Ultimate Precision Limits for Noisy Frequency Estimation

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Quantum metrology protocols allow us to surpass precision limits typical to classical statistics. However, in recent years, no-go theorems have been formulated, which state that typical forms of uncorrelated noise can constrain the quantum enhancement to a constant factor and, thus, bound the error to the standard asymptotic scaling. In particular, that is the case of time-homogeneous (Lindbladian) dephasing and, more generally, all semigroup dynamics that include phase covariant terms, which commute with the system Hamiltonian. We show that the standard scaling can be surpassed when the dynamics is no longer ruled by a semigroup and becomes time inhomogeneous. In this case, the ultimate precision is determined by the system short-time behavior, which when exhibiting the natural Zeno regime leads to a nonstandard asymptotic resolution. In particular, we demonstrate that the relevant noise feature dictating the precision is the violation of the semigroup property at short time scales, while non-Markovianity does not play any specific role.

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Introduction.—Parameter estimation, ranging from the precise determination of atomic transition frequencies to external magnetic field strengths, is a central task in modern physics [1–7]. Quantum probes made up of N entangled particles can attain the so-called Heisenberg limit (HL), where the estimation mean squared error (MSE) scales as $\sim 1/N^2$, as compared with the standard quantum limit (SQL) $\sim 1/N$ of classical statistics [8].

Heisenberg resolution relies on the unitarity of the time evolution. In realistic situations, however, quantum probes decohere as a result of the unavoidable interaction with the surrounding environment [9]. Such interactions can have a dramatic effect on estimation precision—even infinitesimally small uncorrelated dephasing noise, modeled as a semigroup (time-homogeneous-Lindbladian) evolution [10], forces the MSE to eventually follow the SQL [11]. This result was proven to be an instance of the quantum Cramér-Rao bound (QCRB) [12] for generic Lindbladian dephasing and, thus, holds even when using optimized entangled states and measurements [13–16]. The question then arises, what is the ultimate precision limit when the noisy time evolution is not governed by a dephasing dynamical semigroup [13–26]? The SQL has been shown to be surpassable in the presence of time-inhomogeneous (nonsemigroup) dephasing noise [24], noise with a particular geometry [25], and correlated dephasing with [27] and without [28] memory, or when the noise geometry allows for error correction techniques [29,30].

Here, we derive the ultimate lower bounds on the MSE for the noisy frequency estimation scenario depicted in Fig. 1 where probe systems are independently affected by the decoherence. In particular, we focus on uncorrelated phase-covariant noise, that is, noise types commuting with the parameter-encoding Hamiltonian, as these underpin the asymptotic SQL-like precision in the semigroup case [16,25]. Yet, most importantly, we allow for any form of time inhomogeneity and non-Markovian features in the noise. Our results show that, when moving away from the semigroup regime, entanglement generally improves the precision beyond the constant-factor enhancement, so that the SQL is truly overcome. As a special case, we confirm the conjecture made in [24], where, by considering a Ramsey interferometry scheme and nonsemigroup dephasing dynamics, a $1/N^{3/2}$ error scaling was shown to be achievable. This was argued to be a consequence of the Zeno regime at short time scales. The generality of this

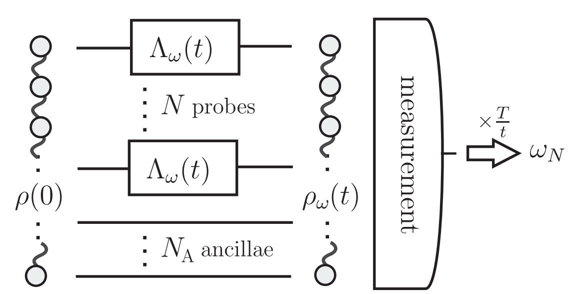

FIG. 1. Noisy frequency estimation scenario. N qubit probes sense a parameter ω following a preparation in a state $\rho(0)$, including an arbitrary number N_A of ancillary particles potentially entangled with the sensing probes. During the evolution, the probes are subject to uncorrelated noise and, after time t, the whole system, in state $\rho_\omega(t)$, is measured. The protocol is repeated T/t times, with $T \gg t$, to construct a frequency estimate ω_N.

