Non-Markovian Quantum Friction of Bright Solitons in Superfluids

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We explore the quantum dynamics of a bright matter-wave soliton in a quasi-one-dimensional bosonic superfluid with attractive interactions. Specifically, we focus on the dissipative forces experienced by the soliton due to its interaction with Bogoliubov excitations. Using the collective coordinate approach and the Keldysh formalism, a Langevin equation of motion for the soliton is derived from first principles. The equation contains a stochastic Langevin force (associated with quantum noise) and a nonlocal in time dissipative force, which appears due to inelastic scattering of Bogoliubov quasiparticles off of the moving soliton. It is shown that Ohmic friction (i.e., a term proportional to the soliton's velocity) is absent in the integrable setup. However, the Markovian approximation gives rise to the Abraham-Lorentz force (i.e., a term proportional to the derivative of the soliton's acceleration), which is known from classical electrodynamics of a charged particle interacting with its own radiation. These Abraham-Lorentz equations famously contain a fundamental causality paradox, where the soliton (particle) interacts with excitations (radiation) originating from future events. We show, however, that the causality paradox is an artifact of the Markovian approximation, and our exact non-Markovian dissipative equations give rise to physical trajectories. We argue that the quantum friction discussed here should be observable in current quantum gas experiments.

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In recent years, solitons and solitonlike textures have been the subject of much interest and research in quantum superfluids [1] and optical fibers [2,3]. The hallmark features of solitons are their remarkable robustness and stability, which stem from the integrability of the underlying nonlinear model. However, solitons in realistic environments experience dissipative forces and eventually decay. Their quasiclassical motion can be well described by Newton's equation [4]

$$M\ddot{X} = -\partial_X U + F[X(t)], \tag{1}$$

where X is the soliton's position, M is its effective mass, and the right-hand side contains external forces due to a confining potential U and the friction force $F[X(t)]$. The standard Ohmic friction with $F[X] = -\gamma\dot{X}$ has been considered before by many researchers [5–13]. It can be shown that the friction coefficient γ is proportional to an integral of the reflection coefficients of the Bogoliubov excitations off the soliton [5]. However, there is a nontrivial caveat in pristine integrable setups, where the soliton represents a *reflectionless* potential for excitations. Hence, if integrability is preserved, Ohmic friction is strictly absent. This has motivated the authors of Refs. [6–9] to introduce integrability-breaking terms to induce a nonzero friction coefficient γ.

Here we revisit this question of soliton dissipation in superfluids, and ask: are there dissipative forces acting on a moving soliton in the perfectly integrable model? Naively, by the above argument there should be none because the soliton appears blind to the surrounding cloud of Bogoliubov excitations. We show that this is not the full story and intrinsic, albeit non-Ohmic, friction does exist even if integrability is not broken. It turns out that this problem has a distant cousin in electrodynamics: if a charged particle is moving in an external potential, it is accelerated by the potential and loses energy by emitting electromagnetic radiation. The corresponding classical equation of motion (EOM) contains the Abraham-Lorentz force, $\mathbf{F}_{\mathrm{AL}} \sim \dddot{\mathbf{R}}$, giving rise to a famous paradox—the solutions to Abraham-Lorentz equations violate causality (see Refs. [14–23] for a modern point of view and historical perspectives). In this work, we show that the problem of a moving soliton is similar and contains both the Abraham-Lorentz paradox and its resolution by accounting for retardation effects [21–23]. There are two key processes, which contribute to intrinsic soliton friction: emission of quasiparticles when accelerating (for dark solitons moving in the presence of gapless phonons) and inelastic scattering of emitted or thermal quasiparticles. In this work, we focus specifically on the simpler case of bright solitons [1,24–26], for which only the latter mechanism plays out. Our main result—the soliton's EOM—is presented below:

$$M\ddot{X}(t) + \int_0^t dt'\eta(t-t')\dot{X}(t') = -\partial_X U + f_s(t), \tag{2}$$

where $\eta(t)$ is the dissipation kernel and $f_s(t)$ is a stochastic Langevin force. Its correlation function,

