Hidden String Order in a Hole Superconductor with Extended Correlated Hopping

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Ultracold fermions in one-dimensional, spin-dependent nonoverlapping optical lattices are described by a nonstandard Hubbard model with next-nearest-neighbor correlated hopping. In the limit of a kinetically constraining value of the correlated hopping equal to the normal hopping, we map the invariant subspaces of the Hamiltonian exactly to free spinless fermion chains of varying lengths. As a result, the system exactly manifests spin-charge separation and we obtain the system properties for arbitrary filling: ground state collective order characterized by a spin gap, which can be ascribed to an unconventional critical hole superconductor associated with finite long range nonlocal string order. We study the system numerically away from the integrable point and show the persistence of both long range string order and spin gap for appropriate parameters as well as a transition to a ferromagnetic state.

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Introduction.—Understanding the collective behavior of many particle systems is the essence of contemporary many body physics. One-dimensional (1D) systems hold a special place in this effort [1], where enhanced quantum fluctuations are particularly important. These systems are amenable to exact, analytical, and precise numerical methods revealing exotic physics, e.g., Luttinger liquids and associated spin-charge separation [2–5], fractional excitations [6], or nonlocal symmetry breaking [7,8].

A particularly interesting lattice model of fermions is $H_H = \sum_{<i,j>,\sigma=\uparrow,\downarrow} c^\dagger_{i\sigma} c_{j\sigma}[-J + X(n_{i\bar\sigma} + n_{j\bar\sigma})] + U\sum_i n_{i\uparrow} n_{i\downarrow}$ ($c_{i\sigma}$ annihilates a spin-σ fermion in the Wannier orbital centered on site i, $n_{i\sigma}$ is the fermion number, and $\bar\sigma$ is the spin opposite to σ). Diagonal and off-diagonal particle interactions lead to the repulsion U and nearest neighbor correlated hopping (CH) X, respectively [9]. $-J$ is the hopping integral. Such CH has been intensively studied, particularly in $d > 1$ dimensions, as a mechanism for unconventional (nonphonon mediated) superconductivity [10,11], especially at weak to moderate values of the CH integral X. Large $X \sim J$ inhibits certain tunnelling processes similarly to large repulsion U [12,13]. For $X = J$, the on-site interaction term is conserved. This facilitates an exact solution in 1D [14–17], revealing nonperturbative effects such as a paramagnetic Mott transition and exclusion statistics [18]. An incommensurate singlet superconducting phase appears in proximity to $X = J$ [19,20]. While in physical systems typically $X \ll J$ with $\mathrm{sgn}(X) \neq \mathrm{sgn}(J)$ [21–23], fast, strong, periodic modulation of the Feshbach-resonance controlled interaction U [24] has been proposed as a path to the extreme regime $X \sim J$ in optical lattice systems.

Recently, exquisite control over single particle dynamics has been demonstrated in shifted, spin-dependent lattices [25–27]. This paves the way to study many body effects in these systems. While hitherto unnoticed (see Supplemental Material [28]), maximally separating the lattices available to the two spin species yields a strong *next-nearest neighbor* CH. We predict numerous exotic properties of this system in particular via an exact solution as well as by numerical means: nonvanishing string order, a gapless lowest excitation spectrum, a spin gap associated with collective hole pairing, and transition to a simple ferromagnetic state.

The model.—A spin dependent optical lattice setup with a red and blue detuned sublattice offset by half a lattice constant each trapping a single neutral fermion spin (or Tonks-Girardeau boson [35–38]) species $\uparrow$ and $\downarrow$, respectively, is described by the Hamiltonian

$$H = \sum_i (-J + X n_{i+1}) c^\dagger_i c_{i+2} + \mathrm{H.c.} + \sum_i V n_i n_{i+1}, \quad (1)$$

where odd (even) sites correspond to the blue (red) sublattice for $\downarrow(\uparrow)$ spins. We consider equal length L sublattices. The two sublattices together define a zigzag lattice (ZL) in Eq. (1). The hopping $J > 0$ connects next nearest neighbors, as does the CH (X) which is conditioned on the occupation of the intervening site, while V is the nearest neighbor repulsion. The derivation of these bare parameters, as well as a protocol facilitating their large

tunability is presented in the Supplemental Material [28]. We note here that unlike in Ref. [24], the bare CH can be strong $X \sim J$ in the system Eq. (1) without external driving.