scaling has been recently verified for pure dephasing noise [31]. We formally prove the emergence of non-SQL scaling for any nonsemigroup phase covariant noise. We demonstrate that it is solely the short-time expansion of the effective noise parameters that determines the ultimate attainable precision. In particular, any memory (non-Markovian) effects, which may be displayed by the system at later times, are irrelevant for the asymptotic N limit.

Noisy frequency estimation.—In a typical frequency estimation setting, a parameter ω is unitarily encoded on N sensing particles (probes), specifically qubits, over the interrogation time t during which the probes are also independently disturbed by the decoherence [11,13]. As depicted in Fig. 1, we generalize such a setup to allow for an arbitrary number N_{A} of ancillary particles, that can be initially entangled with the probes and measured at the end of the protocol. Hence, the combined final state of the system reads

$$\rho_\omega(t) = \Lambda_\omega(t)^{\otimes N} \otimes \mathbb{1}^{\otimes N_{\mathrm{A}}}[\rho(0)], \tag{1}$$

with $\rho(0)$ being the initial state, and $\Lambda_\omega(t)$ a completely positive and trace preserving (CPTP) linear map [32] representing the identical, but independent, evolution of each probe [33]. We assume full control and noise-free evolution for the ancillae, so as to allow for single-step error-correction protocols [30]. The N dependent parameter estimate, ω_N, relies on sufficiently large statistical data after performing T/t repetitions, provided the total experimental time $T \gg t$.

We quantify the performance of the estimation protocol by the MSE, $\Delta^2\omega_N$—describing the average deviation of the estimate from the true value. Crucially, requiring unbiasedness and consistency for the estimate, the QCRB directly provides us with the ultimate lower bound on the MSE that is optimized over all potential measurement strategies [12]. Hence, possessing, also, the freedom to adjust the single-shot duration time t, the ultimate attainable precision can be written as

$$\Delta^2\omega_N T \geq \min_t \frac{t}{F_{\mathrm{Q}}[\rho_\omega(t)]}, \tag{2}$$

where $F_{\mathrm{Q}}[\rho_\omega(t)]$ is the quantum Fisher information (QFI) with respect to the estimated parameter ω encoded in the final state. Importantly, the t minimizing the rhs in Eq. (2), i.e., the optimal single-shot duration, generally depends on the system size, and thus, we denote it as $t_{\mathrm{opt}}(N)$.

Phase covariant dynamics.—The frequency parameter ω is unitarily encoded within the phase, ωt, accumulated during the free evolution of the qubit probe, which, in the Bloch ball picture, corresponds to a rotation around a known direction—z in Fig. 2. We consider systems exhibiting uncorrelated forms of noise that commute with such rotations, which formally correspond to the so-called phase covariant qubit maps [34]. Such noise types are known to most severely limit the attainable precision in the case of semigroup dynamics, for which they constrain the

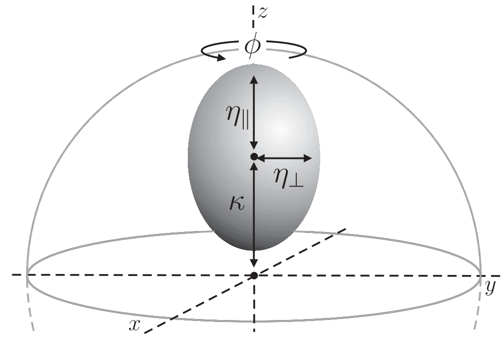

FIG. 2. Phase covariant quantum maps. Bloch ball representation of a qubit noisy evolution, $\Lambda_\omega(t)$, that commutes with the rotation about the z axis by an angle ωt, which represents the parameter encoding. The overall effect of the noise is to shrink the ball by factors $\eta_\parallel(t)$, $\eta_\perp(t)$ in the vertical direction and the horizonal plane, respectively, as well as to displace its the center by $\kappa(t)$ and to further rotate it about the z axis, so that the global rotation angle is $\phi = \omega t + \theta$.