$C_s(t) = \langle f_s(t) f_s(0) \rangle$, and $\eta(t)$ are linked through the same spectral function via fluctuation-dissipation theorem:

$$\eta(t) = \frac{2}{\pi} \int_0^\infty d\omega \frac{J(\omega)}{\omega} \cos(\omega t), \tag{3}$$

$$C_s(t) = \frac{2\hbar}{\pi} \int_0^\infty d\omega J(\omega) \coth\left(\frac{\hbar\omega}{2T}\right) \cos(\omega t). \tag{4}$$

An analytic expression for the spectral function of Bogoliubov excitations is derived in Eq. (15) (see also Fig. 1). The Markovian limit [i.e., where $\eta(t-t')$ is approximated by a local-in-time delta function or its derivatives] of the dissipation force in Eq. (2) contains no Ohmic friction, but gives rise to the Abraham-Lorentz-type force, $F[X] \sim \dddot{X}$, and noncausal soliton trajectories. However, solutions of Eq. (2) with the full non-Markovian dissipative force contain no causality paradox.

Our starting point is a $(1+1)$-dimensional field theory, describing a Bose gas with attraction:

$$L = \int dx \left[\phi^* i\hbar \partial_t \phi - \frac{\hbar^2}{2m} |\nabla \phi|^2 + \mu |\phi|^2 - \frac{g_1}{2} |\phi|^4 \right], \tag{5}$$

where m is the mass of the atoms and $\mu < 0$ is the chemical potential. In the context of realistic (quasi-)one-dimensional experiments, the interaction parameter $g_1 = 2\pi\hbar^2 a / m l_\perp^2$, where a is the 3D scattering length, $l_\perp = \sqrt{\hbar / m\omega_\perp}$, and $\omega_\perp$ is the transverse harmonic confinement frequency [27].

Importantly, in both 1D and quasi-1D, the interaction part scales as $1/L$ with system size (L) and balances the kinetic energy ($\sim 1/L^2$). This implies that (in contrast to higher dimensions) the attractive Bose gas in one dimension is stable against collapse [1,28]. The attractive nonlinear mean-field interaction energy favors aggregation of particles and counteracts the dispersion of the wave packet. This leads to the formation of a *bright soliton*, where a Bose-Einstein condensate is localized in a lump of matter with a size set by the coherence length $\xi = \hbar / \sqrt{2m|\mu|}$. The bright soliton solution $\phi_0(x)$ is obtained by minimizing the Lagrangian [Eq. (5)]; i.e., it solves the Gross-Pitaevskii equation. For a soliton with N particles, the wave function is given by [29,30]

$$\phi_0(x) = \sqrt{\frac{N}{2\xi}} e^{i\theta} \mathrm{sech}\left(\frac{x - X}{\xi}\right). \tag{6}$$

Here, θ and X are the phase and the coordinate of the soliton. Note that the energy of a static soliton is independent of θ and X. We consider small-amplitude fluctuations on top of the soliton background, and write the field $\phi(x,t) = \phi_0(x) + \delta\phi(x,t)$, where the soliton wave function ϕ_0 is defined in Eq. (6). The linear correction vanishes, as ϕ_0 solves the Gross-Pitaevskii equation. The quadratic correction to the Lagrangian is $\delta L = 1/2 \int dx \Psi^\dagger [i\hbar\sigma_3 \partial_t - K_{\mathrm{BdG}}] \Psi$, where we define $\Psi = (\delta\phi, \delta\phi^*)^T$ and K_{BdG} is the positive semidefinite Bogoliubov–de Gennes kernel:

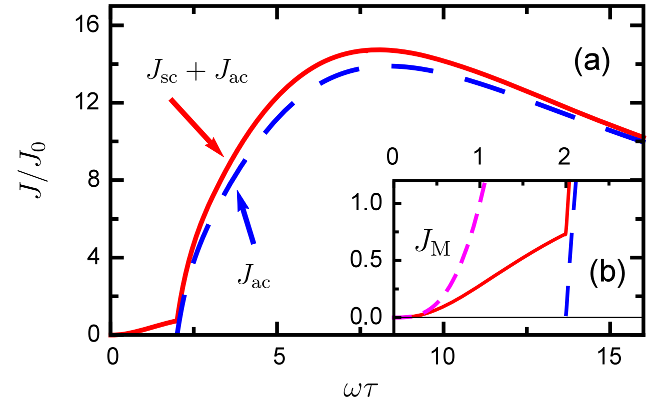

FIG. 1. (a) Spectral function of the bath formed by Bogoliubov quasiparticles $J(\omega) = J_{sc}(\omega) + J_{ac}(\omega)$ (red line) and the contribution of creation or annihilation processes $J_{ac}(\omega)$ to it (dashed blue line). (b) The low-frequency part of (a) with the asymptotics $J_M(\omega)$ (dashed purple line) given by Eq. (16). J_0 is a constant defined after Eq. (16).

$$K_{\mathrm{BdG}} = \begin{pmatrix} -\frac{\hbar^2\nabla^2}{2m} - \mu + 2g_1|\phi_0|^2 & g_1\phi_0^2 \\ g_1\phi_0^{*2} & -\frac{\hbar^2\nabla^2}{2m} - \mu + 2g_1|\phi_0|^2 \end{pmatrix}. \tag{7}$$

The diagonalization proceeds in a similar way as for trapped Bose-Einstein condensates [29,31,32]: finite-energy excitations solve the BdG equation $K_{\mathrm{BdG}}|k\rangle = \sigma_3 \varepsilon_k |k\rangle$ with energy $\varepsilon_k = \hbar^2 k^2 / 2m + |\mu|$ and wave function $|k\rangle = (u_k, -v_k)^T$ given by [33,34]

$$\begin{pmatrix} u_k \\ -v_k \end{pmatrix} = \frac{e^{ikx}}{(k^2\xi^2 + 1)} \begin{pmatrix} e^{-i\theta}[k\xi + i\tanh(x/\xi)]^2 \\ -e^{i\theta}\mathrm{sech}^2(x/\xi) \end{pmatrix}. \tag{8}$$

Here we assume N and μ to be given. Formally, BdG equations have eigenvalues with negative energies $-\varepsilon_k$ and wave functions $\overline{|k\rangle} = (-v_k^*, u_k^*)^T$. In addition, there are two zero modes given by $|\theta\rangle = (\phi_0, -\phi_0^*)$ and $|X\rangle = -\xi\partial_x(\phi_0, \phi_0^*)$, corresponding to a small change in the phase and the soliton position, respectively. In the following, we neglect the phase degree of freedom since we are interested in only the soliton dynamics. The zero modes cannot be treated as a small perturbation. The correct way to treat them nonperturbatively is via the collective coordinate method [35,36].

Since the zero modes have vanishing norm, they cannot be included in the basis set that diagonalizes K_{BdG}. Instead, the space of BdG excitations is supplemented by adjoint modes with nonzero norm, chosen as

$$K_{\mathrm{BdG}}|X^a\rangle = \frac{\hbar^2}{M\xi^2}\sigma_3|X\rangle, \tag{9}$$

where the mass M is chosen such that the adjoint modes have unit overlap with the corresponding zero modes. For the zero mode of soliton spatial translations, the adjoint mode is $|X^a\rangle = -x/N\xi(\phi_0, -\phi_0^*)$ with mass $M = mN$. Now we promote the soliton coordinate to a quantum dynamical variable $X(t)$ and present the bosonic field $\phi(x,t)$ in terms of the complete basis set (quasiparticle eigenmodes and the adjoint to the zero mode of translations) as follows:

$$\phi(x,t) = \phi_0[x - X(t)] + i\frac{\xi\pi_0}{\hbar}u_X^a[x - X(t)] + \sum_k\{c_k(t)u_k[x - X(t)] - c_k^*(t)v_k^*[x - X(t)]\}. \quad (10)$$

