

Should Entanglement Measures be Monogamous or Faithful?

Cécilia Lancien,^{1,2} Sara Di Martino,¹ Marcus Huber,^{3,1,7} Marco Piani,⁴ Gerardo Adesso,⁵ and Andreas Winter^{1,6}

¹*Física Teòrica: Informació i Fenòmens Quàntics, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain*

²*Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 Boulevard du 11 Novembre 1918, 69622 Villeurbanne Cedex, France*

³*Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland*

⁴*SUPA and Department of Physics, University of Strathclyde, Glasgow G4 0NG, United Kingdom*

⁵*School of Mathematical Sciences, The University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom*

⁶*ICREA—Institució Catalana de Recerca i Estudis Avançats, Passeig Lluís Companys 23, ES-08010 Barcelona, Spain*

⁷*Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmannngasse 3, A-1090 Vienna, Austria*

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“Is entanglement monogamous?” asks the title of a popular article [B. Terhal, *IBM J. Res. Dev.* **48**, 71 (2004)], celebrating C. H. Bennett’s legacy on quantum information theory. While the answer is affirmative in the qualitative sense, the situation is less clear if monogamy is intended as a quantitative limitation on the distribution of bipartite entanglement in a multipartite system, given some particular measure of entanglement. Here, we formalize what it takes for a bipartite measure of entanglement to obey a general quantitative monogamy relation on all quantum states. We then prove that an important class of entanglement measures fail to be monogamous in this general sense of the term, with monogamy violations becoming generic with increasing dimension. In particular, we show that every additive and suitably normalized entanglement measure cannot satisfy any nontrivial general monogamy relation while at the same time faithfully capturing the geometric entanglement structure of the fully antisymmetric state in arbitrary dimension. Nevertheless, monogamy of such entanglement measures can be recovered if one allows for dimension-dependent relations, as we show explicitly with relevant examples.

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Introduction.—Entanglement is a quintessential manifestation of quantum mechanics [1,2]. The study of entanglement and its distribution reveals fundamental insights into the nature of quantum correlations [3], on the properties of many-body systems [4,5], and on possibilities and limitations for quantum-enhanced technologies [6]. A particularly interesting feature of entanglement is known as *monogamy* [7], that is, the impossibility of sharing entanglement unconditionally across many subsystems of a composite quantum system.

In the clearest manifestation of monogamy, if two parties A and B with the same (finite) Hilbert space dimension are maximally entangled, then their state is a pure state $|\Phi\rangle_{AB}$ [8], and neither of them can share any correlation—let alone entanglement—with a third party C , as the only physically allowed pure states of the tripartite system ABC are product states $|\Phi\rangle_{AB} \otimes |\Psi\rangle_C$. Consider now the more realistic case of A and B being in a mixed, partially entangled state ρ_{AB} . It is then conceivable for more parties to get a share of such entanglement. Namely, a state ρ_{AB} on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is termed “ n -shareable” with respect to subsystem B if it admits a symmetric n -extension, i.e., a state $\rho'_{AB_1\dots B_n}$ on $\mathcal{H}_A \otimes \mathcal{H}_B^{\otimes n}$ invariant under permutations of the subsystems B_1, \dots, B_n and such that the marginal state of A and any B_j amounts to ρ_{AB} . While even an entangled state can be shareable up to some number of extensions, a seminal result is that a state ρ_{AB} is n -shareable for all $n \geq 2$ if and only if it is separable, that is, no entangled state can be infinitely shareable [7,9–12]. This statement formalizes exactly the monogamy of

entanglement (in an asymptotic setting), and has many important implications, including the equivalence between asymptotic quantum cloning and state estimation [13,14], the emergence of objectivity in the quantum-to-classical transition [15], the security of quantum key distribution [16–19], and the study of frustration and topological phases in many-body systems [20–24].