quantum enhancement to a constant factor above the SQL [13–16]. Although such negative conclusions cannot be drawn for other less severe but still semigroup noises, i.e., purely transversal [25] and correlated [28], the phase covariant noise, if present, no matter how weak, will always asymptotically dominate and limit the ultimate quantum improvement to a constant factor [16,25]. Thus, in what follows, we focus on the frequency estimation scenario of Fig. 1 in the presence of general independent, identical, and phase covariant (IIC) noise, where each single probe at any instance of time may be described by the action of a map $\Lambda_\omega(t) = \mathcal{U}_\omega(t)\circ\Gamma(t) = \Gamma(t)\circ\mathcal{U}_\omega(t)$, with $\mathcal{U}_\omega(t)$ and $\Gamma(t)$ being its unitary encoding and ω-independent dissipative parts, respectively. We fix $\mathcal{U}_\omega(t)[\varrho] = e^{-(i/2)\omega\sigma_z t}\varrho e^{(i/2)\omega\sigma_z t}$ and use the affine representation of qubit maps [35–39] to express the most general phase covariant $\Lambda_\omega(t)$ as a matrix [33]

$$\Lambda_\omega(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \eta_\perp(t)\cos\phi(t) & -\eta_\perp(t)\sin\phi(t) & 0 \\ 0 & \eta_\perp(t)\sin\phi(t) & \eta_\perp(t)\cos\phi(t) & 0 \\ \kappa(t) & 0 & 0 & \eta_\parallel(t) \end{pmatrix}, \tag{3}$$

which acts on a four-component Bloch vector. As depicted in Fig. 2, the qubit evolution amounts then to: a rotation around the z axis by an angle ϕ containing the parameter encoding $[\phi(t) = \omega t + \theta(t)]$, a contraction in the xy plane by a factor $0 \leq \eta_\perp \leq 1$, a contraction in the z direction by a factor $0 \leq |\eta_\parallel| \leq 1$ ($\eta_\parallel < 0$ case corresponds to an additional reflection with respect to the xy plane), and a displacement in the z direction by $-1 \leq \kappa \leq 1$. The map in Eq. (3) fulfils the CPTP condition as long as $\eta_\parallel \pm \kappa \leq 1$ and $1 + \eta_\parallel \geq \sqrt{4\eta_\perp^2 + \kappa^2}$.

It is important to stress that all phase covariant dynamics $\varrho(t) = \Lambda_\omega(t)[\varrho(0)]$ can be put on physical grounds by considering the corresponding time-local master equation, which is of the form

$$\frac{d}{dt}\varrho(t) = -\frac{i}{2}[\omega + h(t)][\sigma_z, \varrho(t)] + \gamma_z(t)[\sigma_z\varrho(t)\sigma_z - \varrho(t)] + \gamma_+(t)\left(\sigma_+\varrho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \varrho(t)\}\right) + \gamma_-(t)\left(\sigma_-\varrho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \varrho(t)\}\right). \quad (4)$$

The proof and the link between $\theta, \eta_\perp, \eta_\parallel, \kappa$ and $h, \gamma_z, \gamma_+, \gamma_-$ are given in [33]. Phase covariant dynamics describe any physical evolution that may arise from combinations of time-varying dephasing, absorption, and emission processes, as well as Lamb shift corrections to the free Hamiltonian [9]. Moreover, Eq. (4) allows for the quantitative characterization of general non-Markovian effects [40,41]. In the special case of positive constant rates (time homogeneity), Eq. (4) provides the generator of any phase covariant CPTP semigroup [42].