Here π_0 is the bare momentum of the soliton without Bogoliubov quasiparticles. After substitution of Eq. (10) to the original Lagrangian Eq. (5) and integrating out π_0, we get [37]

$$L = \frac{M\dot{X}^2}{2} + \pi_{\rm qp}\dot{X} + \sum_k c_k^*[i\hbar\partial_t - \epsilon_k]c_k. \quad (11)$$

Here $\pi_{\rm qp}$ is the total momentum of Bogoliubov quasiparticles, while the momentum of the soliton in their presence is given by $\pi_s = M\dot{X} + \pi_{\rm qp}$. The explicit form of $\pi_{\rm qp}$ is given by

$$\pi_{\rm qp} = \frac{1}{2}\sum_{k,k'}(c_k^*, c_k)\begin{pmatrix} \langle k|\sigma_z\hat{p}|k'\rangle & -\langle k|\sigma_z\hat{p}|\overline{k'}\rangle \\ -\overline{\langle k|\sigma_z\hat{p}|k'\rangle} & \overline{\langle k|\sigma_z\hat{p}|\overline{k'}\rangle} \end{pmatrix}\begin{pmatrix} c_{k'} \\ c_{k'}^* \end{pmatrix}, \quad (12)$$

where $\hat{p} = -i\hbar\partial_x$ is the momentum operator. Diagonal components $\pi^{\rm sc}$ correspond to scattering of quasiparticles, while nondiagonal components $\pi^{\rm ac}$ correspond to their annihilation and creation. Using the explicit form of wave functions Eq. (8), they can be found as follows:

$$\pi_{k'k}^{\rm sc} = \frac{\pi\hbar}{3\xi}\frac{(k^2 - k'^2)(k'^2 + k'k + k^2 + k_\xi^2)}{(k'^2 + k_\xi^2)(k^2 + k_\xi^2)\sinh[\frac{\pi}{2}\xi(k' - k)]}, \quad (13)$$

$$\pi_{k'k}^{\rm ac} = \frac{\pi\hbar}{3\xi}\frac{(k + k')^2(k'^2 - k'k + k^2 + k_\xi^2)}{(k'^2 + k_\xi^2)(k^2 + k_\xi^2)\sinh[\frac{\pi}{2}\xi(k' + k)]}, \quad (14)$$

where we introduce the wave vector scale $k_\xi = \xi^{-1}$. It should be noted that, due to the integrability of the original problem, the backscattering is suppressed: $\pi_{k,-k}^{\rm sc} = 0$.

The Lagrangian Eq. (11) describes a motion of the soliton and Bogoliubov quasiparticles coupled with each other. The coupling term, $L_{\rm int} = \pi_{\rm qp}\dot{X}$, is a new and important result of our work. The soliton is a quasiclassical entity while Bogoliubov quasiparticles can be treated as a quantum bath. The coupling with the bath leads to the friction and Langevin force in the EOM of the soliton Eq. (2). To derive such quasiclassical dissipative dynamics is an old, fundamental problem, which arises in the context of Brownian motion and the general Caldeira-Leggett model [43]. However, since the coupling of the collective soliton coordinate to the bath is quadratic here, the problem at hand is more complicated than the Caldeira-Leggett model (where the coupling to the bath is linear and the model is exactly solvable; see also Refs. [44,45]). We have derived the quasiclassical EOM as the saddle point of a one-loop effective action in the Keldysh formalism [46,47]. Formally, this corresponds to an expansion of the full action in terms of the soliton velocity, $\dot{X}/c \ll 1$, where $c = \hbar/m\xi$ is a characteristic velocity scale in the model. Detailed technical calculations are presented in Supplemental Material [37], while here we present the main results—Eq. (2), which represents the quasiclassical Langevin EOM for the soliton. It is in effect Newton's second law for the soliton in the potential $U(X)$, supplemented with a retarded friction force, $F[X(t)] = -\int_0^t dt'\eta(t - t')\dot{X}(t')$, and a stochastic force, $f_s(t)$. The dissipation kernel $\eta(t)$ and correlation function $\langle f_s(t)f_s(0)\rangle$ are related via the fluctuation-dissipation theorem and are expressed [see Eqs. (3) and (4)] through the same spectral function $J(\omega)$ given by

