Phase Structure of Driven Quantum Systems

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Clean and interacting periodically driven systems are believed to exhibit a single, trivial "infinite-temperature" Floquet-ergodic phase. In contrast, here we show that their disordered Floquet many-body localized counterparts can exhibit distinct ordered phases delineated by sharp transitions. Some of these are analogs of equilibrium states with broken symmetries and topological order, while others—genuinely new to the Floquet problem—are characterized by order *and* nontrivial periodic dynamics. We illustrate these ideas in driven spin chains with Ising symmetry.

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Introduction.—Extending ideas from equilibrium statistical mechanics to the nonequilibrium setting is a topic of perennial interest. We consider a question in this vein: Is there a sharp notion of a phase in driven, interacting quantum systems? We find an affirmative answer for Floquet systems [1,2] whose Hamiltonians depend on time t periodically, $H(t+T) = H(t)$. Unlike in equilibrium statistical mechanics, disorder turns out to be an essential ingredient for stabilizing different phases; moreover, the periodic time evolution allows for the existence (and diagnosis) of phases without any counterparts in equilibrium statistical mechanics.

Naively, Floquet systems hold little promise of a complex phase structure. In systems with periodic Hamiltonians, not even the basic concept of energy survives, being replaced instead with a quasienergy defined up to arbitrary shifts of $2\pi/T$. Indeed, interacting Floquet systems should absorb energy indefinitely from the driving field, as suggested by standard linear response reasoning wherein any nonzero frequency exhibits dissipation. This results in the system heating up to "infinite temperature," at which point all static and dynamic correlations become trivial and independent of starting state—thus exhibiting a maximally trivial form of ergodicity [3–5].

To get anything else requires a mechanism for energy localization wherein the absorption from the driving field saturates, and the long-time state of the system is sensitive to initial conditions. The current dominant belief is that translationally invariant interacting systems cannot generically exhibit such energy localization [3–5], although there are computations that suggest otherwise [6–8]. The basic intuition is that spatially extended modes in translationally invariant systems interact with and transfer energy between each other.

This can be different when disorder spatially localizes the modes, with individual modes exhibiting something like Rabi oscillations while interacting only weakly with distant modes. While the actual situation is somewhat more involved, several pieces of work [9–11] have made a convincing case for the existence of Floquet energy localization exhibiting a set of properties closely related to those exhibited by time-independent many-body localized [12] (MBL) systems [13].

In the following, we show that such Floquet-MBL systems can exhibit multiple phases. Some of these are driven cousins of MBL phases characterized by broken symmetries and topological order. Remarkably, others are genuinely new to the Floquet setting, characterized by order *and* nontrivial periodic dynamics. Our analysis identifies a key feature of the Floquet problem, the existence of Floquet eigenstates, which permits us to extend the notion of eigenstate order [14–18] to time-dependent Hamiltonians. Our work also builds on the discovery of topologically nontrivial Floquet single particle systems and recent advances in their classification [19–27]. As we will explain, nontrivial single-particle drives can yet lead to trivial many-body (MB) periodic dynamics even without interactions. Thus, the full framework of disorder and interactions is required for the MB problem.

In the following sections, we briefly review the Floquet formalism and describe the notion of eigenstate order before generalizing it to the Floquet setting. We then illustrate our ideas for driven Ising spin chains. We first show that there are two Floquet phases, paramagnet (PM) and spin glass (SG), that connect smoothly to phases in the undriven systems. We then identify two *new* phases that do *not*, which we term the Floquet 0π-PM and the Floquet π-SG. Along the way we note that, reformulated as fermion problems, these yield instances of Floquet topological order.

Floquet formalism.—We consider periodic, interacting Hamiltonians which are local in space. The time-dependent Schrödinger equation

$$i\frac{d|\psi(t)\rangle}{dt} = H(t)|\psi(t)\rangle \tag{1}$$

has special solutions [1,2]:

$$|\psi_\alpha(t)\rangle = e^{-i\epsilon_\alpha t}|\phi_\alpha(t)\rangle \tag{2}$$

defined by periodic states $|\phi_\alpha(t)\rangle = |\phi_\alpha(t+T)\rangle$ and quasienergies ϵ_α defined modulo $2\pi/T$.