Over the last two decades, the goal to formalize monogamy of entanglement in precise quantitative terms and for a finite number of parties has attracted increasing interest. The concept of monogamy became synonymous with the validity of an inequality due to Coffman, Kundu, and Wootters (CKW) [25]. Given any tripartite state ρ_{ABC} , and choosing a bipartite entanglement measure E , the CKW inequality reads [26]

$$E_{A:BC}(\rho_{ABC}) \geq E_{A:B}(\rho_{AB}) + E_{A:C}(\rho_{AC}), \quad (1)$$

with $\rho_{AB} = \text{Tr}_C[\rho_{ABC}]$ and $\rho_{AC} = \text{Tr}_B[\rho_{ABC}]$. Intuitively, Eq. (1) means that the sum of the individual pairwise entanglements between A and each of the other parties B or C cannot exceed the entanglement between A and the remaining parties grouped together. Equation (1) was originally proven for arbitrary states of three qubits, adopting the squared concurrence as entanglement measure [25]. Variations of the CKW inequality and generalizations to n parties have been established for a number of entanglement measures in discrete as well as continuous variable systems [20,27–39]. In particular, the squashed entanglement [40] and the one-way distillable entanglement fulfill

Eq. (1) in composite systems of arbitrary dimension [28]. Hybrid CKW-like inequalities involving entanglement and other forms of correlations have also been proven [28,41], while measures of quantum correlations weaker than entanglement generally violate the CKW inequality [42]. To some extent, therefore, Eq. (1) does capture the spirit of monogamy as a distinctive property of entanglement.

The main problem with CKW inequalities, however, is that their validity is not universal, but rather depends on the specific choice of E . Perhaps counterintuitively, several prominent entanglement monotones, such as the entanglement of formation or the distillable entanglement [3,43,44], do not obey the constraint formalized by Eq. (1), unless one introduces *ad hoc* rescalings (see, e.g., Ref. [45]). Since entanglement as a concept is monogamous in the n -shareability sense [7], one is led to raise the following key question: Should any valid *entanglement measure* be monogamous in a CKW-like sense?

In this Letter we address the question in general terms. Given an entanglement measure E , we shall say that it is *monogamous* if there exists a nontrivial function $f: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that the generalized monogamy relation

$$E_{A:BC}(\rho_{ABC}) \geq f(E_{A:B}(\rho_{AB}), E_{A:C}(\rho_{AC})), \quad (2)$$

is satisfied for *any* state ρ_{ABC} on *any* tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Recalling that E is an entanglement monotone (which in turn implies that it is nonincreasing under partial traces), the function f in Eq. (2) is without loss of generality such that $f(x, y) \geq \max(x, y)$. Thus, to give rise to a nontrivial constraint, we need $f(x, y) > \max(x, y)$ for at least some range of values of x and y . One might further impose that $f(x, y)$ is monotonic in both its arguments, but we will not require this here. We will, however, require that f is continuous in general.

While the CKW form Eq. (1) of a monogamy relation [which is recovered for the particular choice $f(x, y) = x + y$] implicitly presumes some kind of additivity of the entanglement measure in question, our general form Eq. (2) transcends this and can be applied to recognize any entanglement measure E as *de facto* monogamous, based on the intuition that it should obey *some* trade-off between the values of $E_{A:B}$ and $E_{A:C}$ for a given $E_{A:BC}$, see Fig. 1. Oppositely, if the only possible choice in Eq. (2) were $f(x, y) = \max(x, y)$, then the measure E would fail

monogamy in the most drastic fashion: given a state ρ_{ABC} , having $E_{A:BC} > 0$ *a priori* would not imply that $E_{A:B}$ and $E_{A:C}$ have to constrain each other in the interval $[0, E_{A:BC}]$.

Quite remarkably, we rigorously show in the following that the entanglement of formation E_F [43] and the relative entropy of entanglement E_R [46], which are two of the most important entanglement monotones for mixed states [3,44], *cannot* satisfy a nontrivial monogamy relation in the sense of Eq. (2), with violations becoming generic [47] with increasing Hilbert space dimension. We further show that a whole class of additive entanglement measures, including the entanglement cost E_F^∞ [43,48] and the regularized relative entropy of entanglement E_R^∞ [49,50], also fail monogamy as captured by Eq. (2). The latter result is proven by a constructive argument which exploits the peculiar properties of the maximally antisymmetric state on $\mathbb{C}^n \otimes \mathbb{C}^n$, which is $(n-1)$ -shareable yet far from separable [51], and has hence been dubbed the “universal counterexample” in quantum information theory [52]. Specifically, any additive entanglement measure which is geometrically *faithful* in the sense of being lower-bounded by a quantity with a sub-polynomial dimensional dependence on the antisymmetric state, cannot be monogamous in general.