Bounding the ultimate precision.—Having fully characterized the class of qubit IIC dynamics, we can now state the main result of the Letter. Given N qubit probes and N_A ancillae evolving according to Eq. (1), with the single qubit dynamics given by phase covariant maps $\Lambda_\omega(t)$ as in Eq. (3), and provided that at all times ($\forall t > 0$) $\eta_\perp(t) < 1$, the MSE in estimating the frequency ω is asymptotically determined by the short-time expansion of the noise parameters

$$\eta_\perp(t) = 1 - \alpha_\perp t^{\beta_\perp} + o(t^{\beta_\perp}), \qquad \kappa(t) = \alpha_\kappa t^{\beta_\kappa} + o(t^{\beta_\kappa}), \qquad \eta_\parallel(t) = 1 - \alpha_\parallel t^{\beta_\parallel} + o(t^{\beta_\parallel}), \quad (5)$$

and it satisfies the following inequality:

$$\lim_{N\to\infty} \frac{\Delta^2\omega_N T}{N^{-(2\beta_\perp - 1)/\beta_\perp}} \geq \frac{\alpha^{1/\beta_\perp}\beta_\perp}{(\beta_\perp - 1)^{(\beta_\perp - 1)/\beta_\perp}} = D, \quad (6)$$

where $D > 0$ and

$$\alpha = \begin{cases} 2\alpha_\perp & \beta_\perp < \beta_\parallel; \\ 2\alpha_\perp - \frac{\alpha_\parallel}{2} & \beta_\perp = \beta_\parallel < \beta_\kappa; \\ \max\left\{2\alpha_\perp - \frac{\alpha_\parallel}{2} - \frac{|\alpha_\kappa|}{2}, \frac{|\alpha_\kappa|}{4}\right\} & \beta_\perp = \beta_\parallel = \beta_\kappa; \end{cases} \quad (7)$$

Crucially, as (see below) the bound in Eq. (6) is always attainable up to a constant factor, the asymptotic precision is fully determined by the short-time expansion of the radius in the plane perpendicular to the rotation axis, which fixes the asymptotic scaling to $1/N^{(2\beta_\perp - 1)/\beta_\perp}$. For semigroup dynamics ($\beta_\perp = 1$), one accordingly recovers the SQL-like $1/N$ limit, while, with increasing $\beta_\perp$, one finds a progressively more favorable scaling that tends to HL for unrealistic $\beta_\perp \to \infty$. Besides the assumption of having IIC noise, the only condition assuring the bound (6) to be valid is $\eta_\perp(t) < 1$. In fact, if $\eta_\perp(t) = 1$, then by the CPTP-property $\eta_\parallel(t) = 1$ and $\kappa(t) = 0$. In other words, a "full revival" of the Bloch vector length occurs, and the only effect of the interaction with the environment is a rotation about the z axis by an angle θ. Not surprisingly, the best estimation strategy is, then, to measure the frequency at such pseudonoiseless time, when the HL is attainable. However, such a behavior is quite unlikely when dealing with open systems subject to realistic sources of noise [9].

The sketch of the proof is given below, while a complete version is in [33]. First, we fix the evolution time t (and omit it for simplicity), to use the finite-N channel extension method [15,19], which provides us with an upper bound on the QFI already optimized over all the initial states

$$\max_{\rho(0)} F_Q[\Lambda_\omega^{\otimes N} \otimes \mathbb{1}^{\otimes N_A}[\rho(0)]] \leq 4N \min_{\{K_i\}} \{\|A\| + (N-1)\|B\|^2\} = F^\uparrow. \quad (8)$$

The minimization above is performed over the Kraus representations of the channel $\Lambda_\omega[\varrho] = \sum_i K_i \varrho K_i^\dagger$, which refers to a single probe; $\|\cdot\|$ denotes the operator norm, whereas $A = \sum_i \dot{K}_i^\dagger \dot{K}_i$ and $B = \sum_i \dot{K}_i^\dagger K_i$ with $\dot{K}_i \equiv dK_i/d\omega$. Identifying the optimal Kraus representation is usually nontrivial; however, the numerical semidefinite programing (SDP) methods introduced in [15] automatically provide the correct ansatz, with which one may then proceed analytically.