$$J(\omega) = 2\pi\sum_{kk'}\Big[\underbrace{2|\pi_{kk'}^{\rm sc}|^2(f_{k'} - f_k)(\varepsilon_{kk'}^-)^2\delta(\varepsilon_{kk'}^- - \hbar\omega)}_{\text{scattering processes}} + \underbrace{|\pi_{kk'}^{\rm ac}|^2(1 + f_{k'} + f_k)(\varepsilon_{kk'}^+)^2\delta(\varepsilon_{kk'}^+ - \hbar\omega)}_{\text{annihilation and creation processes}}\Big], \quad (15)$$

where $\varepsilon_{kk'}^\pm = \varepsilon_k \pm \varepsilon_{k'}$ and $f_k = f_B(\varepsilon_k)$ is the Bose-Einstein distribution. The first term $J_{\rm sc}$ in Eq. (15) corresponds to the scattering of Bogoliubov quasiparticles, while the second term $J_{\rm ac}$ originates from their annihilation and creation processes. Their dependencies on frequency are presented in Fig. 1. The ω dependence of $J_{\rm ac}$ weakly depends on temperature T and has a threshold $2|\mu|/\hbar$, which is the minimal energy to create a pair of Bogoliubov quasiparticles in the superfluid of attractive bosons. At low frequencies *only* $J_{\rm sc}$ survives. It does not have an Ohmic component (which would be linear in frequency), but is super-Ohmic:

$$J_M(\omega) = \frac{16}{9\pi}\frac{m\hbar\omega^3}{|\mu|}e^{-|\mu|/T} = J_0(\omega\tau)^3, \quad (16)$$

where $\tau = \hbar/|\mu|$ and $J_0 = 16m/9\pi\tau^2\exp[-|\mu|/T]$. Note that a super-Ohmic spectral function also appears for an impurity embedded to a bosonic superfluid [10,48]. To get better insight into the origin of this effect (absence of Ohmic friction), we rewrite the linear-in-ω part of the spectral function $J_{\rm sc}(\omega)$ as follows:

$$J_{\rm sc}(\omega) = \int d\varepsilon\nu(\varepsilon)\nu(\varepsilon + \omega)[f_B(\varepsilon) - f_B(\varepsilon + \omega)] \times S(\varepsilon + \omega, \varepsilon) \approx \omega\int_{-\infty}^{\infty} d\varepsilon\nu^2(\varepsilon)S(\varepsilon, \varepsilon)\left(-\frac{\partial f_B}{\partial\varepsilon}\right), \quad (17)$$

where $\nu(\varepsilon) = 1/\pi\xi\sqrt{|\mu|(\epsilon - |\mu|)}$ is the density of states of Bogoliubov quasiparticles and we introduced

$$S = \frac{2\pi}{\nu(\varepsilon_1)\nu(\varepsilon_2)} \sum_{k'k} |\pi^{\mathrm{sc}}_{kk'}|^2 (\epsilon^-_{k,k'})^2 \delta(\epsilon_k - \varepsilon_1)\delta(\epsilon_{k'} - \varepsilon_2),$$

which can be interpreted as the probability of scattering of quasiparticles with energies ε_1 and ε_2. From Eq. (17), we see that Ohmic friction comes exclusively from elastic scattering, which in 1D is equivalent to backscattering. However, as can be seen from Eq. (13), integrability ensures that $\pi^{\mathrm{sc}}_{k,-k} = 0$. Hence, backscattering is forbidden—and there is no Ohmic friction.