Exact solution at $X = J$, $V = 0$ for open boundary conditions (OBCs).—For $X = J$, the free hopping and CH cancel for the interchange of a nearest neighbor pair of $\uparrow$ and $\downarrow$ spins [$-J + X = 0$ in Eq. (1)]. For a system of N_t particles, define a spin sequence $\Sigma_i = (\sigma_1, \sigma_2, \ldots, \sigma_{N_t})_i$ as the order of the spins of the particles ($\sigma_l = \uparrow$ or $\downarrow$ is the lth particle's spin) from left to right on the ZL. The impossibility of switching $\uparrow$ and $\downarrow$ fermions and OBC, i.e., hard wall boundaries, implies conservation of each sequence Σ_i. Hence, (i) for any N_t, the Hamiltonian H can be diagonalized in each invariant Σ_i sector, i.e., $H = \oplus_{\Sigma_i} H_{\Sigma_i}$, and (ii) for $V = 0$, each H_{Σ_i}, i.e., H restricted to configurations on the ZL with fixed Σ_i, acts as constrained hopping: a particle can hop with amplitude $-J$ onto available empty sites on its sublattice until it reaches the next particle on the ZL.

To illustrate the action of H_{Σ_i}, note that there are only two types of generic segments, $O1$ (where $\sigma_l = \sigma_{l+1}$) and $O2$ (where $\sigma_l \neq \sigma_{l+1}$), between two consecutive particles l and $l+1$ in configurations on the ZL. In the example of $O1$ below, the number of holes between the particles is initially $d = 5$. The hopping H_{Σ_i} allows $h = (d-1)/2 = 2$ hops of particle 1 towards particle 2 before being blocked by the latter:

$$O1.\quad |\bullet\circ\circ\circ\circ\bullet\rangle \to |\circ\circ\bullet\circ\circ\bullet\rangle \to |\circ\circ\circ\bullet\circ\bullet\rangle$$

In the example of $O2$ below, $d = 4$ holes initially separate the particles. The hopping H_{Σ_i} allows $h = d/2 = 2$ hops of particle 1 towards particle 2 after which it is blocked.

$$O2.\quad |\bullet\circ\circ\circ\bullet\rangle \to |\circ\circ\bullet\circ\bullet\rangle \to |\circ\circ\circ\bullet\bullet\rangle$$

Hence effectively the spacing (number of holes) between the two particles is reduced under the action of H_{Σ_i}. More generally, note that the ZL structure dictates that each allowed single hop of a particle towards or away from another particle (or edge) changes the separation between them by 2 on the ZL. So, the number of holes d is always odd for $O1$ type segments, and always even for $O2$ type segments on the ZL. Generalizing the example above, the number of holes d_l on the ZL between particles l, $l+1$ maps to the corresponding effective number of holes h_l available for hopping as

$$d_l \mapsto h_l = (d_l/2)\delta_{\sigma_l,\bar{\sigma}_{l+1}} + \delta_{\sigma_l,\sigma_{l+1}}(d_l - 1)/2, \qquad (2)$$

where $\delta_{a,b}$ is the Kronecker Delta and $l = 1, 2, \ldots, N_t - 1$. The segments of particle 1 with $\sigma_1 = \uparrow(\downarrow)$ from the left edge of the ZL and particle N_t with $\sigma_{N_t} = \downarrow(\uparrow)$ from the right edge, have odd (even) number of holes [see Figs. 1(a) and 1(b)] and so are of type $O1$ ($O2$). These edge cases are consistently treated in Eq. (2) as $d_0 \mapsto h_0$ and $d_{N_t} \mapsto h_{N_t}$ using two *fixed* auxiliary boundary variables $\sigma_0 = \uparrow$ and $\sigma_{N_t+1} = \downarrow$.

By replacing the set of holes $\mathbf{d} = \{d_l\}$ in ZL configurations with the corresponding effective number of holes $\mathbf{h} = \{h_l\}$, one obtains configurations defined on a fictitious lattice—the charge lattice (CL). From Eq. (2), the total number of holes in the CL is $(D_0 - M_0)/2$ where D_0 is the number of holes on the ZL and $M_0(\Sigma_i) = \delta_{\uparrow\sigma_1} + \sum_{l=1}^{N_t-1} \delta_{\sigma_l,\sigma_{l+1}} + \delta_{\sigma_{N_t},\downarrow}$ is the number of segments of type $O1$ determined by the sequence Σ_i. The *total* number of holes and particles yields

$$L_{\text{eff}}(\Sigma_i) = N_t + (D_0 - M_0)/2 = L + (N_t - M_0)/2 \qquad (3)$$

where $D_0 = 2L - N_t$, for N_t particles on $2L$ sites of the ZL. L_{eff} is hence a conserved property of Σ_i and defines the length of the CL for this sequence.

