Operational Resource Theory of Coherence

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We establish an operational theory of coherence (or of superposition) in quantum systems, by focusing on the optimal rate of performance of certain tasks. Namely, we introduce the two basic concepts —"coherence distillation" and "coherence cost"—in the processing quantum states under so-called incoherent operations [Baumgratz, Cramer, and Plenio, Phys. Rev. Lett. 113, 140401 (2014)]. We, then, show that, in the asymptotic limit of many copies of a state, both are given by simple single-letter formulas: the distillable coherence is given by the relative entropy of coherence (in other words, we give the relative entropy of coherence its operational interpretation), and the coherence cost by the coherence of formation, which is an optimization over convex decompositions of the state. An immediate corollary is that there exists no bound coherent state in the sense that one would need to consume coherence to create the state, but no coherence could be distilled from it. Further, we demonstrate that the coherence theory is generically an irreversible theory by a simple criterion that completely characterizes all reversible states.

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Introduction.—The universality of the superposition principle is the fundamental nonclassical characteristic of quantum mechanics: Given any configuration space X, its elements x label an orthogonal basis $|x\rangle$ of a Hilbert space, and we have all superpositions $\sum_x \psi_x |x\rangle$ as the possible states of the system. In particular, we could choose a completely different orthonormal basis to express the superpositions. However, often a basis is distinguished, be it the eigenbasis of an observable or of the system's Hamiltonian, so that conservation laws or even superselection rules apply. In such a case, the eigenstates $|x\rangle$ are distinguished as "simple" and superpositions are "complex." Indeed, in the presence of conservation laws, structured superpositions of eigenstates can serve as so-called "reference frames," which are resources to overcome the conservation laws [1–3]. Based on this idea, Åberg [4] and, more recently, Baumgratz *et al.* [5] have proposed considering any nontrivial superposition as a resource and creating a theory in which computational basis states and their probabilistic mixtures are free, and operations preserving these "incoherent" states are free as well. This suggests that coherence theory can be regarded as a resource theory.

Let us briefly recall the general structure of a quantum resource theory (QRT) and basic questions that should be asked in a QRT through entanglement theory, a well-known QRT. QRT has three ingredients: (1) free states (separable states), (2) resource states (entangled states), and (3) the restricted or free operations [local operations and classical communication (LOCC)]. A prerequisite for a consistent QRT is that no resource state can be created from any free state under any free operation. QRT is then the study of interconversion between resource states under free operations. The pure resource states play a special role and are much more preferable because, usually, they are used to circumvent the restriction on operations (Bell states are used in teleportation to overcome LOCC operations). So the conversions between pure resource states and mixed ones are a major focus of QRTs. A standard unit resource measure can be constructed if the conversions between pure states are asymptotically reversible (entropy of entanglement of a pure entangled state, with a Bell state as the unit). Then, there are two basic processes that are well motivated: one is the so-called resource distillation, the transformation from a mixed resource state to the unit resource, and the other is resource formation, the reverse transformation from the unit resource to a mixed state (entanglement distillation and entanglement formation). Because of the reversibility in the pure state conversion, we need not worry about what kind of pure state is the target, as they are equivalent up to the transformation rate between them. Thus, two well-motivated quantities arise from the two basic processes, distillable resource and resource cost, which have a clear operational interpretation (distillable entanglement and entanglement cost). The principal objective of the theory is the characterization of these two quantities. This is often a highly complex problem, but resource monotones yield various limits on possible transformations and achievable rates. Another basic question in any QRT is to ask whether the theory is reversible or not. If the conversion between pure states is reversible, then the reversibility problem is reduced to the question regarding whether or not the optimal conversion rate in the formation process is equal to that in the distillation. If a QRT is reversible, a unique resource measure exits, quantifying the conversion rate between different states, so that everything about resource transformations is clear and simple. However, if a QRT is irreversible, the phenomena are ample, and further interesting questions can be asked, for example, whether so-called bound resource states exist as an analogue of bound entangled states [6,7], in the sense that, from them, no resource can be distilled, but for which, in order to create them, a nonzero resource is required. Several QRTs were constructed along these lines, some of them, indeed, irreversible: entanglement theory (with respect to LOCC) [8,9], thermodynamics (with respect to energy-conserving operations and thermal states) [10,11], and reference frames (with respect to group-covariant operations) [12], etc.

