Topological Septet Pairing with Spin-$\frac{3}{2}$ Fermions: High-Partial-Wave Channel Counterpart of the ^{3}He-B Phase

Wang Yang,[1] Yi Li,[2] and Congjun Wu[1]
[1]*Department of Physics, University of California, San Diego, California 92093, USA*
[2]*Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218, USA*
(Received 1 February 2016; revised manuscript received 20 March 2016; published 10 August 2016)

We systematically generalize the exotic ^{3}He-B phase, which not only exhibits unconventional symmetry but is also isotropic and topologically nontrivial, to arbitrary partial-wave channels with multicomponent fermions. The concrete example with four-component fermions is illustrated including the isotropic f-, p-, and d-wave pairings in the spin septet, triplet, and quintet channels, respectively. The odd partial-wave channel pairings are topologically nontrivial, while pairings in even partial-wave channels are topologically trivial. The topological index reaches the largest value of N^2 in the p-wave channel (N is half of the fermion component number). The surface spectra exhibit multiple linear and even high order Dirac cones. Applications to multiorbital condensed matter systems and multicomponent ultracold large spin fermion systems are discussed.

DOI: 10.1103/PhysRevLett.117.075301

Superconductivity and paired superfluidity of neutral fermions possessing unconventional symmetries are among the central topics of condensed matter physics. If Cooper pairs formed by spin-$\frac{1}{2}$ fermions carry nonzero spin, their orbital symmetries are usually in the odd partial-wave channels, except in the case of odd-frequency pairings [1,2]. The p-wave paired superfluidity [3,4] includes the ^{3}He A phase exhibiting point nodes [5], the fully gapped B phase [6], and the recently reported polar state with linear nodes [7,8]. The p-wave superconductivity has also been extensively investigated in heavy fermion systems including UGe_2, URhGe, UCoGe [9]. The p-wave superfluid ^{3}He and superconductors exhibit rich topological structures of vortices and spin textures under rotations or in external magnetic fields, respectively [10,11]. In addition, experimental signatures of the possible nodal f-wave superconductivity have also been reported in UPt_3 [12,13].

Among unconventional pairing phases, the pairing structure of the ^{3}He-B phase is distinct. In spite of its non-s-wave symmetry and nontrivial spin configuration, the overall pairing structure remains isotropic: It is invariant under the combined spatial and spin rotations carrying total angular momentum zero. It shows the relative spin-orbit symmetry breaking from $SO_L(3)\otimes SO_S(3)$ to $SO_J(3)$ [3] where L, S, and J represent the orbital, spin, and total angular momentum, respectively. The relative spin-orbit symmetry breaking has also been studied in the context of Pomeranchuk instability termed as unconventional magnetism leading to the dynamic generation of spin-orbit coupling [14,15].

Furthermore, the ^{3}He-B phase possesses nontrivial topological properties [16–18]. Topological states of matter have become a major research focus since the discovery of the integer quantum Hall effect [19–21]. Recently, the study of topological band structures has extended from time-reversal (TR) breaking systems to TR invariant systems [22–24], from two to three dimensions [17,25,26], and from insulators to superconductors [16–18,27–30]. The ^{3}He-B phase is a 3D TR invariant topological Cooper pairing state. Its bulk Bogoliubov spectra are analogous to the 3D gapped Dirac fermions belonging to the DIII class characterized by an integer-valued index [17]. The nontrivial bulk topology gives rise to the gapless surface Dirac spectra of the midgap Andreev-Majorana modes [31]. Evidence of these low energy states has been reported in recent experiments [32].

Because the electron Cooper pair can only be either spin singlet or triplet, the p-wave ^{3}He-B phase looks like the only choice of the unconventional 3D isotropic pairing state. In this Letter, we will show that actually there are much richer possibilities of this exotic class of pairing in all the partial-wave channels of $L\geq 1$. We consider multicomponent fermions in both orbital-active solid state systems and ultracold atomic systems with large spin alkali and alkaline-earth fermions, both of which have recently attracted a great deal of attention [33–41]. For simplicity, below we introduce an effective spin s to describe the multicomponent fermion systems with the component number expressed as $2N=2s+1\geq 4$. Compared with the 2-component case, their Cooper pair spin structures are greatly enriched [34,42]. For example, the 4-component spin-$\frac{3}{2}$ systems can support the f-wave septet, p-wave triplet, and d-wave quintet pairings, all of which are fully gapped and rotationally invariant. Nevertheless, only the odd partial-wave channel ones, i.e., the p- and f-wave pairings, are topologically nontrivial. Their topological properties are analyzed both from calculating the bulk indices and surface Dirac cones of the Andreev-Majorana modes. For the p-wave case, the topological indices from all the helicity channels add up leading to a large value of N^2. Correspondingly the surface spectra exhibit the coexistence of 2D Dirac cones of all the orders from 1 to $2N-1$.