In [33], we deal explicitly with the general phase covariant qubit maps. We additionally prove the convexity of the bound (8) with respect to the mixing of quantum channels, which allows us to analytically apply Eq. (8) to any map Λ_ω of the form in Eq. (3), after adequately decomposing it into an optimal mixture of unital ($\kappa = 0$) and amplitude damping channels ($\eta_\parallel = \eta_\perp^2 = 1 - \kappa$). Here, for simplicity, we focus on unital channels: the general upper bound (8) reduces to [33]

$$F^\uparrow_{\eta_\parallel,\eta_\perp} = \frac{t^2 N^2}{1 + N\ell(t)}, \qquad \ell(t) = \frac{1 + \eta_\parallel(t) - 2\eta_\perp(t)^2}{2\eta_\perp(t)^2}. \quad (9)$$

Thus, substituting into Eq. (2), we obtain the precision bound

$$\Delta^2\omega_N T \geq \min_t \frac{1 + N\ell(t)}{tN^2}, \quad (10)$$

which, in the case of semigroup dynamics, coincides with the asymptotically tight limit derived in [16].

First, beating the SQL-like scaling necessarily requires $\lim_{N\to\infty} t_{\mathrm{opt}}(N) = 0$. Assume, on the contrary, that the optimal evolution time attains some $t_\infty > 0$ as $N \to \infty$. Then, inspecting Eq. (9), one sees that the "no full-revival" assumption $\eta_\perp(t_\infty) < 1$, along with the CPTP constraints, implies $\ell(t_\infty) > 0$, and hence, Eq. (10) directly restricts the precision to asymptotically follow $1/N$. As a consequence, we can focus on the short-time regime and expand $\eta_\perp(t)$,

$\eta_\parallel(t)$ as in Eq. (5) to get $\ell(t) = 2\alpha_\perp t^{\beta_\perp} - \frac{1}{2}\alpha_\parallel t^{\beta_\parallel} + o(t^{\beta_\perp})$. From CPTP constraints, it follows that $\beta_\perp \le \beta_\parallel$, and if $\beta_\perp = \beta_\parallel$, then, additionally, $\alpha_\parallel \le 2\alpha_\perp$. Hence, up to the leading order: $\ell(t) = \alpha_l t^{\beta_\perp} + o(t^{\beta_\perp})$, with $\alpha_l = 2\alpha_\perp$ if $\beta_\perp < \beta_\parallel$, and $\alpha_l = 2\alpha_\perp - \alpha_\parallel/2$ if $\beta_\perp = \beta_\parallel$. Plugging the above expansion into Eq. (10), we find that the minimum is reached for

$$t_{\mathrm{opt}}(N) \overset{N\to\infty}{=} [\alpha_l(\beta_\perp - 1)N]^{-1/\beta_\perp}, \tag{11}$$

which yields the bound (6) with $\alpha = \alpha_l$, correctly coinciding with Eq. (7) for $\kappa = 0$.

Attaining the ultimate precision.—As Eq. (8) provides us only with an upper limit on the QFI, we still must investigate the tightness of bound (6). Yet, also, the QCRB (2), itself, is guaranteed to be saturable only in the limit of infinite independent experimental repetitions $T/t \to \infty$. This issue is particularly important in the noiseless case, when the minimization of the MSE (2) over t yields $t_{\mathrm{opt}} = T$, indicating that a single experimental shot consuming all time resources should be performed [13]. The QCRB is, then, not saturable, as can be proved by means of a rigorous Bayesian approach [43]. Fortunately, in the presence of IIC noise, the optimal single-shot duration, t_{opt}, is independent of T and decays as $N^{-1/\beta_\perp}$ with N, see Eq. (11), so that T/t always diverges as $N \to \infty$.