The dynamics of the soliton at macroscopic time scales $t \gg \tau$ is determined by the low-frequency part of the Fourier transform of the dissipation kernel $\eta(\omega)$. Its low-frequency asymptotics $\omega\tau \to 0$ correspond to Markovian approximation and *local*-in-time EOM. The real part $\eta'(\omega) = J(|\omega|)/|\omega| \approx J_0\omega^2\tau^3$, which is even and breaks time-reversal invariance at the quasiclassical level, is responsible for friction and originates from the scattering contribution to the spectral function J_{sc}. The imaginary part of the dissipation kernel $\eta''(\omega) = -i\delta M\omega$ is odd in frequency and is responsible for the mass renormalization $M \to M + \delta M$. The mass renormalization, however, is small with $\delta M/M \sim N^{-1}$ and can be neglected. The resulting EOM of the soliton in a trap with frequency ω_t is in the Markovian approximation given by

$$\ddot{X} - \tau_{\mathrm{AL}}\dddot{X} + \omega_t^2 X = f_s(t)/M. \qquad (18)$$

The term $F[X] = \tau_{\mathrm{AL}} M\dddot{X}$ (with $\tau_{\mathrm{AL}} = J_0\tau^3/M \approx 16\tau/9\pi N$ here) can be recognized as the Abraham-Lorentz friction force, originally derived in the context of electrodynamics (where it describes the backreaction of electromagnetic radiation emitted by a charged particle on its motion). In our context, inelastic scattering of Bogoliubov quasiparticles play the role of the radiation. The Abraham-Lorentz equation is plagued by spurious "runaway" solutions and its regularization has been a controversial and long-standing problem [16,17]. The equation violates causality, which can be seen by calculating the response function $\chi^{-1}(\omega) = M[\omega_t^2 - \omega^2(1 + i\tau_{\mathrm{AL}}\omega)]$ of the soliton coordinate to the stochastic or external force $X(\omega) = \chi(\omega)f_s(\omega)$. The response function is supposed to be analytical in the upper half-plane, while it has the spurious pole $\omega_{\mathrm{AL}} \approx i\tau_{\mathrm{AL}}^{-1}$, as is depicted in Fig. 2(b).

Note as a crucial point that the breaking of causality is an artifact of the Markovian approximation. Indeed, the location of the unphysical pole $|\omega_{\mathrm{AL}}|\tau \approx 9\pi N/16 \gg 1$ is beyond the applicability of the Markovian approximation, which requires $\omega\tau \ll 1$. For regularization, consider the spectral function $J^*_M(\omega) = J_0(\omega\tau)^3 e^{-\omega\tau_*}$, where we parametrize its high-frequency part by a memory time τ_*. The spectral function qualitatively captures the time dependence of the exact dissipation kernel, as is presented in Fig. 2(a). The exact expression for the response function $\chi(\omega)$, corresponding to the spectral function $J^*_M(\omega)$, is