We begin with an f-wave spin septet Cooper pairing Hamiltonian in a 3D isotropic system of spin-$\frac{3}{2}$ fermions

$$H = \sum_{\vec{k}} \epsilon_{\vec{k}} c_\alpha^\dagger(\vec{k}) c_\alpha(\vec{k}) - \frac{g}{V_0} \sum_{\vec{k},\vec{k}',m,\nu} P_{m,\nu}^\dagger(\vec{k}) P_{m,\nu}(\vec{k}'), \quad (1)$$

in which $\epsilon_{\vec{k}} = (\hbar^2 k^2/2m) - \mu$ and μ is the chemical potential. $\alpha = \pm\frac{3}{2}, \pm\frac{1}{2}$ is the spin index, g is the pairing interaction strength, and V_0 is the system volume. The pairing operator is defined as $P_{m,\nu}^\dagger(\vec{k}) = c_\alpha^\dagger(\vec{k}) Y_{3m}(\hat{k}) [S^{3\nu} R]_{\alpha\beta} c_\beta^\dagger(-\vec{k})$ where $\hat{k} = \vec{k}/k$, $Y_{3m}(\hat{k})$'s with $-3 \le m \le 3$ are the third order spherical harmonic functions, and $S^{3\nu}$ with $-3 \le \nu \le 3$ are the rank-3 spherical tensors based on the spin operator $\vec{S}$ in the spin $\frac{3}{2}$-representation, where ν is the eigenvalue of S_z. For later convenience, $Y_{3m}(\hat{k})$ are normalized according to $\sum_m |Y_{3m}(\hat{k})|^2 = 1$. R is the charge conjugation matrix defined as $R_{\alpha\beta} = (-)^{\alpha+\frac{1}{2}} \delta_{\alpha,-\beta}$ satisfying $R\vec{S}^T R^{-1} = -\vec{S}$ such that $R_{\alpha\beta} c_\beta^\dagger$ transforms in the same way under rotation as c_α does. The expressions for spherical harmonic functions and spin tensors are presented in the Supplemental Material I of [43].

After the mean-field decomposition, Eq. (1) becomes

$$\frac{H_{\mathrm{MF}}}{V} = \frac{1}{V} \sum_{\vec{k}}{}' \Psi^\dagger(\vec{k}) H(\vec{k}) \Psi(\vec{k}) + g \sum_{m,\nu} \Delta_{m,\nu}^* \Delta_{m,\nu}, \quad (2)$$

in which $\vec{k}$ is summed over half of momentum space; $\Psi(\vec{k}) = (c_{\vec{k},\alpha}, c_{-\vec{k},\alpha}^\dagger)^T$ is the Nambu spinor; the order parameter $\Delta_{m,\nu}$ is defined through the self-consistent equation as

$$\Delta_{m,\nu} = \frac{g}{V} \sum_{\vec{k}} \langle G | c_\gamma(-\vec{k}) Y_{3m}^*(\hat{k}) R^\dagger S^{3\nu,\dagger} c_\delta(\vec{k}) | G \rangle \quad (3)$$

with $\langle G|\ldots|G\rangle$ meaning the ground state average. The matrix kernel $H(\vec{k})$ in Eq. (2) is expressed as

$$H(\vec{k}) = \epsilon(\vec{k}) \tau_3 \otimes I_{4\times4} + \hat{\Delta}(\vec{k}) \tau_+ + \hat{\Delta}(-\vec{k}) \tau_-, \quad (4)$$

where τ_3 and $\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$ are the Pauli matrices acting in the Nambu space. $\hat{\Delta}(\vec{k})$ is defined in the matrix form in spin space as

$$\hat{\Delta}(\vec{k}) = \sum_\nu (S^{3\nu} R) d^{*,\nu}(\vec{k}), \quad (5)$$

where $d^{*,\nu}(\vec{k}) = \Delta_{m,\nu} Y_{3m}(\hat{k})$ and is dubbed as the *d tensor* in analogy to the d vector in ^{3}He. The usual d vector is represented in its three Cartesian components, while here, the d tensor is a rank-3 complex spherical tensor.