These replace the eigenstates of the time-independent problem; in them observables have periodic expectation values, and they form a complete basis. The "Floquet Hamiltonian" H_F is defined via the time evolution operator over a full period, $U(T) = e^{-iH_F T}$. The $|\phi_\alpha(0)\rangle$ are eigenstates of both $U(T)$ and H_F, with eigenvalues $e^{-i\epsilon_\alpha T}$ and ϵ_α, respectively.

The question of energy localization relates to the action of $U(nT) = [U(T)]^n$ as $n \to \infty$. An equivalent formulation of the Floquet-MBL regime is that there exists an H_F which is local and exhibits the generic properties of any fully MBL Hamiltonian, such as a full set of conserved local operators [28–32] (l bits), area law eigenstates, and failure of the eigenstate thermalization hypothesis (ETH) [33].

Eigenstate order.—Traditionally, phase transitions at nonzero energy densities are considered in the framework of quantum statistical mechanics, signaled by singularities in thermodynamic functions or observables computed in the $T > 0$ Gibbs state. Work on MBL has led to the realization that this viewpoint is too restrictive [14–18]—instead the MB eigenstates and eigenvalues can directly exhibit singular changes as a parameter is varied. Such transitions have been termed eigenstate phase transitions. This distinction is irrelevant for Hamiltonians obeying ETH, but when ETH fails it becomes important. The passage from ergodicity to localization is an example of an eigenstate transition undetectable by standard ensembles [33–35]—it can take place at $T = \infty$ and yet the individual MB eigenstates are sharply different in their entanglement properties, with the eigenvalue distributions exhibiting different statistics. Moreover, eigenstate phase transitions can take place between two ETH violating phases [14] and they may even involve a singular rearrangement of the eigenvalues alone.

We now generalize this to the Floquet-MBL regime, leading us to find multiple ordered phases whose existence can be detected in the Floquet-MB eigensystem. In general, this will require examination of the full periodic solutions $|\psi_\alpha(t)\rangle$ for sharply different characteristics. We will demonstrate our results in the simple interacting setting of a one dimensional disordered spin chain with Ising symmetry,

$$H = \sum_i J_i \sigma_i^x \sigma_{i+1}^x + \sum_i h_i \sigma_i^z + J_z \sum_i \sigma_i^z \sigma_{i+1}^z. \tag{3}$$

Carrying out a Jordan-Wigner transformation on only the first two terms gives a p-wave superconducting free-fermion model, whereas the final term is a density-density interaction in the fermion language. The paramagnetic and symmetry-broken ferromagnetic phases of the Ising model are related by a well-known duality.

Floquet paramagnet and spin glass.—We begin with the two phases that *do* exist in undriven systems and demonstrate the stability of these to being (not too strongly) driven. Starting with the noninteracting limit $J_z = 0$, we choose the J_i and the h_i to be log-normally distributed with a tunable mean $\overline{\log(J_i)} \equiv \overline{\log} J$, fixed $\overline{\log(h_i)} \equiv \overline{\log}\, h = 0$, and two fixed and equal standard deviations $\delta \log(h_i) = \delta\log(J_i) = 1$. Work on random, noninteracting Ising models culminating in Ref. [36] finds a ground state phase diagram which is a paramagnet for $\overline{\log} J < \overline{\log}\, h$ and a Z_2 breaking ferromagnet for $\overline{\log} J > \overline{\log}\, h$, separated by an infinite disorder fixed point at $\overline{\log} J = \overline{\log}\, h$. The work on eigenstate order has shown that, with disorder and localization, both phases exist at *all* energies with the symmetry-breaking phase exhibiting SG order in individual eigenstates instead of ferromagnetism. The eigenstates are also eigenstates of parity $P = \prod \sigma_i^z$, and deep in the PM phase, they (roughly) look like frozen spins along the z direction $|\uparrow\downarrow\downarrow\cdots\uparrow\rangle$ while deep in the SG phase they look like global superposition/cat states with spins in the x direction with frozen domain walls $|\pm\rangle = (1/\sqrt{2})(|\rightarrow\leftarrow\rightarrow\cdots\rightarrow\rangle \pm |\leftarrow\rightarrow\leftarrow\cdots\leftarrow\rangle)$.