The above map can easily be uniquely reversed for any configuration on the CL using Σ_i:

$O1$. If $\sigma_l = \sigma_{l+1}$, then $d_l = 2h_l + 1$. h_l holes on the σ_l sublattice follow the σ_l particle and $h_l + 1$ holes are placed on the other sublattice between these particles. For $\sigma_1 = \uparrow$, $d_0 = 2h_0 + 1$ with h_0 holes preceding it on the $\uparrow$ sublattice and the first $h_0 + 1$ sites occupied by holes on the $\downarrow$ sublattice. The case $\sigma_{N_t} = \downarrow$ is analogous.

$O2$. If $\sigma_l \neq \sigma_{l+1}$, $d_l = 2h_l$ where h_l holes follow the σ_l particle on the σ_l sublattice and h_l holes precede the σ_{l+1} particle on the σ_{l+1} sublattice. For $\sigma_1 = \downarrow$, $d_0 = 2h_0$ with h_0 holes preceding it on the $\downarrow$ sublattice and h_0 holes occupy the beginning of the $\uparrow$ lattice. Analogously, one obtains the map for $\sigma_{N_t} = \uparrow$.

So, the CL particle positions $(p_1, p_2, \ldots, p_{N_t})$ transform to ZL positions $(q_1, q_2, \ldots, q_{N_t})$ as

$$q_{l+1} = q_l + A_l^{l+1}\delta_{\sigma_l\sigma_{l+1}} + B_l^{l+1}\delta_{\sigma_l\bar{\sigma}_{l+1}} + 1 \qquad (4)$$

where the number of holes d_l between particles l and $l+1$ are $A_l^{l+1} = 2(p_{l+1} - p_l) - 1$ or $B_l^{l+1} = 2(p_{l+1} - p_l) - 2$ and $q_1 = (2p_1 - 1)\delta_{\sigma_1,\downarrow} + 2p_1\delta_{\sigma_1,\uparrow}$.

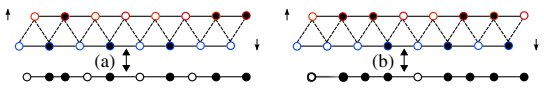

FIG. 1. Examples of the configurational mapping from a $2L = 16$-site zigzag lattice (ZL) to the charge lattice (CL) for OBC. $\downarrow$ ($\uparrow$) particles are filled circles on the lower (upper) rail of the ZL. Arrows beyond the edges of the ZL are the auxiliary fixed boundary variables needed for the classification of boundary segments in the mapping. (a) $N_t = 6$ particles in the sequence $\Sigma = (\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow)$. There are $\mathbf{d} = (2, 0, 2, 3, 2, 1, 0)$ holes interjected by particles from the left to right edge on the ZL. So $M_0 = 2$ and the length $L_{\text{eff}} = 10$. From Eq. (2), $\mathbf{h} = (1, 0, 1, 1, 1, 0, 0)$ holes on the CL. (b) $N_t = 7$ and $\Sigma = (\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow)$ with $\mathbf{d} = (3, 1, 0, 3, 0, 1, 0, 1)$ holes. So $L_{\text{eff}} = 9$ ($M_0 = 5$). Using Eq. (2), $\mathbf{h} = (1, 0, 0, 1, 0, 0, 0, 0)$.

Note that the particles in configurations on the CL bear no reference to the spin flavor—the spin information is solely in the sequence Σ_i. The arguments in $O1$ and $O2$, and the 1-to-1 nature of the mapping implies that the action of each H_{Σ_i} on the ZL is identical to a free nearest neighbor hopping Hamiltonian (with tunnelling J) on a CL of length $L_{\rm eff}(\Sigma_i)$. Hence our mapping manifests exact spin-charge separation. The many body eigenstates (eigenenergies) correspond to all distributions of N_t fermions over the $L_{\rm eff}$ single particle eigenstates (energy levels) of the free hopping model. The mapping Eq. (4) transforms these into the original ZL coordinates. Mapping each H_{Σ_i} to its CL thus exactly solves $H = \oplus_{\Sigma_i} H_{\Sigma_i}$.