We consider the isotropic pairing with total angular momentum $J = 0$, which is a generalization of the p-wave ^{3}He-B phase. Similarly, it is fully gapped, and thus conceivably energetically favorable within the mean-field theory. Its $d^\nu(\vec{k})$ can be parametrized as $d^\nu(\vec{k}) = c_f \Delta_f (k/k_f)^3 Y_{3\nu}(\hat{k})$, where c_f is an overall normalization factor given in Supplemental Material II of [43], Δ_f is the complex gap magnitude, or, equivalently,

$$\hat{\Delta}(\vec{k}) = \Delta_f \left(\frac{k}{k_f}\right)^3 K_f(\hat{k}) R \quad (6)$$

in which $K_f = c_f U(\hat{k}) S^{30} U^\dagger(\hat{k})$; $U(\hat{k})$ rotates the z axis to $\hat{k}$ as defined in the following gauge $U(\hat{k}) = e^{-i\phi_k s_z} e^{-i\theta_k s_y}$ in which θ_k and ϕ_k are polar and azimuthal angles of $\hat{k}$, respectively. The explicit form of $\hat{\Delta}(\hat{k})$ and the corresponding spontaneous symmetry-breaking pattern are presented in the Supplemental Material II and III of [43], respectively. The pairing matrix $K_f(\hat{k})$ and that of the isotropic p-wave pairing $K_p(\hat{k})$ are depicted in Figs. 1(a) and 1(b), respectively.

With the help of the helicity operator $h(\hat{k}) = \hat{k} \cdot \vec{S}$, $K_f(\hat{k})$ can be further expressed in an explicitly rotational invariant form as

$$K_f(\hat{k}) = -\frac{5}{2} h^3(\hat{k}) + \frac{41}{8} h(\hat{k}), \quad (7)$$

which is diagonalized as $U^\dagger(\vec{k}) K_f(\hat{k}) U(\hat{k}) = [-\frac{5}{2} S_z^3 + (41/8) S_z]$. For a helicity eigenstate with the eigenvalue λ, the corresponding eigenvalue ξ_λ of $K_f(\hat{k})$ reads $\xi_\lambda = -\frac{3}{4}, \frac{9}{4}, -\frac{9}{4}, \frac{3}{4}$ for $\lambda = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, respectively. The Bogoliubov quasiparticle spectra are $E_\lambda(\vec{k}) = \sqrt{\epsilon^2(\vec{k}) + |\Delta_f|^2 (k/k_f)^6 \xi_\lambda^2}$ satisfying $E_\lambda(\vec{k}) = E_{-\lambda}(\vec{k})$ due to the parity symmetry.

Next we study the pairing topological structure. The pairing Hamiltonian Eq. (4) in the Bogoliubov–de Gennes (B-deG) formalism possesses the particle-hole symmetry $C_p H(\vec{k}) C_p^{-1} = -H^*(-\vec{k})$ with $C_p = \tau_1 \otimes I_4$. Furthermore, the isotropic pairing state described by Eq. (6) is TR invariant satisfying $C_T H(\vec{k}) C_T^{-1} = H^*(-\vec{k})$ with $C_T = I_2 \otimes R$, and thus it belongs to the DIII class. The associated topological index is integer valued which will be calculated following the method in Ref. [30]. $H(\vec{k})$ is transformed with only two off-diagonal blocks as $\epsilon(\vec{k})\tau_1 + \Delta_f (k/k_f)^3 K(\hat{k}) \tau_2$. The singular-value-decomposition to its up-right block yields $U(\hat{k}) L(k) \Lambda(k) U^\dagger(\hat{k})$, in which $L(k)$ and $\Lambda(k)$ are two diagonal matrices only dependent on the magnitude of k defined as $L_{\lambda\lambda}(k) = E_\lambda(k)$ and $\Lambda_{\lambda\lambda}(k) = e^{i\theta_\lambda(k)}$, respectively. The angles satisfy $\tan\theta_\lambda(\vec{k}) = -(\Delta_f \xi_\lambda / \epsilon_{\vec{k}})(k/k_f)^3$ and for simplicity Δ_f is set as positive. The k^3 dependence of the pairing amplitude is regularized: Beyond a cutoff k_c, Δ_f vanishes.

The topological index is calculated through the SU(4) matrix $Q_{\vec{k}} = U(\hat{k}) \Lambda(k) U^\dagger(\hat{k})$ as

$$N_w = \frac{1}{24\pi^2} \int d^3\vec{k} \epsilon^{ijl} \mathrm{Tr}[Q_{\vec{k}}^\dagger \partial_i Q_{\vec{k}} Q_{\vec{k}}^\dagger \partial_j Q_{\vec{k}} Q_{\vec{k}}^\dagger \partial_l Q_{\vec{k}}], \quad (8)$$

which is integer valued characterizing the homotopic class of the mapping, i.e., $\pi_3(SU(4)) = \mathbb{Z}$. Nevertheless, N_w is only well defined up to a sign: After changing $\Delta_f \to -\Delta_f$, N_w flips the sign. As shown in the Supplemental Material IV of [43], at $\mu > 0$ N_w is evaluated as

$$N_w = \sum_{\lambda=\pm\frac{3}{2},\pm\frac{1}{2}} \lambda \mathrm{sgn}(\xi_\lambda). \tag{9}$$