Now, we show that the scaling exponent in Eq. (6) is always correct, and it is only the constant D that, in some cases, may be underestimated. Consider a GHZ state $|\psi_{\mathrm{GHZ}}\rangle = 1/\sqrt{2}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$. Thanks to its simple structure, the expression for its QFI with respect to the estimated ω can be analytically derived

$$F_{\mathrm{Q}}[\Lambda_\omega^{\otimes N}(\psi_{\mathrm{GHZ}}^N)] = \frac{t^2 N^2 \eta_\perp^{2N}}{2^{-1-N}(A_{-,-}^N + A_{+,-}^N + A_{-,+}^N + A_{+,+}^N)}, \tag{12}$$

with $A_{\pm,\pm} = 1 \pm \eta_\parallel \pm \kappa$. Focusing, again for simplicity, on unital maps, expanding the above formula for short times and using optimal $t_{\mathrm{GHZ}}(N) = 1/(\alpha_l \beta_\perp N)^{1/\beta_\perp}$ that minimizes asymptotically the QCRB for the GHZ-based scenario, we arrive at

$$\lim_{N\to\infty} \frac{\Delta^2 \omega_N^{\mathrm{GHZ}} T}{N^{-(2\beta_\perp - 1)/\beta_\perp}} = (\alpha_l \beta_\perp e)^{1/\beta_\perp}. \tag{13}$$

For the semigroup case ($\beta_\perp = 1$), the asymptotic coefficient (13) differs by a factor e from D of Eq. (6)—a known fact for the pure dephasing model [11,13] which may be remedied by replacing GHZ with spin-squeezed states [17] —yet the discrepancy decreases with increasing $\beta_\perp$. Crucially, Eq. (13) proves that the $1/N^{(2\beta_\perp - 1)/\beta_\perp}$ scaling of the MSE predicted by Eq. (6) is always achievable when $\kappa = 0$; even more, such a claim can be proved to apply to all phase covariant maps [33].

Role of non-Markovianity and Zeno regime.—We have shown that, by going beyond the semigroup regime, one can overcome the SQL for a relevant class of open system dynamics. A natural question is whether non-Markovian features are of some relevance. Since non-Markovianity is typically associated with backflow of information to the system of interest [40,41], one may think that such recovered information (also about the estimated parameter) could be advantageous for metrological purposes, possibly leading to improved scalings of the precision. Our results clearly indicate that this is not the case. As any measurement strategy outside the short-time regime will be asymptotically bounded by a $1/N$ scaling, to beat the SQL one must perform measurements on shorter and shorter time scales as $N \to \infty$, whatever the subsequent memory effects are. The attainable asymptotic precision is then fully dictated by the time-inhomogeneous, i.e., nonsemigroup, nature of the dynamics.

The characterization of nonsemigroup dynamics is a complex task, which calls for a detailed knowledge of the environmental properties, as well as the interaction mechanism [9]. Yet, a general property of any evolution derived exactly from the global (system + environment) unitary dynamics is the quadratic decay of the survival probability at short time scales—the emergence of the so-called quantum Zeno regime [44]. For any phase covariant $\Lambda_\omega(t)$, such a quadratic decay implies that $\beta_\perp = 2$ [33] and Eq. (6) then reduces to $\lim_{N\to\infty} \Delta\omega_N^2 T N^{3/2} \ge \sqrt{\alpha}$. Thus, we can conclude that the ultimate $1/N^{3/2}$ precision scaling—provably attainable—is a general feature of any reduced dynamics exhibiting the Zeno regime and phase covariance. If we restrict to the specific case of pure dephasing, we provide a further confirmation of the conjecture made in [24] and, also, recently proved in [31].