Our analysis then reveals that entanglement measures divide into two main categories: monogamous (yet geometrically unfaithful) ones, like the squashed entanglement [28,40], and geometrically faithful (yet nonmonogamous) ones, like E_F , E_R , and their regularizations. Finally, we show that this dilemma can be resolved if one relaxes the definition Eq. (2) to introduce monogamy relations for any *fixed* dimension of $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Explicitly, we prove that E_F and E_R^∞ are retrievable as monogamous for any finite dimension, by providing dimension-dependent choices of f in Eq. (2), which only reduce to the trivial one in the limit of infinite dimension.

Result (1), generic nonmonogamy for entanglement of formation and relative entropy of entanglement.—We begin by defining the measures employed in our analysis [3,43,44,46]. The entanglement of formation E_F is the convex roof extension of the entropy of entanglement, $E_F(\rho_{A:B}) = \inf_{\{p_i, |\psi_i\rangle_{AB}\}} \sum_i p_i S(\text{Tr}_B[|\psi_i\rangle\langle\psi_i|_{AB}])$, where $S(\rho) = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy, and $\log \equiv \log_2$. On the other hand, the relative entropy of

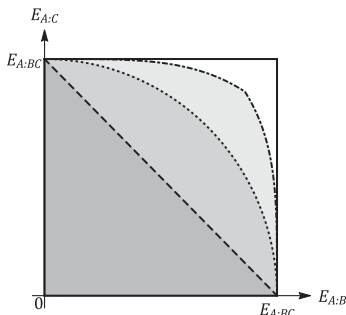


FIG. 1. For any tripartite state ρ_{ABC} , an entanglement measure E obeys monogamy if, given the global entanglement $E_{A:BC}$, the pairwise terms $E_{A:B}$ and $E_{A:C}$ are nontrivially constrained. We formalize these constraints via a function $f(E_{A:B}, E_{A:C})$ in Eq. (2). Choosing $f(x, y) = x + y$, one gets the CKW inequality Eq. (1), which limits the pairwise terms to the triangular darker region with dashed boundary. The other depicted shaded regions correspond to $f(x, y) = \sqrt{x^2 + y^2}$ (dotted boundary) and $f(x, y) = \max(x + cy^4, y + cx^4)$ with c a constant (dot-dashed boundary). Any measure E is termed *monogamous* if, for all tripartite states, the ensuing entanglement distribution can be confined to a region strictly smaller than the white square with solid boundary. The latter denotes the trivial choice $f(x, y) = \max(x, y)$, which is satisfied *a priori* by any entanglement monotone E .

entanglement E_R quantifies the distance from the set of separable states, $E_R(\rho_{A:B}) = \inf_{\sigma_{AB}} S(\rho_{AB} || \sigma_{AB})$, where the minimization is over all separable σ_{AB} , and $S(\rho || \sigma) = \text{Tr}[\rho \log \rho - \rho \log \sigma]$ is the relative entropy.

The fact that both E_F and E_R violate the CKW inequality Eq. (1) may be traced to their subadditivity, meaning that there exist states ρ_{AB} and $\sigma_{A'C}$ such that $E(\rho_{A:B} \otimes \sigma_{A':C}) < E(\rho_{A:B}) + E(\sigma_{A':C})$, with E denoting either E_F [53] or E_R [54]. We now show that these measures fail monogamy even in the general sense of Eq. (2). These results are based on *random induced states*, defined as follows. Given $n, s \in \mathbb{N}$, a random mixed state ρ on \mathbb{C}^n is induced by \mathbb{C}^s if $\rho = \text{Tr}_{\mathbb{C}^s} |\psi\rangle\langle\psi|$ for $|\psi\rangle$ a uniformly distributed random pure state on $\mathbb{C}^n \otimes \mathbb{C}^s$. Note that if $s \leq n$, this is equivalent to ρ being uniformly distributed on the set of mixed states of rank at most s on \mathbb{C}^n . Here we focus on the (balanced) bipartite Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$, and aim to determine, given $d, s \in \mathbb{N}$, what is the typical value of $E_F(\rho_{A:B})$ and $E_R(\rho_{A:B})$ for ρ_{AB} a random state on $\mathbb{C}^d \otimes \mathbb{C}^d$ induced by an environment \mathbb{C}^s . The answer is as follows:

$$\mathbb{P}\left(\left|E_F(\rho_{A:B}) - \left[\log d - \frac{1}{2 \ln 2}\right]\right| \leq t\right) \geq 1 - e^{-cd^2t^2/\log^2 d}, \quad (3)$$

$$\mathbb{P}\left(\left|E_R(\rho_{A:B}) - \left[\log\left(\frac{d^2}{s}\right) + \frac{1}{2 \ln 2} \frac{s}{d^2}\right]\right| \leq t\right) \geq 1 - e^{-cs^2t^2}, \quad (4)$$

where $s \leq Cd^2t^2/\log^2(1/t)$ for any fixed $t > 0$ in Eq. (3), and $Cd \log(1/t)/t^2 \leq s \leq d^2$ for any fixed $0 < t < 1$ in Eq. (4), while $c, C > 0$ denote universal constants in both equations. While Eq. (3) was established in Ref. [47] (Theorem V.1), although with a looser dependence on the parameters, Eq. (4) is an entirely original result of independent interest. Both proofs are mathematically quite involved, and are relegated to Ref. [55].

Importantly, failure of monogamy is then retrieved as a *generic* trait of entanglement quantified by these measures. Namely, the main result of this section is that there exist states $\rho_{ABC}^{(x)}$ on Hilbert spaces $\mathcal{H}_A^{(x)} \otimes \mathcal{H}_B^{(x)} \otimes \mathcal{H}_C^{(x)}$ such that, as $x \rightarrow \infty$,

$$E_{A:BC}(\rho_{ABC}^{(x)}) \leq x, \text{ while } E_{A:B}(\rho_{AB}^{(x)}) \sim E_{A:C}(\rho_{AC}^{(x)}) \sim x, \quad (5)$$

for E denoting either E_F or E_R . To sketch the proof [55], set $d = \lfloor 2^x \rfloor$ and $\mathcal{H}_A^{(x)} \equiv \mathcal{H}_B^{(x)} \equiv \mathcal{H}_C^{(x)} \equiv \mathbb{C}^d$. Next, consider $\rho_{ABC}^{(x)}$ a random state on $\mathcal{H}_A^{(x)} \otimes \mathcal{H}_B^{(x)} \otimes \mathcal{H}_C^{(x)}$, induced by some $\mathcal{H}_E^{(x)} \equiv \mathbb{C}^s$, with $s \sim \log d$. In that way, $\rho_{AB}^{(x)}$ and $\rho_{AC}^{(x)}$ are random states on $\mathbb{C}^d \otimes \mathbb{C}^d$, induced by some $\mathbb{C}^s \otimes \mathbb{C}^d$, with s and d satisfying both the conditions for Eqs. (3) and (4) to apply. We then have: $E_{\{F,R\}}(\rho_{A:BC}^{(x)}) \leq \log d \leq x$, while $E_F(\rho_{A:B}^{(x)})$ and $E_F(\rho_{A:C}^{(x)})$ are both equal to

$\log d - O(1) \sim \log d \sim x$ with a probability greater than $1 - 2e^{-cd^2/\log^2 d}$, and $E_R(\rho_{A:B}^{(x)})$ and $E_R(\rho_{A:C}^{(x)})$ are both equal to $\log d - \log(\log d) - O(1) \sim \log d \sim x$ with a probability greater than $1 - 2e^{-cd \log d}$. \square

Result (2), nonmonogamy for a whole class of additive entanglement measures.—We now show that a class of additive entanglement measures also fail monogamy in the sense of Eq. (2). A key role in this result is played by the antisymmetric state, defined as follows [48,51]. Given a subsystem A with (finite-dimensional) Hilbert space \mathcal{H}_A , the (maximally) antisymmetric state α_{A^n} on $\mathcal{H}_A^{\otimes n}$ is the normalized projector onto the antisymmetric subspace of $\mathcal{H}_A^{\otimes n}$. A crucial property of α_{A^n} is that its reduced state on any group of k subsystems, for any $0 \leq k \leq n$, is α_{A^k} , i.e., the antisymmetric state on $\mathcal{H}_A^{\otimes k}$.