In this Letter, we establish an operational coherence theory in the framework proposed in [5] and [4]. Namely, first, we show that the conversion between the pure coherent states is asymptotically reversible so that the standard unit coherence measure exists. Then, we introduce the basic transformation processes: coherence distillation and coherence formation, from which two basic coherence measures naturally arise: distillable coherence and coherence cost, with operational interpretations. Remarkably, both are given by single-letter formulas that, hence, make these quantities computable. These results in turn allow us to formulate a simple criterion to decide whether a given state is reversible or not and to show that there is no bound coherence. Although the main results are in the asymptotic setting, we also get the single copy conversion of pure states along the way. In the following, we state and discuss our results carefully, while all proofs are found in [13], Sec. B.

Coherence as a resource theory.—We follow the framework of coherence theory by Baumgratz *et al.* [5]. Let $\{|i\rangle\}$ be a fixed basis in the finite dimensional Hilbert space. The free states called incoherent states are those whose density matrices are diagonal in the basis, being of the form $\sum_i p_i|i\rangle\langle i|$ where p_i is a probability distribution, and the set of incoherent states is denoted as Δ. The resource states called coherent states are those not of this form. Quantum operations are specified by a set of Kraus operators $\{K_\ell\}$ satisfying $\sum_\ell K_\ell^\dagger K_\ell = \mathbb{1}$; a quantum operation can have many different Kraus representations. The free operations, called incoherent operations (IC), are those for which there exist a Kraus representation $\{K_\ell\}$ such that $(1/\mathrm{Tr}\rho K_\ell^\dagger K_\ell)K_\ell\rho K_\ell^\dagger \in \Delta$ for all ℓ and all $\rho\in\Delta$. Such a restriction guarantees that, even if one has access to individual measurement outcomes ℓ of the instrument $\{K_\ell\}$, one cannot generate coherent states from an incoherent state. Under this restriction, each Kraus operator is of the form $K_\ell = \sum_i c(i)|j(i)\rangle\langle i|$ where $j(i)$ is a function from the index set of the basis, and $c(i)$ are coefficients; we call such Kraus operators incoherent, too. If not only K_ℓ, but also $K_\ell^\dagger$ is incoherent, we call it strictly incoherent, and the corresponding operation a strictly incoherent operation. Strictly incoherent K_ℓ are characterized by a one-to-one $j(i)$ function. Another equivalent form of a general incoherent Kraus operator is $K=\sum_j|j\rangle\langle\gamma_j|$, with $|\gamma_j\rangle\in\mathrm{span}\{|i\rangle : i\in S_j\}$ for a partition $[d]=\dot{\bigcup}_j S_j$.

We introduce some notation: $\rho\overset{\mathrm{IC}}{\mapsto}\sigma$ means there is an incoherent operation T such that $\sigma = T(\rho) = \sum_\ell K_\ell\rho K_\ell^\dagger$; if T is strictly incoherent, we write $\rho\overset{\mathrm{IC}_0}{\mapsto}\sigma$. If the transformation is obtained probabilistically, i.e., if $\sigma\propto K_\ell\rho K_\ell^\dagger$ and $\mathrm{Tr}K_\ell\rho K_\ell^\dagger\neq 0$ for some ℓ, we write $\rho\overset{\mathrm{pIC}}{\mapsto}\sigma$ and $\rho\overset{\mathrm{pIC}_0}{\mapsto}\sigma$, for probabilistic incoherent and probabilistic strictly incoherent transformations, respectively. We define the decohering operation $\Delta(\rho)=\sum_i\langle i|\rho|i\rangle|i\rangle\langle i|$, i.e., the diagonal part of the density matrix. This makes the incoherent states Δ the image of the map Δ, thus, justifying the slight abuse of notation.