Its dependence on $\mathrm{sgn}(\xi_\lambda)$ is because $\theta_\lambda(\vec{k})$ varies from $0 \to \frac{\pi}{2} \to \pi$ at $\xi_\lambda > 0$ but from $\pi \to \frac{\pi}{2} \to 0$ as k varies from 0 to k_f to $+\infty$. A similar form of Eq. (9) was obtained in Ref. [30] in which the Fermi surface Chern number plays the role of λ in Eq. (9). For two helicity pairs of $\lambda = \pm\frac{3}{2}$ and $\lambda = \pm\frac{1}{2}$, their contributions are with opposite signs, and thus $N_w = 2$.

The nontrivial bulk topology gives rise to gapless surface Dirac cones. Because of the pairing isotropy, without loss of generality, an open planar boundary is chosen at $z = 0$ with $\mu(z) = \epsilon_f > 0$ at $z < 0$ and $-\infty$ at $z > 0$. The mean-field Hamiltonian becomes $H(\vec{k}_\parallel, z)$ in which $\vec{k}_\parallel = (k_x, k_y)$ remains conserved while the translation symmetry along the z axis is broken. The symmetry on the boundary is $C_{v\infty}$ including the uniaxial rotation around the z axis and the reflection with respect to any vertical plane. $C_{v\infty}$ is also the little group symmetry at $\vec{k}_\parallel = 0$, then the four zero Andreev-Majorana modes at $\vec{k}_\parallel = 0$ are s_z eigenstates denoted as $|0_{\alpha,f}\rangle$. The associated creation operators $\gamma_\alpha^\dagger$ are solved as

$$\gamma_\alpha^\dagger = \int_{-\infty}^{0} dz[e^{i(\frac{\varphi}{2}+\frac{\pi}{4})} c_\alpha^\dagger(\vec{k}_\parallel = 0, z) + e^{-i(\frac{\varphi}{2}+\frac{\pi}{4})} c_{-\alpha}(\vec{k}_\parallel = 0, z)] u_\alpha(z), \tag{10}$$

where φ is the phase of Δ; $u_{f,\alpha}(z)$ is the zero mode wave function exponentially decaying along the z axis, and its expression is presented in the Supplemental Material V of [43].

The surface zero modes $|0_{\alpha,f}\rangle$ at $\vec{k}_\parallel = 0$ possess an important property that the gapped bulk modes do not have: They are chiral eigenmodes satisfying $C_{\mathrm{ch}}|0_{\alpha,f}\rangle = (-)^{\nu_\alpha}|0_{\alpha,f}\rangle$ with $\nu_\alpha = 0$ for $\alpha = \frac{3}{2}, -\frac{1}{2}$ and $\nu_\alpha = 1$ for $\alpha = \frac{1}{2}, -\frac{3}{2}$, respectively, in which the chiral operator is defined as $C_{\mathrm{ch}} = iC_p C_T = i\tau_1 \otimes R$. The mean-field Hamiltonian $H(\vec{k}_\parallel, z)$ is in the DIII class satisfying the particle-hole and TR symmetries, and it transforms as $C_{\mathrm{ch}} H(\vec{k}_\parallel, z) C_{\mathrm{ch}}^{-1} = -H(\vec{k}_\parallel, z)$. Thus C_{ch} is a symmetry only for zero modes. For a nonzero mode $|\psi_n\rangle$ and its chiral partner $|\psi_{\bar{n}}\rangle = C_{\mathrm{ch}}|\psi_n\rangle$, their energies are opposite to each other, i.e., $\epsilon_{\bar{n}} = -\epsilon_n$. If a perturbation δH remains in the DIII class, then $C_{\mathrm{ch}} \delta H C_{\mathrm{ch}}^{-1} = -\delta H$. δH can *only* mix two zero modes with opposite chiral indices because $\langle 0_{\alpha,f}|\delta H|0_{\beta,f}\rangle = (-)^{\nu_\alpha+\nu_\beta+1}\langle 0_{\alpha,f}|\delta H|0_{\beta,f}\rangle$, and it is nonzero only if $\nu_\alpha \neq \nu_\beta$.