With weak interactions, $0 < J_z \ll 1$, the strongly localized PM and SG phases remain MB localized [37,38]. The fate of the SG-PM transition is more sensitive to the inclusion of interactions. It was suggested that it would remain localized [14] and exhibit the same scaling as the noninteracting fixed point [37]; we comment on the analogous question in the Floquet setting below.

We consider a periodic binary drive—computationally much simpler than a monochromatic modulation—switching between two static H s with $\overline{\log} J$ differing by 1:

$$H(t) = \sum_i f_s(t) J_i \sigma_i^x \sigma_{i+1}^x + \sum_i h_i \sigma_i^z + J_z \sum_i \sigma_i^z \sigma_{i+1}^z,$$

$$f_s(t) = \begin{cases} 1 & \text{if } 0 \le t < \frac{T}{4} \quad \text{or} \quad \frac{3T}{4} < t \le T \\ e & \text{if } \frac{T}{4} \le t \le \frac{3T}{4}. \end{cases} \tag{4}$$

We set $J_z = 0.1$ in the following. For $-1 \le \overline{\log} J \le 0$ the drive straddles the undriven phase transition, up to small corrections to its location due to the interaction.

Drives consistent with Floquet localization require both small interactions and not too small frequencies. We arrange the latter by defining, for each set of $(\overline{\log} J, \overline{\log}\, h)$ parameters, an effective "single-particle bandwidth," $W = \max(\sigma_J, \sigma_h)$, where σ_h and σ_J are the standard deviations of h_i and J_i determined from the underlying log-normal distributions. The period is then defined by $\omega = 2\pi/T = 2W$. This choice ensures a roughly constant ratio of ω/W for different $\overline{\log} J - \overline{\log}\, h$ values and thus isolates the effect of tuning the means through the phase diagram.

The lowest frequency in our drives is bigger than the estimated single-particle bandwidth but much smaller than the MB bandwidth, so that localization is not a foregone conclusion. In Fig. 1 we characterize the quasienergy spectrum $\epsilon_n \in [0, 2\pi)$ using the level statistics of H_F. We define quasienergy gaps by $\delta_n = \epsilon_{n+1} - \epsilon_n$ and the

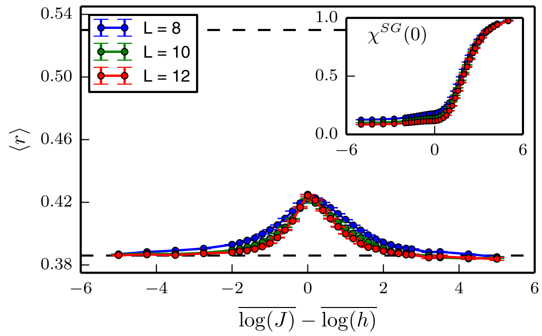

FIG. 1. Disorder averaged level statistics $\langle r \rangle$ of H_F for the driven, disordered Ising model (4). $\langle r \rangle$ approaches the Poisson limit of 0.386 with increasing L deep in the PM and SG phases, showing that these remain well localized. There is a peak in $\langle r \rangle$ near the noninteracting critical point at $\overline{\log} J = \overline{\log} h$ indicating partial delocalization, although the value still remains well below the COE value of 0.527. (inset): The SG diagnostic χ^{SG} defined in (6) goes to 0 in the PM and approaches a nonzero value in the SG phase. All data is averaged over $2000 - 10^5$ samples depending on L.

level-statistics ratio $r = \min(\delta_n, \delta_{n+1}) / \max(\delta_n, \delta_{n+1})$. Away from the critical region which—given the weak interactions and large frequency of the drive—is close to the undriven, noninteracting transition point $\overline{\log} J = \overline{\log} h$, the disorder averaged $\langle r \rangle$ approaches the Poisson limit of 0.386 with increasing system size L, signaling a lack of level repulsion and hence MBL. In the interacting critical region we find a peak in $\langle r \rangle$ which does not grow with system size and is much less than the delocalized Circular Orthogonal Ensemble (COE) value of 0.527; we return to this below.