When we consider composite systems, we simply declare as incoherent the tensor product basis of the local bases; the incoherent operations are then defined with respect to the tensor product basis. Notice that there are several special incoherent transformations: phase and permutation unitaries. In particular, in a two-qudit system, controlled NOT (CNOT): $|i\rangle|j\rangle\mapsto|i\rangle|(i+j)\mathrm{mod}\, d\rangle$ is an incoherent operation, as it is simply a permutation of the tensor product basis vectors [28,29]. In [13], Sec. C, we discuss a slightly more general and flexible model.

Pure state transformations.—We start by developing the theory of pure state transformations; the main result, in this context, is that, in the asymptotic setting of many copies, this becomes reversible, the rates governed by the entropy of the decohered state, a quantity we dub entropy of coherence. First, however, we review the situation for exact single-copy transformations.

Observe a simple fact on ranks: Let φ be transformed to ψ by an incoherent, or more generally, a probabilistic incoherent operation, $|\psi\rangle\propto K|\varphi\rangle\neq 0$, for $K=\sum_i c_i|j(i)\rangle\langle i|$. The rank r of $\Delta(\varphi)$ is precisely the number of nonzero diagonal entries of $\Delta(\varphi)$, which is the number of nonzero terms in $|\varphi\rangle = \sum_{i\in R}\varphi_i|i\rangle$, $|R|=r$. Thus, $|\psi\rangle\propto K|\varphi\rangle = \sum_{i\in R}\varphi_i c_i|j(i)\rangle$ has at most $r=|R|$ terms. This proves the following.

Lemma 1.—If $\varphi\overset{\mathrm{pIC}}{\mapsto}\psi$, then $\mathrm{rank}\Delta(\psi)\le\mathrm{rank}\Delta(\varphi)$, i.e., the rank of the diagonal part of pure states cannot increase under incoherent operations.

Theorem 2.—(cf. Du *et al.* [30].) For two pure states ψ and φ, if $\Delta(\psi)\succ\Delta(\varphi)$, then there is an incoherent (in fact, a strictly incoherent) operation: $\varphi\overset{\mathrm{IC}_0}{\mapsto}\psi$. Conversely, if $\varphi\overset{\mathrm{IC}_0}{\mapsto}\psi$, or if $\varphi\overset{\mathrm{IC}}{\mapsto}\psi$ and, in addition, $\mathrm{rank}\Delta(\varphi)=\mathrm{rank}\Delta(\psi)$, then $\Delta(\psi)\succ\Delta(\varphi)$.

Here, the majorization relation for matrices $\rho\succ\sigma$, means that $\mathrm{spec}(\rho)=\vec{p}=(p_1\ge\cdots\ge p_d)$ and $\mathrm{spec}(\sigma)=\vec{q}=(q_1\ge\cdots\ge q_d)$ are in majorization order [31–33]: $\forall\ t<d$, $\sum_{i=1}^t p_i\ge\sum_{i=1}^t q_i$, and $\sum_{i=1}^d p_i=\sum_{i=1}^d q_i$.

As a consequence of Theorem 2, just as for pure state entanglement [33], there is catalysis for pure state incoherent transformations, cf. [34]; in [35], examples, such that initial and final states have equal rank, are given. An immediate corollary of Theorem 2 is the following.

Corollary 3.—(Baumgratz *et al.* [5]) Let ρ be a state in $\mathbb{C}^d$, and $\Phi_d = |\Phi_d\rangle\langle\Phi_d| = (1/d)\sum_{i,j=0}^{d-1}|i\rangle\langle j|$. Then, there is an incoherent operation transforming Φ_d to ρ.