Although the quasiparticles are noninteracting spinless fermions in each sequence, the interplay of the spectra of various spin sequences determine the system's properties. We characterize the ground state and excitations below.

(i) For a given sequence Σ_i, the lowest energy is obtained by filling the Fermi sea of the CL model with N_t particles on $L_{\rm eff}$ sites, yielding the thermodynamic limit value $E_0/L_{\rm eff} = -(2J/\pi)\sin(\pi N_t/L_{\rm eff})$. The lowest energy per ZL site is $e_0 = E_0/L = -2J\sin[2\pi\rho\mathcal{L}]/(\pi\mathcal{L})$, where $\rho = N_t/(2L)$ is the original fermion density and $\mathcal{L} = L/L_{\rm eff}$.

(ii) While all considerations can be carried out for arbitrary fillings, we restrict to even $N_t = 2N$ from now on. The *global* ground state energy of H is obtained by minimizing the ground state energies of all possible sequences Σ_i. The global ground state is that of the sequence $\Sigma_0 = |\downarrow\uparrow\downarrow\uparrow\ldots\downarrow\uparrow\rangle$ with perfect antiferromagnetic spin order, since one obtains the lowest possible CL particle density for maximal $L_{\rm eff} = L + N$ ($M_0 = 0$). Lack of reflection symmetry about a horizontal line through the middle of the ZL, makes the order not degenerate with respect to flipping all spins. The Z_2 transformed sequence $|\uparrow\downarrow\uparrow\downarrow\uparrow\ldots\downarrow\rangle$ yields a shorter $L_{\rm eff} = L + N - 1$ ($M_0 = 2$) with higher energy due to higher CL density.

(iii) Antiferromagnetic order in the spin sector implies ideal hidden nonlocal string order in the real lattice [39]: each fermion is *always* followed by a different spin fermion separated arbitrarily. This is analogous to string order in spin-1 Heisenberg chains [7]. We stress that there is no local symmetry breaking of the density wave type accompanying this string order—the ground state is translationally invariant (modulo OBC) since it corresponds to a free nearest neighbor hopping chain.

(iv) The manifest spin-charge separation implies two types of excitations w.r.t the global ground state (ii). The first are charge excitations above the Fermi sea in the spinless fermion model for fixed sequence. The second are spin excitations, obtained by flipping spins. These excitations correspond to domain walls in the spin sequence in analogy to well-known excitations of the 1D Heisenberg antiferromagnetic model. A domain wall is present between two like consecutive spins in a sequence. Similarly there is a domain wall before (or after) the first (last) spin if it is $\sigma_1 = \uparrow$ ($\sigma_{N_t} = \downarrow$). This equips the quantum number M_0 in Eq. (3) with a physical meaning as the number of domain walls in an excited spin sequence.

(v) While the charge spectrum is gapless for each sequence Σ_i, there is a spin gap—the energy cost associated with flipping one spin with respect to the ground state sequence Σ_0. A single spin flip shortens the CL to $L_{\rm eff} = L + N - 1$ ($M_0 = 2$). This change shifts all single particle energies leading, in the thermodynamic limit, to a nonzero spin gap $\Delta_S = 2J(\sin r - r\cos r)/\pi$, (see Supplemental Material [28]) for arbitrary filling densities where $r = 2\pi\rho/(\rho+1)$. Because of the same length change, exchanging a pair of nearest fermions in Σ_0 entails the same energy cost.

(vi) A revealing aspect of the sequence Σ_0 containing the *global* ground state is that in any configuration *all* holes are unambiguously bound into nearest neighbor pairs [see examples in Figs. 2(a) and 2(b)]. Because of this, single particle motion is equivalent to the coherent motion of paired holes in the opposite direction. Furthermore, the finite spin gap Δ_S (v) is in the paired hole picture the energy of breaking a hole-pair [see Fig. 2(c)]. Such a spin gap for *all* densities is a feature of superconductors, so this suggests that the ground state of H is an exotic hole superconductor.