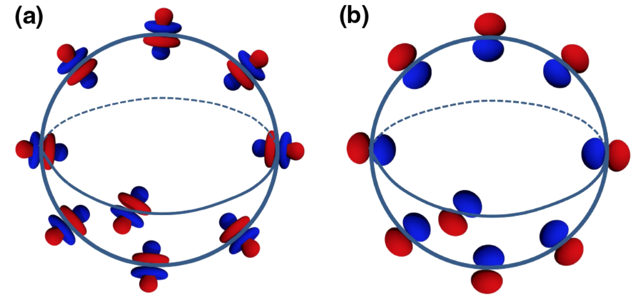

FIG. 1. Pictorial representations of the pairing matrices over the Fermi surfaces of (a) the isotropic f-wave septet pairing and (b) the isotropic p-wave triplet pairing with spin-$\frac{3}{2}$ fermions. Intuitively, the f-wave matrix kernels $U(\hat{k})S^{30}U^\dagger(\hat{k})$ and the p-wave ones $U(\hat{k})S^{10}U^\dagger(\hat{k})$ for each wave vector $\vec{k}$ are depicted in their orbital counterpart harmonic functions in (a) and (b), respectively.

As moving away from $\vec{k}_\parallel = 0$, the zero modes evolve to the midgap states developing energy dispersions. At $k_\parallel \ll k_f$, these midgap states can be solved by using the $k \cdot p$ perturbation theory within the subspace spanned by the zero modes $|0_{\alpha,f}\rangle$ at $\vec{k}_\parallel = 0$. By setting $\delta H = H(\vec{k}_\parallel, z) - H(0, z)$, the effective Hamiltonian to the linear order of $k_\parallel$ is

$$H_{\mathrm{mid}}^f(\vec{k}_\parallel) = \frac{9\Delta_f}{4k_f} \begin{pmatrix} 0 & -ik_- & 0 & O(k_-^3) \\ ik_+ & 0 & -2ik_- & 0 \\ 0 & 2ik_+ & 0 & -ik_- \\ O(k_+^3) & 0 & ik_+ & 0 \end{pmatrix}, \tag{11}$$

where $k_\pm = k_x \pm ik_y$. The matrix elements in the same chiral sector are exactly zero, and the elements at the order of $O(k_\pm^3)$ are neglected. The solutions consist of two sets of 2D surface Dirac cone spectra represented by $E_\pm^{a(b)}(\vec{k}_\parallel) = \pm v_{a(b)} k_\parallel$. The velocities are solved as $v_{a(b)} = \frac{9}{4}(|\Delta_f|/k_f)(\sqrt{2} \pm 1)$. We also develop a systematic method beyond the $k \cdot p$ theory to solve the midgap spectra for all the range of $k_\parallel$ as presented in the Supplemental Material VIII of [43], and the results are plotted in Fig. 2(a). In addition to the Dirac cones, there also exists an additional zero energy ring not captured by Eq. (11), which is located at $k/k_f = (\sqrt{3}/2)$ as analyzed in the Supplemental Material V of [43].

Now we move to other unconventional isotropic pairings of spin-$\frac{3}{2}$ fermions in the p- and d-wave channels. The p-wave triplet one is topologically nontrivial, and the analysis can be performed in the same way as above. The pairing matrix is $\hat{\Delta}_p(\vec{k}) = \sum_{\nu=0,\pm1}(S^{1\nu}R) d_p^{*,\nu}(\vec{k}) = \Delta_p(k/k_f) K_p(\hat{k}) R$, where $S^{1\nu}$ is the rank-1 spin tensor, $d_p^\nu = \Delta_p(k/k_f) Y_{1\nu}(\hat{k})$, and $K_p(\hat{k}) = \hat{k} \cdot \vec{S}$ is just the helicity operator. The quasiparticle spectra are fully gapped

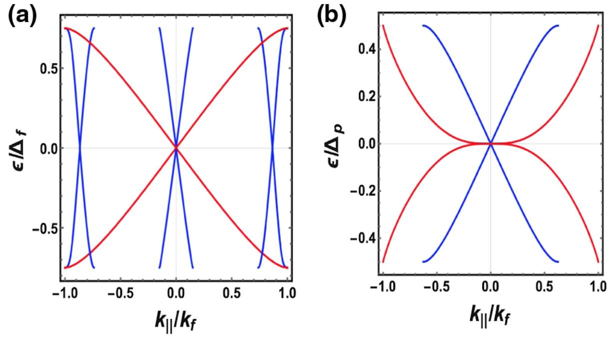

FIG. 2. The gapless surface spectra for the isotropic f-wave septet pairing in (a) and for the isotropic p-wave triplet pairing in (b) with spin-$\frac{3}{2}$ fermions.

as $E_\lambda(\vec{k}) = \sqrt{\epsilon^2(\vec{k}) + |\Delta_p|^2(k/k_f)^2\lambda^2}$, and the topological index of this pairing can be evaluated based on Eq. (9) by replacing the eigenvalues of $K_f(\hat{k})$ with those of $K_p(\hat{k})$. The contributions from two helicity pairs of $\lambda = \pm\frac{3}{2}$ and $\pm\frac{1}{2}$ add up leading to a high value $N_w = 4$. In comparison, the topological index of the ^{3}He-B phase is only 1, and thus their topological sectors are different in spite of the same pairing symmetry.