With localization established, we turn to distinguishing the phases. Consider a pair of Z_2 invariant correlators (with $A = x$ or y)

$$C^{\alpha}_{AA}(ij;t) = \langle \phi_\alpha(t) | \sigma^A_i \sigma^A_j | \phi_\alpha(t) \rangle \tag{5}$$

for $i - j \gg 1$ in any given Floquet eigenstate. We find that for $\overline{\log} J < \overline{\log} h$ both correlators vanish with increasing system size L at all t, signaling a PM. For $\overline{\log} J > \overline{\log} h$, both are generically nonzero, though of random sign varying with eigenstate and location, signaling SG order. For the SG, our parameters give $|C^{\alpha}_{xx}(ij;t)| \gg |C^{\alpha}_{yy}(ij;t)|$ at all t although the more general signature is that $|C^{\alpha}_{xx}(ij;t)|$ and $|C^{\alpha}_{yy}(ij;t)|$ do not cross for $0 < t < T$. For our parameters, it suffices to compute

$$\chi^{\mathrm{SG}}_\alpha(t) = \frac{1}{L^2} \sum_{i,j=1}^{L} [\langle \phi_\alpha(t) | \sigma^x_i \sigma^x_j | \phi_\alpha(t) \rangle]^2 \tag{6}$$

for $t = 0$. In Fig. 1 (inset) we plot the disorder averaged $\chi^{\mathrm{SG}}_\alpha(0)$; the trend with system size indicates that $\chi^{\mathrm{SG}}(t) > 0$ in the spin glass and $\chi^{\mathrm{SG}}(t) \to 0$ in the paramagnet.

Three comments are in order. First, recall that it would be sufficient to establish the existence of the Floquet PM—the SG can be obtained by duality [39]. Second, in chains with uniform couplings, both spin and dual spin order vanish in all but one of the Floquet eigenstates (the notion of the "ground state" is not well defined in a Floquet system) even without interactions—this is the Landau-Peierls prohibition against discrete symmetry breaking in disguise. Localization is essential to avoid this. Third, the $|\pm\rangle$ MB Floquet eigenstates in the localized SG phase come in conjugate, almost degenerate pairs with different parity but with similar domain wall configurations. In the fermionic formulation of the problem, the PM is topologically trivial while the SG is nontrivial. The noninteracting SG phase has zero energy edge Majorana modes in open chains, and the twofold degeneracy of the many-body SG spectrum (in this language) stems from the occupation or unoccupation of the bilocal Dirac mode formed from the edge Majorana modes. With interactions, the edge mode remains coherent only in the MBL setting [17,18]. Thus, the degenerate Floquet eigenstates can be connected by either (i) spectrum-generating operators localized near the edges which toggle the state of the coherent edge mode (fermionic language) or (ii) any spin operator that flips the parity of the eigenstates. Concretely, the spectral function of σ^+_i, the spin raising operator on any site i, in the Floquet eigenbasis

$$\mathcal{A}(\omega) = \frac{1}{2^L} \sum_{\alpha\beta} \langle \phi_\alpha(0) | \sigma^+_i | \phi_\beta(0) \rangle \delta(\omega - (\epsilon_\alpha - \epsilon_\beta)) \tag{7}$$

is a delta function peaked at $\omega = 0$ (this phase will hence also be labeled the "0" phase below). Finally, we note that the SG displays long-range string order in all eigenstates regardless of boundary conditions. Without disorder, the string order vanishes even in the many-body eigenstates of free-fermion chains—despite the nontrivial momentum space topology present in their Hamiltonians.

