throughout, we will take $J = h^z = 1$. First, we use the time-evolving block decimation (TEBD) scheme [58] to compute the time evolution of the short-range correlated initial state $[\cos(\pi/8)|\uparrow\rangle + \sin(\pi/8)|\downarrow\rangle]^{\otimes L}$ for system size $L = 200$ and $h = 0.3$ and $t_0 = 1$. The top panel of Fig. 1 shows the expectation values of the Pauli spin operators, averaged over 146 disorder configurations and over the spatial interval $i \in [50, 150]$. The TEBD calculations were done with Trotter step $0.01T$ and bond dimension $\chi = 50$. The spin-flip part of the Floquet operator is applied instantaneously, which explains why the oscillation appears to be steplike. After an initial transient, the expectation values oscillate at frequency π/T, half the drive frequency.

Lest a skeptic wonder whether such oscillations continue to much later times or decay just beyond the times accessible by TEBD, we analyze smaller systems by numerical exact diagonalization (ED) of the Floquet operator. To extract the time on which the magnetization decays, we consider the time evolution of the magnetization starting from random initial product states that are polarized in the z direction, and compute the average $Z(t) = \overline{(-1)^t \langle\sigma_i^z(t)\rangle \mathrm{sgn}(\langle\sigma_i^z(0)\rangle)}$ over 500 disorder realizations



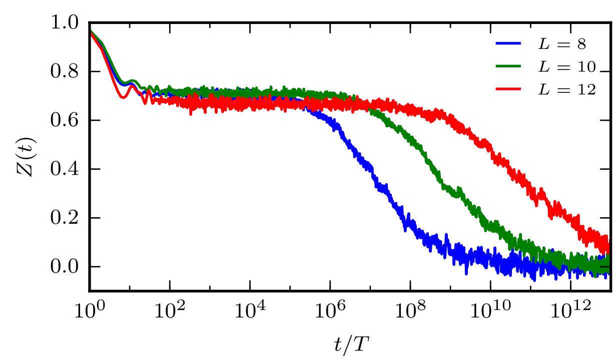


FIG. 1. The time evolution of a short-range correlated initial state satisfies TTSB-1 for $h = 0.3$. Top panel: the time dependence of the disorder-averaged $\langle\sigma_i^x\rangle$, $\langle\sigma_i^y\rangle$, and $\langle\sigma_i^z\rangle$ show that the former two decay rapidly while the latter displays persistent oscillations. (The spin-flip part of the Floquet operator is here taken to be applied instantaneously.) Bottom panel: The decay of the disorder-averaged magnetization, $Z(t)$, as defined in the main text, is found to decay zero on a time scale that diverges exponentially in the system size.

and for a fixed position i. As shown in the bottom panel of Fig. 1, there is an initial decay of this quantity, which for the parameters chosen here occurs around $t/T = 10$, and then a plateau that extends up to a time that diverges exponentially in the system size, and even for these small system sizes, reaches times comparable to the inverse floating point precision. In the Supplemental Material [55], we explore these time scales in more detail and describe ways in which signatures of TTSB can be observed for individual disorder configurations (without disorder averaging).

We now turn to the ED of the Floquet operator to verify that TTSB-2 holds. We diagonalize U_f for $L = 6, 8, 10, 12$ sites and 3200 disorder realizations and compute the mutual information between the left- and rightmost n sites, labeled F_{nn}. We find that the mutual information obeys the scaling form $F_{nn}(h, L) = F_{nn}(g, \infty) + c_n \exp[-L/\xi(h)]$. We expect that $F_{nn}(h, \infty) = 0$ in the TTS-invariant phase, $h > h_c$, and $F_{nn}(g, \infty) > 0$ in the TTSB phase, $h < h_c$, with $F_{nn}(g, \infty) \to \ln 2$ as $n \to \infty$. The results in Fig. 2 are consistent with this form, with $h_c \gtrsim 1$. It is remarkable that scaling holds even for such small systems, and that $F_{22} \approx F_{33} \approx \log 2$ for $h < 0.3$; evidently, $L = 12$ and $n = 2, 3$ are not so far from the thermodynamic limit.