The surface spectra of the isotropic p-wave pairing with spin-$\frac{3}{2}$ fermions are interesting: They exhibit a cubic Dirac cone in addition to a linear one. Consider the same planar boundary configuration as before, similarly for each spin component α there exists one zero mode at $\vec{k}_\parallel = 0$ labeled by $|0_{\alpha,p}\rangle$. Again we perform the $k\cdot p$ analysis at $k_\parallel \ll k_f$ in the subspace spanned by $|0_{\alpha,p}\rangle$ with respect to $\delta H = H(\vec{k}_\parallel, z) - H(0,z)$. The chiral eigenvalue of $|0_{\alpha,p}\rangle$ is $(-)^{\nu_\alpha} = \mathrm{sgn}(\alpha)$, which leads to a different structure of effective Hamiltonian from that of the f-wave one. Only $|0_{\pm\frac{1}{2},p}\rangle$ can be directly coupled by δH, which leads to a linear Dirac cone. In contrast, the pair of states $|0_{\pm\frac{3}{2},p}\rangle$ are not directly coupled, rather $|0_{\frac{3}{2},p}\rangle$ and $|0_{-\frac{1}{2},p}\rangle$ are coupled through the second order perturbation theory, and so are $|0_{-\frac{3}{2},p}\rangle$ and $|0_{\frac{1}{2},p}\rangle$. Consequently, $|0_{\pm\frac{3}{2},p}\rangle$ are coupled at the order of $(\delta H)^3$ developing a cubic Dirac cone as shown in the Supplemental Material VII of [43]. The above analysis is confirmed by the solution based on the nonperturbative method in the Supplemental Material VIII of [43] as plotted in Fig. 2(b).

In contrast, the d-wave spin quintet isotropic pairing of spin-$\frac{3}{2}$ fermions is topologically trivial. By imitating the analyses above, we replace $K_f(\hat{k})$ with $K_d(\hat{k}) = 2(\hat{k}\cdot\vec{S})^2 - \frac{5}{2}I_4$. Different from the kernels K_p and K_f in odd partial-wave channels, K_d's eigenvalues are even with respect to the helicity index, i.e., $\xi^d_\lambda = \xi^d_{-\lambda}$, such that N_w vanishes. This result agrees with the fact that 3D TR invariant topological superconductors should be parity odd as shown in Ref. [44]. The explicit calculation of the surface spectra in the Supplemental Material VII of [43] confirms this point showing the absence of zero modes.

The above analysis can be straightforwardly applied to multicomponent fermion systems with a general spin value $s = N - \frac{1}{2}$. The spin tensors at the order of l are denoted as S^{lm} with $0 \le l \le 2S$ and $-l \le m \le l$. For each partial-wave channel $0 \le l \le 2S$, there exists an isotropic pairing with the pairing matrix $\hat{\Delta}(\hat{k}) = \Delta_l(k/k_f)^l K_l(\hat{k})R$ in which $K_l = U(\hat{k})S^{l0}U^\dagger(\hat{k})$, whose topological index $N_w(l)$ is determined by the sign pattern of the elements of the diagonal matrix S^{l0}. For even and odd values of l, $S^{l0}_{\alpha\alpha} = \pm S^{l0}_{-\alpha-\alpha}$, respectively, and thus N^l_w vanishes when l is even, while for odd values of l,

$$N_w(l) = \sum_{\lambda>0} 2\lambda\,\mathrm{sgn}(S^{l0}_{\lambda\lambda}), \tag{12}$$

in which $S^{l0}_{\alpha\alpha} = (-)^{\alpha+\frac{1}{2}}\langle S\alpha, S-\alpha|SS; l0\rangle$ up to an overall factor. The largest value of N_w is reached for the p-wave case: Since $S^{10} \propto S_z$, contributions from all the components add together leading to $N_w = N^2$. The ^{3}He-B phase of spin-$\frac{1}{2}$ fermions and the isotropic p-wave pairing with spin-$\frac{3}{2}$ fermions are two examples. As for the surface zero modes $|0_{\alpha,p}\rangle$ at $\vec{k}_\parallel = 0$, their chiral indices equal $\mathrm{sgn}(\alpha)$. As a result, similar to the spin-$\frac{3}{2}$ case, when performing the $k\cdot p$ analysis for midgap states within the subspace spanned by $|0_{\alpha,p}\rangle$, only $|0_{\pm\frac{1}{2},p}\rangle$ are directly coupled leading to a linear Dirac cone, and other pairs of $|0_{\pm\alpha,p}\rangle$ are indirectly coupled at the order of $(\delta H)^{2\alpha}$ leading to high order Dirac cones.