We now focus on entanglement monotones E satisfying the following conditions: (a) Normalization: For any state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, $E_{A:B}(\rho_{AB}) \leq \min(\log d_A, \log d_B)$; (b) Lower-boundedness on the bipartite antisymmetric state: Denoting by $\alpha_{AA'}$ the antisymmetric state on $\mathcal{H}_A \otimes \mathcal{H}_{A'}$ (with $d_A = d_{A'}$), $E_{A:A'}(\alpha_{AA'}) \geq c/(\log d_A)^t$, where $c, t > 0$ are universal constants; (c) Additivity on product states: For any state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, $E_{A^m:B^m}(\rho_{AB}^{\otimes m}) = mE_{A:B}(\rho_{AB})$; (d) Linearity on mixtures of locally orthogonal states: For any $0 \leq \lambda \leq 1$, and any states ρ_{AB}, σ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $\text{Tr}[\rho_A \sigma_A] = \text{Tr}[\rho_B \sigma_B] = 0$, $E_{A:B}[\lambda \rho_{AB} + (1 - \lambda) \sigma_{AB}] = \lambda E_{A:B}(\rho_{AB}) + (1 - \lambda) E_{A:B}(\sigma_{AB})$.

Important examples of entanglement measures fulfilling the above requirements are the regularized versions of the entanglement of formation (a.k.a. entanglement cost) E_F^∞ [43,48] and of the relative entropy of entanglement E_R^∞ [49,50]. Indeed, condition (c) holds by construction for any regularized entanglement measure, defined as $E_{A:B}^\infty(\rho_{AB}) = \lim_{n \rightarrow \infty} (1/n) E_{A^n:B^n}(\rho_{AB}^{\otimes n})$. Furthermore, in the case of E_F^∞ and E_R^∞ , conditions (a) and (d) are inherited as they hold for E_F and E_R . Finally, condition (b) can be seen as some kind of *faithfulness* (or geometry-preserving) property: given that the antisymmetric state has constant trace distance from the set of separable states, one may wish for an entanglement measure to stay bounded away from 0 on the antisymmetric state, dimension-independently as well (or with a sub-polynomial dependence). For E being E_F^∞ or E_R^∞ , a condition stronger than (b) in fact holds, namely, $E_{A:A'}(\alpha_{AA'}) \geq c$, where $c > 0$ is a universal constant [51].

What we show here is that any entanglement measure E , obeying properties (a)–(d), cannot satisfy a nontrivial monogamy relation in the sense of Eq. (2). We first establish the following result. Let $n \in \mathbb{N}$, $d = 2^n + 1$, and set $\mathcal{H}_{A_j} \equiv \mathbb{C}^d$ for each $0 \leq j \leq 2^n$. Assume next that E satisfies conditions (a) and (b). Then, there exists $0 \leq k \leq n - 1$ such that



FIG. 2. Schematic of the antisymmetric state $\alpha_{A^{2^{k+1}}}$ partitioned into three subsystems, $A \equiv A_0$, $B \equiv A_1 \cdots A_{2^k}$, and $C \equiv A_{2^{k+1}} \cdots A_{2^{k+1}}$.

$$\begin{aligned} E_{A_0:A_1 \cdots A_{2^k}}(\alpha_{A^{2^{k+1}}}) &= E_{A_0:A_{2^{k+1}} \cdots A_{2^{k+1}}}(\alpha_{A^{2^{k+1}}}) \\ &\geq \left(1 - \frac{\ln(n^{t+1}/c)}{n}\right) \\ &\quad \times E_{A_0:A_1 \cdots A_{2^{k+1}}}(\alpha_{A^{2^{k+1}+1}}). \end{aligned} \quad (6)$$

To prove Eq. (6), consider a partition of the antisymmetric state $\alpha_{A^{2^{k+1}+1}}$ as illustrated in Fig. 2, and set $g_k = E_{A_0:A_1 \cdots A_{2^k}}(\alpha_{A_0 \cdots A_{2^k}})$ for each $0 \leq k \leq n$. Then,

$$\frac{c}{n^t} \approx \frac{c}{(\log d)^t} \leq g_0 \leq g_1 \leq \cdots \leq g_{n-1} \leq g_n \leq \log d \approx n. \quad (7)$$