This motivates the name maximally coherent state for Φ_d. In addition to enabling the creation of arbitrary d-dimensional coherent superpositions by incoherent means, Φ_d also allows the implementation of arbitrary unitaries $U \in \mathrm{SU}(d)$ [5]. Then, fixing the qubit maximally coherent pure state $\Phi_2 = \frac{1}{2}\sum_{ij=0}^{1}|i\rangle\langle j|$ as a unit reference, we are ready to consider asymptotic pure state transformations with vanishing error as the number of copies goes to infinity. As in information theory, entanglement theory, and other similar cases (cf. [36,37]), this simplifies the picture dramatically. Special cases are coherence concentration, the transformation from a nonmaximally coherent pure state to the unit coherent state, and coherence dilution, the reverse one. To express our result, we introduce the entropy of coherence for pure states as $C(\psi) = S(\Delta(\psi))$. Here, $S(\rho) = -\mathrm{Tr}\rho\log\rho$ is the von Neumann entropy, where logarithms are to base 2. Note, the maximum value of this functional among d-dimensional states is attained on Φ_d: $C(\Phi_d) = \log d$, in particular $C(\Phi_2) = 1$ for the unit coherence resource.

Theorem 4.—(Yuan *et al.* [38]) For two pure states ψ and φ and a rate $R \geq 0$, the asymptotic incoherent transformation $\psi^{\otimes n} \overset{\mathrm{IC}}{\mapsto} \overset{1-\epsilon}{\approx} \varphi^{\otimes nR}$ as $n \to \infty, \epsilon \to 0$, is possible if $R < [C(\psi)/C(\varphi)]$ and impossible if $R > [C(\psi)/C(\varphi)]$.

In particular, ψ can be asymptotically reversibly transformed into Φ_2, and vice versa, at the optimal rate $C(\psi)$. Here, $\rho \overset{1-\epsilon}{\approx} \sigma$ signifies that the two states have high fidelity: $F(\rho,\sigma) \geq 1-\epsilon$ (see [13], Sec. A, for details).

Now, we are ready to introduce two fundamental tasks for arbitrary mixed states, namely asymptotic distillation of $\rho^{\otimes n}$ to $\Phi_2^{\otimes nR}$ and the reverse the process of formation $\rho^{\otimes n}$ from $\Phi_2^{\otimes nR}$. Note, in this respect, the fundamental importance of Theorem 4, which shows that we could equivalently select any pure coherent state ψ as a unit reference for formation and distillation, and all rates would change by the same factor $[1/C(\psi)]$. It turns out that both quantities have single-letter, additive expressions: the former is given by the relative entropy of coherence, the latter by the coherence of formation; both are additive. This is in marked contrast to other resource theories, perhaps most prominently, entanglement theory, in which the basic operational tasks are only characterized by regularized formulas, and the fundamental quantities, such as entanglement of formation [39], relative entropy of entanglement [40], etc., are not additive [41,42].

Distillable coherence.—The distillation process is the process that extracts pure coherence from a mixed state by incoherent operations. The distillable coherence of a state is the maximal rate at which Φ_2 can be obtained from the given state.

Definition 5.—The distillable coherence of a state ρ is $C_d(\rho) = \sup R$, subject to $\rho^{\otimes n} \overset{\mathrm{IC}}{\mapsto} \overset{1-\epsilon}{\approx} \Phi_2^{\otimes nR}$ as $n \to \infty, \epsilon \to 0$.

By definition, C_d naturally has an operational meaning as the optimal rate of performance at a natural task. Theorem 6 shows that the distillable coherence is given by a closed-form expression.

Theorem 6.—For any state ρ, the distillable coherence is given by the relative entropy of coherence: $C_d(\rho) = C_r(\rho) := \min_{\sigma\in\Delta} S(\rho\|\sigma) = S(\Delta(\rho)) - S(\rho)$.