Multicomponent fermion systems are not rare in nature. In solid state systems, many materials are orbital active including semiconductors, transition metal oxides, and heavy fermion systems. Because of spin-orbit coupling, their band structures are denoted by electron total angular momentum j and in many situations $j > \frac{1}{2}$. For example, in the hole-doped semiconductors, the valence band carries $j = \frac{3}{2}$ as described by the Luttinger model [45]. Superconductivity has been discovered in these systems including hole-doped diamond and Germanium [46–48]. Although in these materials, the Cooper pairings are mostly of the conventional s-wave symmetry arising from the electron-phonon interaction, it is natural to further consider unconventional pairing states in systems with similar band structures but stronger correlation effects. The p-wave pairing based on the Luttinger model has been studied in Ref. [40]. In ultracold atom systems, many alkali and alkaline-earth fermions often carry large hyperfine spin values $F > \frac{1}{2}$, and thus their Cooper pair spin structures are enriched taking values from 0 to $2F$ not just singlet and triplet as in the spin-$\frac{1}{2}$ case [34,36,42].

In multicomponent solid state systems, there often exists spin-orbit coupling. For example, the Luttinger model describing hole-doped semiconductors [45], contains an isotropic spin-orbit coupling $H_{so} = \gamma_2 k^2 (\hat{k} \cdot \vec{S})^2$. Since H_{so} is diagonalized in the helicity eigenbasis, we only need to update the kinetic energy with $\epsilon_{k\lambda} = \epsilon_k + \gamma k^2 \lambda^2$ in the mean-field analysis, which satisfies $\epsilon_{k\lambda} = \epsilon_{k,-\lambda}$, and the pairing structure described by Eq. (6) is not affected. The topological properties are the same as analyzed before because the index formula Eq. (9) remains valid and the surface midgap state calculation can be performed qualitatively similarly. Nevertheless, the symmetry-breaking pattern is changed. The relative spin-orbit symmetry is already explicitly broken by the H_{so}. The spin-orbit coupled Goldstone modes in ^{3}He-B become gapped pseudo-Goldstone modes with the gap proportional to the spin-orbit coupling strength γ_2.

In summary, we have found that multicomponent fermion systems can support a class of exotic isotropic pairing states analogous to the ^{3}He-B phase with unconventional pairing symmetries and nontrivial topological structures. High-rank spin tensors are entangled with orbital partial waves at the same order to form isotropic gap functions. For the spin-$\frac{3}{2}$ case, the odd partial-wave channel pairings carry topological indices 2 and 4 for the f- and p-wave pairings, respectively, while the d-wave channel pairing is topologically trivial. The surface Dirac cones of midgap modes are solved analytically which exhibit two linear Dirac cones in the f-wave case, and the coexistence of linear and cubic Dirac cones in the p-wave case. Generalizations to systems with even more fermion components can be performed straightforwardly. This work provides important guidance in the search for novel nontrivial topological pairing states in both condensed matter and ultracold atom systems.

W. Y. and C. W. are supported by the NSF DMR-1410375 and AFOSR FA9550-14-1-0168. Y. L. is grateful to the KSAS General Fund Start-up at Johns Hopkins. C. W. acknowledges the supports from the CAS/SAFEA International Partnership Program for Creative Research Teams of China, and the President's Research Catalyst Awards CA-15-327861 from the University of California Office of the President.

Note added.—Recently, evidence for the septet pairing with spin-$\frac{3}{2}$ fermions was reported in the rare earth-based half-Heusler superconductors [49].