The last inequality is by property (a), because $d_{A_0} = d = 2^n + 1$. The first inequality is by property (b), because $g_0 = E_{A_0:A_1}(\alpha_{A_0 A_1}) \geq c/(\log d)^t \approx c/n^t$. And the middle inequalities are by monotonicity of E under discarding of subsystems, because for each $0 \leq k \leq n-1$, $\alpha_{A_0 \cdots A_{2^k}}$ is the reduced state of $\alpha_{A_0 \cdots A_{2^{k+1}}}$. Now, Eq. (7) implies that there exists $0 \leq \bar{k} \leq n-1$ such that $g_{\bar{k}}/g_{\bar{k}+1} \geq 1 - \ln(n^{t+1}/c)/n$. Indeed, otherwise we would have $(g_0/g_n) = \prod_{k=0}^{n-1} (g_k/g_{k+1}) < \{1 - [\ln(n^{t+1}/c)/n]\}^n \leq (c/n^{t+1})$, which contradicts Eq. (7). Then, as we have on one hand $g_{\bar{k}+1} = E_{A_0:A_1 \cdots A_{2^{\bar{k}+1}}}(\alpha_{A^{2^{\bar{k}+1}+1}})$, and on the other hand $g_{\bar{k}} = E_{A_0:A_1 \cdots A_{2^{\bar{k}}}}(\alpha_{A^{2^{\bar{k}+1}}}) = E_{A_0:A_{2^{\bar{k}+1}} \cdots A_{2^{\bar{k}+1}}}(\alpha_{A^{2^{\bar{k}+1}}})$, Eq. (6) is proven.

The main result of this section then follows immediately. Namely, once again, there exist states $\rho_{ABC}^{(x)}$ on Hilbert spaces $\mathcal{H}_A^{(x)} \otimes \mathcal{H}_B^{(x)} \otimes \mathcal{H}_C^{(x)}$ such that Eq. (5) holds, for E now denoting any entanglement measure satisfying conditions (a)–(d).

The proof goes as follows. As E satisfies (a) and (b), we know by Eq. (6) that, for any $d \in \mathbb{N}$, there exists a state ρ_{ABC} on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where $\mathcal{H}_A \equiv \mathbb{C}^d$ and $\mathcal{H}_B \equiv \mathcal{H}_C \equiv (\mathbb{C}^d)^{\otimes 2^k}$ for some $0 \leq k \leq \lfloor \log d \rfloor$ (see Fig. 2), such that $E_{A:B}(\rho_{AB})$ and $E_{A:C}(\rho_{AC})$ are both lower bounded by $\{1 - [\log(\log^{t+1} d/c)/\log d]\} E_{A:BC}(\rho_{ABC}) = [1 - o(1)] E_{A:BC}(\rho_{ABC})$. Now, by property (c), for any $m \in \mathbb{N}$, considering $\rho_{ABC}^{\otimes m}$ instead of ρ_{ABC} (and relabeling $A^{\otimes m}$ into A etc.) will multiply all values of E by a factor m . By property (d), for any $0 \leq \lambda \leq 1$ and any separable state σ_{ABC} (across the cut $A:BC$) which is locally orthogonal to ρ_{ABC} , considering $\lambda \rho_{ABC} + (1-\lambda)\sigma_{ABC}$ instead of ρ_{ABC} will multiply all values of E by a factor λ . Consequently, any value $x > 0$ for $E_{A:BC}(\rho_{ABC})$ is indeed attainable,

on some suitably large Hilbert space $\mathcal{H}_A^{(x)} \otimes \mathcal{H}_B^{(x)} \otimes \mathcal{H}_C^{(x)}$. \square

Recapitulating, we demonstrated that entanglement measures which faithfully capture the geometric properties of the antisymmetric state cannot be monogamous in general. Conversely, there exist relevant entanglement measures for which the desirable condition (b) does not hold—such as the squashed entanglement, which scales as $o(1/d_A)$ on the antisymmetric state $\alpha_{AA'}$ —yet monogamy holds instead, even in the original CKW form Eq. (1) [28,40]. This is the origin of the “monogamy vs faithfulness” dilemma discussed in the introduction.