Here, $S(\rho\|\sigma) = \mathrm{Tr}\rho(\log\rho - \log\sigma)$ is the quantum relative entropy. The relative entropy of coherence $C_r(\rho)$ is introduced and studied in detail in [4,5]: first, it has a closed formula $C_r(\rho) = S(\Delta(\rho)) - S(\rho)$; second, it is convex in the state; third, it is a coherence monotone, meaning that $C_r(\rho) \geq C_r(T(\rho))$ for any incoherent transformation $T(\rho) = \sum_\ell K_\ell \rho K_\ell^\dagger$. In fact, it is even strongly monotonic [5]: $C_r(\rho) \geq \sum_\ell p_\ell C_r(\rho_\ell)$, where $p_\ell\rho_\ell = K_\ell\rho K_\ell^\dagger$. But it is only due to Theorem 6 that we can give it a clear operational interpretation in the distillation process. Note that, in concurrent independent work, Singh *et al.* [43] show that the same quantity, $C_r(\rho)$, also arises as the minimum amount of incoherent noise that has to be applied to the state to decohere it.

Coherence cost.—The formation process is that which prepares a mixed state by consuming pure coherent states under incoherent operations. The coherence cost is the minimal rate at which Φ_2 has to be consumed for preparing the given state.

Definition 7.—The coherence cost of a state is defined as $C_c(\rho) = \inf R$, subject to $\Phi_2^{\otimes nR} \overset{\mathrm{IC}}{\mapsto} \overset{1-\epsilon}{\approx} \rho^{\otimes n}$ as $n \to \infty, \epsilon \to 0$.

The next result shows that the coherence cost has a single-letter formula, involving a simple entropy optimization.

Theorem 8.—For any state ρ, the coherence cost is given by the coherence of formation, $C_c(\rho) = C_f(\rho)$, where $C_f(\rho) := \min\sum_i p_i S(\Delta(\psi_i))$ subject to $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$.

The coherence of formation is introduced in [4], where its convexity and monotonicity were observed (see [13], Lemma 13); however, its additivity was not remarked. *A priori*, one might have expected the cost to be given by the regularization of the coherence of formation, which would have involved infinitely many optimization problems so that the evaluation of the operational cost had become infeasible. Indeed, it is additivity that makes the single-letter formulas in Theorem 6 and Theorem 8 available. Because of its importance, we state it as a theorem.

Theorem 9.—Both C_f and C_r are additive, i.e., $C_f(\rho\otimes\sigma) = C_f(\rho) + C_f(\sigma)$, $C_r(\rho\otimes\sigma) = C_r(\rho) + C_r(\sigma)$.

Theorem 9 also means that the distillable coherence and the coherence cost are also additive, so there are no superadditivity or activation phenomena in the resource theory of coherence, unlike the theory of entanglement [41,42], that of communication via channels [42,44], and many other resource theories, where the yield (cost) of two resources together may be strictly larger (smaller) than the sum of the yields (costs) of the resources processed individually.

Based on the formulas for the distillable coherence and the coherence cost, we are ready to characterize precisely the (ir-)reversible states.

(Ir-)reversibility.—From the definitions of C_d and C_c, it is immediate that $C_d(\rho) \le C_c(\rho)$. A state is reversible if the equality holds, otherwise, it is irreversible. From Theorem 4, we see that all pure states are asymptotically reversible. In the mixed state case, Theorem 10 provides a simple criterion to decide whether the given state is reversible or not that completely characterizes all the reversible states. From this, we conclude that the mixed states are generically irreversible and the coherence theory is an irreversible resource theory. However, in contrast to the irreversibility in entanglement theory [6,7,45], there is no "bound coherence" (as an analogue of "bound entanglement" [6,7]), from which no coherence could be distilled, but for which, in order to create it, nonzero coherence would be required.

Theorem 10.—A mixed state is reversible if and only if its eigenvectors are supported on the orthogonal subspaces spanned by a partition of the incoherent basis. That is, $\rho = \bigoplus_j p_j |\phi_j\rangle\langle\phi_j|$, each $|\phi_j\rangle \in \mathcal{H}_j = \mathrm{span}\{|i\rangle : i \in S_j\}$ and $\mathcal{H} = \bigoplus_j \mathcal{H}_j$.