[1] V. L. Berezinskii, JETP Lett. **20**, 287 (1974).
[2] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Rev. Mod. Phys. **77**, 1321 (2005).
[3] A. J. Leggett, Rev. Mod. Phys. **47**, 331 (1975).
[4] G. E. Volovik, *The Universe in a Helium Droplet* (Clarendon Press, Oxford, 2003).
[5] P. Anderson and P. Morel, Phys. Rev. **123**, 1911 (1961).
[6] R. Balian and N. Werthamer, Phys. Rev. **131**, 1553 (1963).
[7] K. Aoyama and R. Ikeda, Phys. Rev. B **73**, 060504 (2006).
[8] V. V. Dmitriev, A. A. Senin, A. A. Soldatov, and A. N. Yudin, Phys. Rev. Lett. **115**, 165304 (2015).
[9] D. Aoki and J. Flouquet, J. Phys. Soc. Jpn. **81**, 011003 (2012).
[10] K. Scharnberg and R. A. Klemm, Phys. Rev. B **22**, 5233 (1980).
[11] M. M. Salomaa and G. E. Volovik, Rev. Mod. Phys. **59**, 533 (1987).
[12] Y. Machida, A. Itoh, Y. So, K. Izawa, Y. Haga, E. Yamamoto, N. Kimura, Y. Onuki, Y. Tsutsumi, and K. Machida, Phys. Rev. Lett. **108**, 157002 (2012).
[13] E. R. Schemm, W. J. Gannon, C. M. Wishne, W. P. Halperin, and A. Kapitulnik, Science **345**, 190 (2014).
[14] C. Wu and S.-C. Zhang, Phys. Rev. Lett. **93**, 036403 (2004).
[15] C. Wu, K. Sun, E. Fradkin, and S.-C. Zhang, Phys. Rev. B **75**, 115103 (2007).
[16] A. Y. Kitaev, Phys. Usp. **44**, 131 (2001).
[17] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).
[18] S. Ryu, A. Schnyder, A. Furusaki, and A. Ludwig, New J. Phys. **12**, 065010 (2010).
[19] K. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
[20] M. Kohmoto, Ann. Phys. (N.Y.) **160**, 343 (1985).
[21] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[22] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[23] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[24] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[25] J. E. Moore and L. Balents, Phys. Rev. B **75**, 121306 (2007).
[26] R. Roy, New J. Phys. **12**, 065009 (2010).
[27] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, Nat. Phys. **7**, 412 (2011).
[28] J. D. Sau, R. M. Lutchyn, S. Tewari, and S. Das Sarma, Phys. Rev. Lett. **104**, 040502 (2010).
[29] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
[30] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **81**, 134508 (2010).
[31] S. B. Chung and S.-C. Zhang, Phys. Rev. Lett. **103**, 235301 (2009).
[32] Y. Bunkov and R. Gazizulin, arXiv:1504.01711.
[33] C. Wu, Nat. Phys. **8**, 784 (2012).
[34] T.-L. Ho and S. Yip, Phys. Rev. Lett. **82**, 247 (1999).
[35] C. Wu, J.-p. Hu, and S.-c. Zhang, Phys. Rev. Lett. **91**, 186402 (2003).
[36] C. Wu, Mod. Phys. Lett. B **20**, 1707 (2006).
[37] B. J. DeSalvo, M. Yan, P. G. Mickelson, Y. N. Martinez de Escobar, and T. C. Killian, Phys. Rev. Lett. **105**, 030402 (2010).
[38] A. V. Gorshkov, M. Hermele, V. Gurarie, C. Xu, P. S. Julienne, J. Ye, P. Zoller, E. Demler, M. D. Lukin, and A. M. Rey, Nat. Phys. **6**, 289 (2010).
[39] S. Taie, Y. Takasu, S. Sugawa, R. Yamazaki, T. Tsujimoto, R. Murakami, and Y. Takahashi, Phys. Rev. Lett. **105**, 190401 (2010).
[40] C. Fang, B. A. Bernevig, and M. J. Gilbert, Phys. Rev. B **91**, 165421 (2015).
[41] L. Fu and E. Berg, Phys. Rev. Lett. **105**, 097001 (2010).

[42] C. Wu, J. Hu, and S.-C. Zhang, Int. J. Mod. Phys. B **24**, 311 (2010).

[43] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.117.075301 for explanations and technical details.

[44] F. Zhang, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **111**, 056403 (2013).

[45] R. Winkler, *Spin–Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems* (Springer, Berlin, Heidelberg, 2003).

[46] Y. Takano, M. Nagao, I. Sakaguchi, M. Tachiki, T. Hatano, K. Kobayashi, H. Umezawa, and H. Kawarada, Appl. Phys. Lett. **85**, 2851 (2004).

[47] E. A. Ekimov, V. A. Sidorov, E. D. Bauer, N. N. Mel'nik, N. J. Curro, J. D. Thompson, and S. M. Stishov, Nature (London) **428**, 542 (2004).

[48] T. Herrmannsdörfer *et al.*, Phys. Rev. Lett. **102**, 217003 (2009).

[49] H. Kim *et al.*, arXiv:1603.03375.