Result (3), recovering monogamy: dimension-dependent relations.—In the previous two sections, we proved that several important entanglement measures cannot obey a monogamy relation of the form Eq. (2) with f a universal function. Nevertheless, it may become possible to establish such an inequality if we allow the function f to be dimension dependent. For instance, the squared entanglement of formation obeys the CKW inequality for arbitrary three-qubit states [36], which means that choosing $f(x, y) = \sqrt{x^2 + y^2}$ in Eq. (2), as depicted in Fig. 1 (dotted boundary), makes E_F monogamous when restricted to Hilbert spaces with $d_A = d_B = d_C = 2$.

The third main result of this Letter is to show that nontrivial dimension-dependent monogamy relations can be established for E_F and E_R^∞ in any finite dimension. Concretely, for any state ρ_{ABC} on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, it holds

$$\begin{aligned} E_F(\rho_{A:BC}) &\geq \max \left(E_F(\rho_{A:B}) + \frac{c}{d_A d_C \log^8 d_{A,C}} [E_F(\rho_{A:C})]^8, \right. \\ &\quad \left. E_F(\rho_{A:C}) + \frac{c}{d_A d_B \log^8 d_{A,B}} [E_F(\rho_{A:B})]^8 \right), \end{aligned} \quad (8a)$$

$$\begin{aligned} E_R^\infty(\rho_{A:BC}) &\geq \max \left(E_R^\infty(\rho_{A:B}) + \frac{c'}{d_A d_C \log^4 d_{A,C}} [E_R^\infty(\rho_{A:C})]^4, \right. \\ &\quad \left. E_R^\infty(\rho_{A:C}) + \frac{c'}{d_A d_B \log^4 d_{A,B}} [E_R^\infty(\rho_{A:B})]^4 \right), \end{aligned} \quad (8b)$$

where $c, c' > 0$ are universal constants, and we set $d_{A,B} = \min(d_A, d_B)$, $d_{A,C} = \min(d_A, d_C)$. An instance of Eq. (8b) is qualitatively illustrated in Fig. 1 (dot-dashed boundary).

The proof of Eqs. (8) makes use of results from Refs. [50,61–63], and is provided in Ref. [55]. While Eqs. (8) may not be tight [55], they do establish that the involved entanglement measures can be effectively regarded as monogamous according to Eq. (2) in any finite dimension, even though the constraints become trivial in the limit of infinite dimension, in agreement with results (1) and (2). Notice further that Eqs. (8) encapsulate *strict*

monogamy, as the constraining functions satisfy $f(x, y) > \max(x, y)$ for all positive x and y . This implies that if, say, $E(\rho_{A:B}) = E(\rho_{A:BC})$, then $E(\rho_{A:C}) = 0$ for E being either E_F or E_R^∞ , which means that A and C must be unentangled, as both measures vanish only on separable states.

Conclusions.—We addressed, on general grounds, the question of whether entanglement measures should be monogamous in the sense of obeying a quantitative constraint akin to Eq. (1) introduced in Ref. [25]. We showed that paradigmatic measures such as the entanglement of formation and the relative entropy of entanglement, as well as their regularizations, cannot be monogamous in general, as they cannot satisfy any nontrivial general relation of the form Eq. (2) limiting the distribution of bipartite entanglement in arbitrary tripartite states. Monogamy can nonetheless be recovered if the constraints are made dependent on the (finite) dimension of the system.

The present study substantially advances our understanding of entanglement and the complex laws governing its distribution in systems of multiple parties, and paves the way to more practical developments in quantum communication and computation. The concept of monogamy as studied here is of particular physical relevance, as the structure of Eq. (2) lends itself to be applied repeatedly to establish limitations in a many-body scenario [5], allowing one to compare the distribution of entanglement on equal footing across the various parts of a composite system, unlike, e.g., the case of hybrid monogamy relations involving different quantifiers [28].

It will be worth investigating further links between the phenomenon of monogamy and the so-called quantum marginal problem [64] as well as the fact that information cannot be arbitrarily distributed in multipartite quantum states [65–68]. Implications of our study for progress in other fields like condensed matter [5] and cosmology [69], where monogamy of entanglement takes center stage, also deserve further study.

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