Paramagnet-spin glass phase transition.—In the noninteracting problem we have strong evidence that the infinite disorder fixed point continues to control the physics. We have examined H_F and we find that all its eigenstates are localized even at the transition, and its structure differs from the canonical strong-disorder renormalization group form [36] by short ranged, irrelevant, terms. The ultimate fate of the critical region in the interacting driven problem is an interesting open question, but we note for now that our data on $\langle r \rangle$ suggest a partially delocalized interacting critical point.

π spin glass and 0π paramagnet.—We now present two new Ising phases which exist only in the driven system—the π-SG phase and the 0π-PM phase. Existing work on the band topology of translationally invariant Z_2 symmetric free-fermion chains [21,40,41] has shown that the Floquet eigenmodes for such chains with *open* boundary conditions can exhibit edge Majorana modes with $\epsilon_\alpha = \pi/T$ in addition to the better known edge modes with $\epsilon_\alpha = 0$. In the MBL setting in the "π" phase, the MB Floquet eigenstates are long-range ordered and come in $|\pm\rangle$ cat pairs

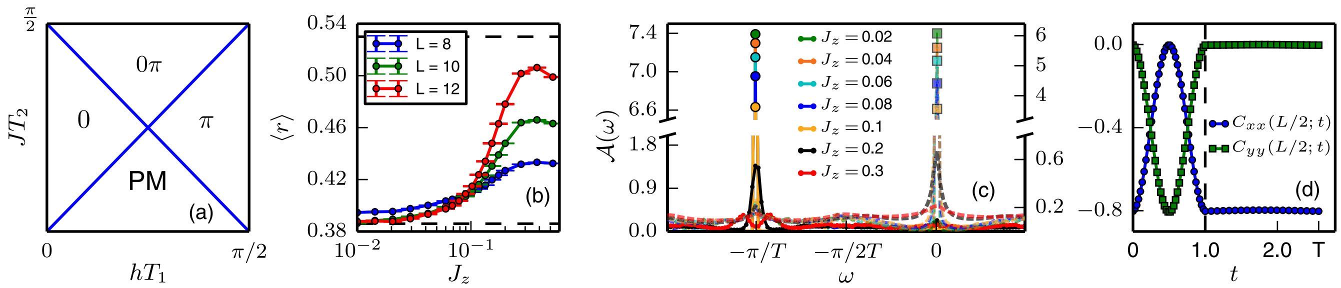

FIG. 2. (a) Phase diagram for the binary Ising drive, Eq. (8) without interactions ($J_z = 0$) and disorder. (b) Level statistics $\langle r \rangle$ of H_F in the π phase with parameters defined in the text and disorder averaged over 2000–100 000 realizations for different Ls. H_F is localized for interaction strengths $J_z \lesssim 0.1$. (c) Disorder averaged spectral function $\mathcal{A}(\omega)$ defined in Eq. (7). Solid (dashed) lines are for the π-SG (0-SG) phase showing a delta function peak at $\omega = \pi/T(0)$ for small interaction strengths which disappears as the interaction is increased. (d) Time dependence of the C_{xx} and C_{yy} correlators defined in Eq. (5) over one period for an eigenstate in the π phase; $L = 10$, $T_1 = 1$, and $J_z = 0.04$. The crossings are robust in the π phase.

separated by quasienergy π/T. These can again be connected by either spectrum-generating operators localized near the two edges (fermion language) or by local parity odd operators (spin language). Thus, the spectral function $\mathcal{A}(\omega)$ (7) now shows a delta function peak at $\omega = \pi/T$.