Note that the criterion is easy to check, and contrast this with the entanglement irreversibility of two-qubit maximally correlated states in [46], where Wootters' formula [47] for calculating the entanglement of formation is used: Here, we do not need the explicit formula of the coherence of formation, which involves an optimization problem itself, and indeed, we do not know such a formula in high dimension. However, the equality constraint is so severe that we can learn the structure of the state.

Theorem 11.—There is no bound coherence: $C_d(\rho) = 0$ implies $C_c(\rho) = 0$. In other words, every state with any coherence (nonzero off-diagonal part) is distillable.

Discussion.—We have shown that the incoherent operations proposed by Baumgratz *et al.* [5] give rise to a well-behaved operational resource theory of coherence. Remarkably, almost all basic questions in this resource theory have simple answers. We saw that it is a theory without bound coherence, but exhibiting generic irreversibility for transformations between pure and mixed states. This should be contrasted with the general abstract framework of Brandão and Gour [48], which applies to the present theory of coherence, but rather than the incoherent operations considered here, requires all completely positive trace-preserving maps $\mathcal{E}$ such that the weaker condition $\mathcal{E}(\Delta) \subset \Delta$ holds. Both the distillable coherence and the coherence cost under this relaxed premise become $C_r(\rho)$, meaning that, while we cannot distill more efficiently with this broader class of operations, formation becomes cheaper.

One curious observation is that the resulting theory of coherence resembles so closely the entanglement theory of maximally correlated states (MCS). Indeed, under the correspondence $\rho = \sum_{ij}\rho_{ij}|i\rangle\langle j| \leftrightarrow \tilde{\rho} = \sum_{ij}\rho_{ij}|ii\rangle\langle jj|$, $C(\psi)$ is identified with $E(\tilde{\psi})$, the entropy of entanglement, $C_c(\rho) = C_f(\rho)$ with $E_c(\tilde{\rho}) = E_f(\tilde{\rho})$, the entanglement cost (which equals the entanglement of formation for MCS [46]), and $C_d(\rho) = C_r(\rho)$ with $E_d(\tilde{\rho}) = E_r(\tilde{\rho})$, the distillable entanglement (which equals the relative entropy of entanglement for MCS [49]). Indeed, this answers all the basic asymptotic questions in the theory, which is much simpler than general entanglement theory. What is missing to elevate this correspondence from an observation to a theoretical explanation (cf. [29]) is a matching correspondence between incoherent operations and LOCC operations that would truly show that the two theories are equivalent. A notable gap in the above correspondence is the non-asymptotic theory of pure states: The diagonal entries of a pure state correspond to the Schmidt coefficients of the associated pure entangled state, and, by Nielsen's theorem [33], a pure state can be transformed by LOCC into another one if and only if the Schmidt vectors are in majorization relation. In the case of incoherent operations on pure states, we only know that majorization is sufficient for transformability but not whether it is necessary.

To study this correspondence further, it might be worth investigating the optimal conversion rates R of incoherent transformations $\rho^{\otimes n} \overset{\mathrm{IC}}{\mapsto} \overset{1-\epsilon}{\approx} \sigma^{\otimes nR}$ for general mixed states, and for which, for the moment, we can get bounds: $[C_r(\rho)/C_f(\sigma)] \le R \le \min\{[C_r(\rho)/C_r(\sigma)], [C_f(\rho)/C_f(\sigma)]\}$.

Given the close resemblance to entanglement theory, we may expect that one-shot coding theorems, as well as finite block length analyses, can be carried out. As in one-shot information theory [50], we expect that min- and max-entropies and relative entropies and Rényi (relative) entropies govern the optimal rates, which now would carry an explicit dependence on the protocol error.

It remains to be seen whether similarly complete theories of asymptotic operational transformations can be carried out for other resource theories, such as that of reference frames [12]. Observe that, as reference frame theories are built on group actions under which the free states are precisely the invariant ones, the present theory of coherence may be viewed as the special case of the group of the diagonal phase unitaries.

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