Shabani-Lidar post-Markovian noise model.—To demonstrate the applicability of our methods to general phase covariant dynamics, we consider the post-Markovian model of Shabani and Lidar [45] (SL), that has been widely used to study nonsemigroup evolutions and their non-Markovian properties [46], showing, e.g., the nonequivalence between the trace distance [40] and the *CP* divisibility [41] definitions of quantum non-Markovianity. The SL model was originally formulated in terms of an integrodifferential equation, relying on a given Lindblad generator and a proper function memory kernel, the latter enclosing the memory effects due to the interaction with the bath. The SL model can also be formulated via a time-local master equation, which has the form as in Eq. (4) and contains all the emission, excitation, and dephasing contributions, yet it is fully described by three parameters: γ_0—dissipation constant, γ—the effective memory rate, and n—the mean number of excitations in the bath. The short-time expansions in Eq. (5) of all the noise parameters is quadratic in t, with coefficients: $\alpha_\parallel = 2\alpha_\perp = (2n+1)\gamma_0\gamma/2$ and $\alpha_\kappa = -\gamma\gamma_0/2$ [33]. The SL model, therefore, exhibits the Zeno regime, which imposes the $1/N^{3/2}$ asymptotic

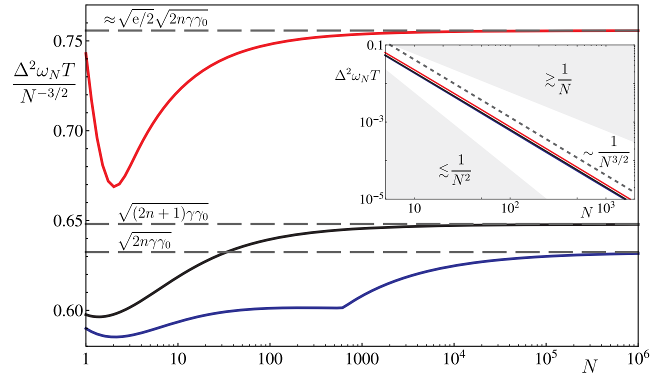

FIG. 3. Attainable precisions for the Shabani-Lidar noise model ($\gamma = 0.2$, $\gamma_0 = 0.1$ and $n = 10$). From bottom to top, we plot the obtained semianalytic (blue) and fully numerical SDP-based (black) precision bounds, as well as the exact MSE attainable with the GHZ states (red). The inset confirms that all three quickly saturate the $1/N^{3/2}$ scaling with N.

precision scaling with $D = \sqrt{2n\gamma\gamma_0}$ in Eq. (6). On the other hand, since the model yields a nonunital map, the GHZ-achievable asymptotic constant has to be computed numerically, and can be approximated by $\sqrt{e/2}D$. The SL model is well suited to verify the performance of our methods at finite N. In [33], we derive the corresponding analytic expressions for the bound in Eq. (10) (generalized to the nonunital case via the convexity of $F^{\uparrow}$) and the GHZ-attainable precision, both of which we numerically optimize over t for each N. We plot the results in Fig. 3 to show their convergence to the correct asymptotic, analytical expressions. Furthermore, we compare the semianalytical bound with its numerically optimized SDP-based version in Eq. (8), which achieves asymptotically only a slightly tighter $\sqrt{(2n+1)\gamma\gamma_0}$ constant. The inset clearly confirms the attainability of the $1/N^{3/2}$ precision scaling.

Conclusions.—We have derived a novel limit on the attainable precision in frequency estimation, which holds for all forms of phase covariant uncorrelated noise. Our results show that, despite the noiseless HL not being within reach, by exploiting the nonsemigroup, time-inhomogeneous system dynamics arising at short times in the Zeno regime, the asymptotic SQL-like scaling of precision can be beaten. Any measurement strategy performed on longer time scales is ultimately limited by the SQL, irrespectively of possible non-Markovian effects exhibited by the evolution. We leave, as an open question, whether the asymptotic precision can be further improved via general active-ancilla assisted schemes [47], where the interplay between multistep unitary operations and memory effects in the probes evolution has to be treated carefully [48].

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