We now establish these phases for the binary periodic drive

$$H(t) = \begin{cases} H_z & \text{if } 0 \le t < T_1 \\ H_x & \text{if } T_1 \le t < T = T_1 + T_2 \end{cases}, \qquad (8)$$

$$H_z = \sum_{i=1}^{L} h_i \sigma_i^z + \sum_{i=1}^{L-1} J_z \sigma_i^z \sigma_{i+1}^z,$$

$$H_x = \sum_{i=1}^{L-1} J_i \sigma_i^x \sigma_{i+1}^x + J_z \sigma_i^z \sigma_{i+1}^z.$$

Figure 2(a) shows the uniform, noninteracting phase diagram with the four possible driven Ising phases. The phases labeled "0" and "π" have edge Majorana modes at quasienergies 0 and π/T, respectively. With disorder and localization, these phases display long-range SG eigenstate order in the correlators (5) for both $A = x, y$. Moreover, in the π-SG phase, the time dependence of the C_{xx} and C_{yy} correlators over the period is nontrivially correlated: their magnitudes must cross twice during a period. Thus, in this phase, the axis of SG order rotates by an angle π about the z axis during the period which can be intuitively understood by thinking semiclassically about the drive (8) at the extremal boundaries of the phase diagram shown in Fig. 2(a). This sign reversal of the order parameter and thus doubling of the period (also found previously in [8]), provides a potential Floquet realization of a time crystal [42,43]. As before, without localization, only one of the Floquet eigenstates (analogous to the ground state) will display long-range order in the $0, \pi$ phases. The other two phases, labeled PM and 0π, have no long-range order and are, respectively, dual to the 0 and π phases.

We now turn to numerically identifying the localized π phase with disorder and interactions. We pick $T = 1$, $T_2 = \pi/2$, and $h_i T_1$ uniformly from the interval (1.512,1.551) and $J_i T_2$ from (0.393,1.492), so that all pairs of values $(h_i T_1, J_j T_2)$ lie in the π/T Majorana region of the free uniform chains. We have confirmed that the free-fermion disordered drive exhibits π/T Majorana modes for open chains while all other modes are localized in the bulk. In Fig. 2(b) we examine stability to interactions via $\langle r \rangle$ and clearly observe a transition around $J_z \approx 0.1$, with the small J_z regime being the MBL phase we seek.

Figure 2(c) shows the appearance of the π/T peak in the disorder averaged spectral function (7) for system size $L = 10$ as the delocalization transition is crossed by decreasing the interaction strength. In contrast, the dashed lines show the spectral function for a similar drive in the 0-SG phase, clearly showing a peak at $\omega = 0$. Finally, Fig. 2(d) displays the anticipated time dependence of the SG order in the C_{xx} and C_{yy} correlators in a single eigenstate of the interacting system with $J_z = 0.04$ and $L = 10$. The crossings in the correlator within the period are robust in the π-SG phase and topologically distinct from the correlators in the 0-SG phase where there are no crossings. We emphasize that this is a true bulk diagnostic of the phase which, unlike the presence of an edge mode, is insensitive to boundary conditions. We also note that the nontrivial spin dynamics captured by it *cannot* be obtained without localization.

Finally, we turn to the 0π-PM which is dual to the π-SG. This is a symmetry protected topological phase with no bulk long-range order, but with coherent edge states. In the fermionic language, there are now two Majorana modes at each edge, one at quasienergy 0 and the other at π/T and thus the MB spectrum is paired into conjugate sets of four MB states—two degenerate pairs of states separated by quasienergy π/T. The eigenstates in this phase do *not* look like global superposition states and the spectral function of bulk spin operators shows no structure. On the other hand, spectral functions of edge operators which toggle the state of the edge modes show a peak at both 0 and π/T.

Summary and open questions.—We have shown that MBL Floquet systems exhibit sharply defined phases bounded by parameter surfaces across which properties of their Floquet eigensystems change in a singular fashion. These phases include the trivial Floquet-ergodic phase and multiple nontrivial nonergodic phases exhibiting various forms of ordering and dynamics, some of which are entirely new to Floquet systems. The net result is something quite striking given the contentious history of nonequilibrium statistical mechanics. Indeed, it is quite likely that Floquet systems constitute the maximal class for which such a definition of phase structure is possible; with generic time dependences it would not be surprising if heating to infinite temperatures is the inevitable result. Going forward, we anticipate a more systematic search for Floquet phases and a better understanding of their phase diagrams. In this context we note two studies that include disorder, but not interactions, in Floquet systems [44,45]. It should also be possible to observe the new localized phases by the methods of Ref. [46].

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