To support this view, we obtain the superconducting correlations (SCs) of the bound hole pairs. While these can be directly calculated via our exact solution [38], we present a more insightful argument for the SC asymptotics. The bound hole pairs occupy edges of the ZL [see Figs. 2(a) and 2(b)]. Nearest neighbor edges of the ZL define its dual lattice (see Fig. 2). Now, any allowed hopping of a single particle in the Hamiltonian H_{Σ_0} on the ZL is effectively the hopping of a hole-pair on the dual lattice. There is an exclusion constraint forbidding the occupation of two bound hole pairs on nearest neighbor sites of the dual lattice as evident from Figs. 2(a) and 2(b). Hence $H_{\Sigma_0} = \sum_i J(\sigma_i^\dagger\sigma_{i+1} + \text{H.c.}) + J_z/4(\sigma_i^z + 1)(\sigma_{i+1}^z + 1)$ takes the form of a spin-1/2 XXZ model at fixed magnetization m on the dual lattice. The spin $\sigma_i^z = 1(-1)$ defines the occupation (lack) of a paired hole, and the repulsion $J_z = \infty$. The magnetization $m = -\rho/2$ (see Supplemental

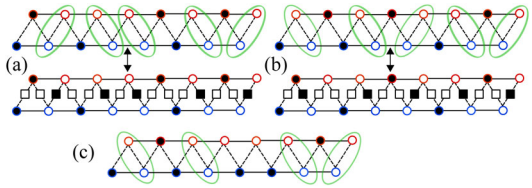

FIG. 2. (a),(b) In the sequence Σ_0 hosting the global ground state, nearest neighbor holes are uniquely paired in all configurations (above). Each single particle hop is equivalent to a hole-pair hop—e.g., two single hops lead from (a) to (b). (Below) The bound hole-pairs (green) are fictitious particles (filled squares) that hop on the edges (squares) of the ZL. Unique pairing of holes (above) implies simultaneously occupied nearest neighbor sites on the dual lattice are forbidden, i.e., infinite repulsion J_z of paired holes. (c) A single spin flip in the sequence Σ_0 breaks a bound hole pair leading to the spin gap.

Material [28]). The ground state of the XXZ model is known to be a Luttinger liquid. The off-diagonal SCs of the paired holes in the ZL clearly correspond to the dual lattice transverse spin correlation function. Using known bosonization results for the latter, we have (see Ref. [1]) for large separations $x \gg 1$: $\langle c_{x+1} c_x c_0^\dagger c_1^\dagger \rangle_{\mathrm{ZL}} = \langle \sigma^+(x)\sigma^-(0)\rangle_{\mathrm{dual}} = C\cos(\pi\rho x)x^{-2K-1/(2K)} + C'\cos(\pi x)x^{-1/(2K)}$, where C and C' are constants. This is in agreement with the lack of explicit symmetry breaking in 1D, where at most quasilong range, i.e., algebraically decaying, SCs can exist. The SC decay is density dependent via the Luttinger parameter K which increases from $1/4$ for $\rho \to 0$ to 1 for $\rho \to 1$ [1].

Robustness of the string ordered phase.—The finite spin gap suggests that the features of the exactly solvable point should be rather general. Note, however, that $V = \infty$ must lead to a phase separated "ferromagnetic" (FM) ground state with two separated domains of like fermions or a single species domain, as this minimizes the interaction energy. To first analyze the transition between $V = 0$ and $V = \infty$ for fixed $X = J$, we introduce the conserved parameter for fixed spin sequences:

$$O = \sum_{j=1}^{2L} \sum_{j'>j} n_j Q_{j,j'} n_{j'} e^{i\pi(j-j'+1)}, \tag{5}$$

where $Q_{j,j'} = \prod_{j'>l>j}(1-n_l)$ for $j' > j+1$ or $Q_{j,j'} = 1$ where j, j' are subsequent sites. The summand assigns $+1(-1)$ to a consecutive pair of the same (different) spins on the lattice. So, $O = 2N-1$ is maximal for ideal string ordered configurations and $O = 3-2N$ for the FM state. Figure 3(a) shows that for arbitrary densities, a large region of stability of ideal string order (and thus collective hole pairing) is followed by a first order transition to the FM state for $V < 2J$ in the ground state of H [Eq. (1)].

For $X \neq J$, the spin sequence is subject to dynamical changes. Moving to the $X \neq J$ regime, we restrict to $X/J \leq 1$ due to the symmetry $X \to 2J - X$ of the model Eq. (1) [38]. Without CH ($X = 0$, $V \neq 0$) an interaction driven, direct liquid to phase-separation jump transition for arbitrary fillings occurs [42]. So, we shall not consider here the similar effect of $V \neq 0$ but rather the fate of the string ordered state for $V = 0$. First, note that the spin gap Δ_S is always open for $X > 0$ [Fig. 3(b)] indicating hole pairing and long range string correlations for arbitrary low values of CH. To probe the long range nonlocal hidden order for general $X \neq J$, we introduce a den Nijs-Rommelse–type string parameter [8,39] O_S in analogy to the string parameter capturing diluted antiferromagnetic order in spin-1 antiferromagntic chains [7]. Grouping pairs of nearest neighbor sites into dimers $(1,2),(3,4),\ldots,(2L-1,2L)$, define the operator $S_i^z = (n_{2i} - n_{2i-1})$ with values $+1, 0, -1$ in analogy with a spin-1 S_z operator, so that

$$O_S(m,m') = \left\langle S_m^z \exp\left(i\pi \sum_{m<l<m'} S_l^z\right) S_{m'}^z \right\rangle. \tag{6}$$