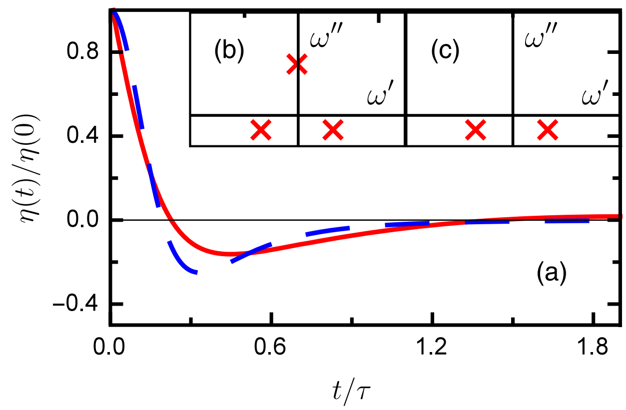

FIG. 2. (a) Time dependence of the friction kernel $\eta(t)$, corresponding to the exact spectral function, given by Eq. (15) (red line), and to the approximate one, given by $J^*_M = J_0(\omega\tau)^3\exp[-\omega\tau_*]$ (dashed blue line), at $T = \mu$ and $\tau_* = 3\tau$. (b) Poles in the complex plane $\omega = \omega' + i\omega''$ of the response function $\chi(\omega)$ in the Markovian approximation leading to the Abraham-Lorentz friction force. In the upper half-plane there is the spurious pole corresponding to causality violation. (c) Poles of the response function $\chi(\omega)$ calculated exactly for the spectral function J^*_M or in the Markovian approximation, if the Abraham-Lorentz force is treated as perturbation. Note that there is no unphysical pole in the upper half-plane of ω.

given by $\chi^{-1}(\omega) = M\{\omega_t^2 - \omega^2 - i\tau_{\mathrm{AL}}\omega^3[\cosh(\omega\tau_*) + 2\pi^{-1}\cosh(\omega\tau_*)\mathrm{Si}(i\omega\tau_*) - 2i\pi^{-1}\sinh(\omega\tau_*)\mathrm{Ci}(-i\omega\tau_*)]\}$, where $\mathrm{Ci}(-i\omega\tau_*)$ and $\mathrm{Si}(i\omega\tau_*)$ are cosine and sine integral functions. The response function is analytical in the upper half-plane, which ensures causality. Note also that the Abraham-Lorentz equation can be regularized by treating super-Ohmic friction as a perturbation, which leads to $\ddot{X} + \tau_{\mathrm{AL}}\omega_t^2\dot{X} + \omega_t^2 X = (f_s + \tau_{\mathrm{AL}}\dot{f}_s)/M$. The resulting response function is analytical in the upper half-plane with poles $\omega \approx \pm\omega_t - i\tau_{\mathrm{AL}}\omega_t^2$ [see Fig. 2(c)]. In the presence of a trap potential the Abraham-Lorentz friction is well approximated by the usual friction, $F[X] = -\tau_{\mathrm{AL}}\omega_t^2 M\dot{X}$. Most importantly, this implies that the effective friction force is very sensitive to the trap frequency ω_t and can be distinguished in this way from regular (extrinsic) Ohmic friction, which appears due to the breaking of integrability [7].

In realistic experiments, the Abraham-Lorentz friction competes with the usual Ohmic friction. The Ohmic friction appears due to the quasi-one-dimensional nature of the trapping potential [7], or due to interactions between Bogoliubov quasiparticles [8]. However, the latter is a two-particle effect and is proportional to $\exp[-2|\mu|/T]$ instead of $\exp[-|\mu|/T]$, which makes it unimportant at low temperatures. For estimates, we use the following parameters corresponding to the experiments in Ref. [24]: particle number in the soliton $N \approx 1.5 \times 10^3$; coherence length $\xi \approx 1.7\ \mu$m; chemical potential and temperature $|\mu| \approx 2T \approx 11$ nK; axial trap frequency $\omega_t \approx 2\pi \times 70$ Hz; transverse confinement and scattering length, $l_\perp = 1.4\ \mu$m and

$a = 0.44$ nm. The resulting decay times of the soliton are of comparable magnitude and given by $\tau_s^{\mathrm{AL}} = 9\pi N\tau \exp[|\mu|/T]/16\pi(\omega_t\tau)^2 \approx 187\,\mathrm{s}$ and $\tau_s^{\mathrm{O}} \approx \pi N\tau \exp[|\mu|/T](|\mu|x^2 + T)^2/2T\mu^2x^4 \approx 400$ s, where $\tau = \hbar/|\mu|$ and $x = Na/l_\perp$. Nevertheless, two mechanisms of friction can be distinguished in the experiment because $\tau_s^{\mathrm{AL}} \sim \omega_t^{-2}$ strongly depends on the axial trap frequency, while τ_s^{O} is sensitive only to the transverse confinement $\omega_\perp$. This implies that the quantum soliton friction predicted in this work is within current experimental capabilities.

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