Implications of TTSB.—In systems exhibiting MBL, it is commonly thought that there exists a complete set of local integrals of motion (LIOMs): that is, there is a set of quasilocal operators τ_i^z which commute with each other and with the Floquet operator U_f (or the Hamiltonian in the static case), and such that the eigenvalues of τ_i^z uniquely specify a state in the Hilbert space [42,44,45]. Systems with TTSB violate this principle. Indeed, in our model, at its soluble point at $h = 0$, the locally indistinguishable states $e^{it_0 E^-(\{s_i\})/2}|\{s_i\}\rangle \pm e^{-it_0 E^-(\{s_i\})/2}|\{-s_i\}\rangle$ are eigenstates, with different quasienergy. No LIOM can

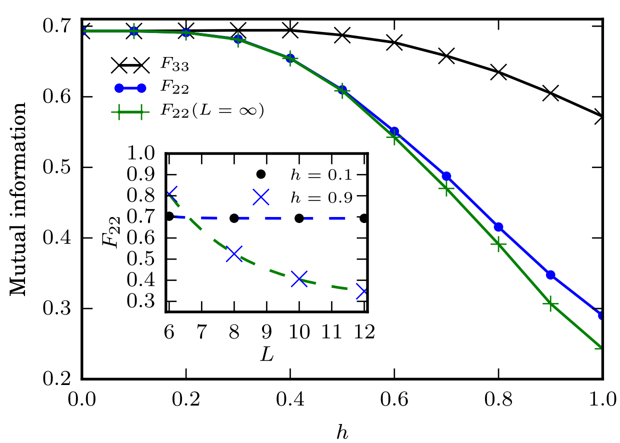


FIG. 2. The mutual information between the n left- and rightmost sites, F_{nn}, for $n = 2$ and $n = 3$. The main panel shows results for $L = 12$, as well as the extrapolated value of F_{22} for $L \to \infty$. To extrapolate, we fit $F_{22}(L) = F_{22}(\infty) + ce^{-L/\xi}$, with $F_{22}(\infty)$, c, and ξ fit parameters. Example fits for $h = 0.1$ and $h = 0.9$ are shown in the inset.

distinguish between these two states, so no set of LIOMs can be complete. [Though the existence of a complete set of LIOMs is sometimes taken as the *definition* of MBL, the TTSB phase is still MBL in the sense of, for example, long-time dynamics, since $(U_f)^2$ does have a complete set of LIOMs]. By a similar argument, one can show that there does not exist a quasilocal effective Hamiltonian $H_{\rm eff}$ such that $U_f = \exp(-iTH_{\rm eff})$, whereas for Floquet-MBL systems without TTSB, this is likely to be the case [25,27].

As noted earlier, the oscillations arise from the occurrence of multiplets of states separated in Floquet eigenvalue by Ω/n, where $\Omega = 2\pi/T$ is the drive frequency. We do not use this to identify the TTSB phase in ED because the states are too closely spaced in energy to pick out such multiplets. However, their existence suggests that the system can radiate at frequency Ω/n. The fact that systems oscillating in time can radiate has been cited as an argument against the existence of TTSB [4,6], since a system maintaining persistent oscillations while simultaneously radiating would be inconsistent with conservation of energy. However, in the Floquet case, this is not an issue since energy is being continually supplied by the drive. (For details, see the Supplemental Material [55].) On the other hand, in a system that breaks continuous TTS, radiation would cause the system to decay to the ground state, which is reason to doubt that continuous TTSB can occur.

Discussion.—The model Eqs. (1) and (2) is soluble at $h = 0$ because the operator $\exp(i(\pi/2)\sum_i \sigma_i^x) = \prod_i i\sigma_i^x$ that is applied at the beginning of each driving cycle maps eigenstates of $H_{\rm MBL}$ to eigenstates of $H_{\rm MBL}$. Analogous soluble models can be constructed for $\mathbb{Z}_n$ spins in which time translation by T is broken down to nT.

Our model has no symmetries, other than discrete time-translation symmetry. Hence, the ln 2 that we find in the mutual information must be a consequence of TTSB; there is no other symmetry to break. However, TTSB can occur in models with other symmetries. For example, in symmetry-protected topological (SPT) phases of Floquet-MBL systems [31–34], TTSB can occur on the boundary.

The definition TTSB-1 naturally suggests an experiment that could observe the phenomenon predicted here. Signatures of MBL have been observed in trapped systems of neutral atoms [59] and trapped ions [60], and signatures of single-particle localization have been seen in coupled superconducting qubits [61]. In any of these systems, one can prepare an arbitrary initial product state, evolve to late times according to a drive in the class considered here, and measure the “spins” in the desired basis. Our prediction is that persistent oscillations will be observed at a fraction of the drive frequency.

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