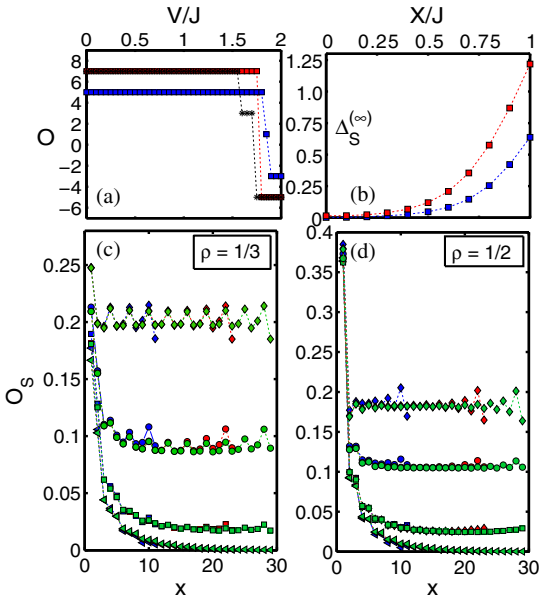

FIG. 3. Exact diagonalization (ED) (a),(b) and TEBD algorithm [40,41] (c),(d) results. (a) O vs V for $X = J$ where $2N = 8$, $2L = 20$ (black), $2N = 8$, $2L = 24$ (red), $2N = 6$, $2L = 24$ (blue). (b) Extrapolated spin gap $\Delta_S^{(\infty)}$ to thermodynamic limit by fitting finite size gaps with quadratic polynomials in $1/L$ (using sizes up to $2L = 30$) for $\rho = 1/2$ (red) and $\rho = 1/3$ (blue) for $V = 0$. (c),(d) Finite size scaling O_S vs distance x under OBC for different densities and $V = 0$. Data are for $X = 1.0J$ (diamonds), $0.6J$ (circles), $0.3J$ (squares), $0.0J$ (triangles). Compared are lattice lengths $2L = 24$ (blue), $2L = 48$ (red) and $2L = 60$ (green) for every X.

Figures 3(c) and 3(d), show that O_S attains a finite value for ideal string order at $X = J$. Reducing X yields a diminishing, yet finite large distance saturation value of O_S for $X \neq 0$ with no evidence of destruction of long range string order before $X = 0$. Results in Figs. 3(c) and 3(d) are for two chosen densities; however, we have checked that this behavior is generic.

Final comments.—Our exact solution, in combination with numerical results, reveals an instability of two fermionic chains, flowing from the extreme coupling point $X = J$ which dominates the physics of the system Eq. (1). Both the phenomenology and exact solution of our model are fundamentally distinct from that of the standard bond charge model H_H [14–20]. The latter for $X = J$ is also solvable by mappings to free fermions within invariant subspaces of H_H labeled by separately conserved sequences of $\uparrow$ and $\downarrow$ fermions, and $\uparrow\downarrow$ (doublons) and 0 (holes). Each spinless fermion lattice is identical to the original lattice, leading to a huge spectral degeneracy with respect to all permutations within these sequences. In *our* solution, the intricate dependence of the effective lattice length (available single particle states) on the conserved sequences substantially reduces analogous degeneracies and is behind the system's properties. Finally, we mention that a supersymmetric model with a *complex valued* 3-site CH was exactly solved recently using the involved nested Bethe-ansatz method [43].

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O. D. derived the model corresponding to the experimental setup leading to large correlated hopping terms. P. R. G. conceived the mathematical construction of the exact solution. The numerical studies of the full model for general parameters were conducted by R. W. C. (ED) and J. S. (TEBD). All authors contributed to the physical interpretation and conceptual understanding of the exact solution and numerical results, with main contributions from P. R. G. and R. W. C. R. W. C. wrote the manuscript with contributions from all authors. M. L. supervised the project.

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