Rigorous Bound on Energy Absorption and Generic Relaxation in Periodically Driven Quantum Systems

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We discuss the universal nature of relaxation in isolated many-body quantum systems subjected to global and strong periodic driving. Our rigorous Floquet analysis shows that the energy of the system remains almost constant up to an exponentially long time in frequency for arbitrary initial states and that an effective Hamiltonian obtained by a truncation of the Floquet-Magnus expansion is a quasiconserved quantity in a long time scale. These two general properties lead to an intriguing classification on the initial stage of relaxation, one of which is similar to the prethermalization phenomenon in nearly integrable systems.

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Introduction.—In periodically driven many-body quantum systems, excited states as well as the ground state participate in the dynamics, and nontrivial macroscopic phenomena can appear. Recent years have witnessed remarkable experimental developments, such as the discoveries of the Higgs mode in the oscillating order parameter of the superconducting material under a terahertz laser [1], and the Floquet topological states in the periodically driven cold atom [2–5]. Periodic driving in isolated quantum systems sometimes generates unexpected dynamical phenomena, even if the instantaneous Hamiltonian at each time step is simple. To name only a few, dynamical localization [6–8], coherent destruction of tunneling [7–9], dynamical freezing [10,11], and dynamical phase transitions [12–14] are remarkable far-from-equilibrium phenomena that cannot be captured within linear-response analysis.

On the other hand, as recently discussed in the context of *thermalization*, careful consideration is necessary on the true steady state in driven many-body systems [15–19]. Thermalization in isolated quantum systems has become one of the critical subjects in modern physics [20–25]. The first study was made by von Neumann early in 1929 [26], and now we are on a new stage by incorporating many concepts, including quantum entanglement [27] and experiments [28]. In the case without driving fields, the notion of the eigenstate thermalization hypothesis (ETH) is a key idea [20,21,23,26] that states that each energy eigenstate is indistinguishable from the microcanonical ensemble with the same energy. As a generalization of ETH to periodically driven systems, the Floquet ETH was proposed, which states that all of the Floquet eigenstates look the same and are indistinguishable from the infinite-temperature (i.e., completely random) state [16,19,29–31]. This leads to the conclusion that, in general, periodically driven many-body systems will eventually reach the steady state of infinite temperature, although several exceptions exist [15,17–19].

The question that follows is on the time scale to reach the steady state. Recent experiments seem to urge us to clarify the general aspects of the time scale especially for the strong amplitude of global driving, where nontrivial transient dynamics is anticipated. We note that most nontrivial dynamical phenomena in driven systems are far-from-equilibrium effects that cannot be analyzed within linear-response analysis. Hence, in this Letter, we for the first time aim to find the universal nature of the relaxation to the steady state under *strong and global driving*. This direction is clearly crucial for a deeper understanding of thermalization and for analyzing the stability of transient dynamics in experiments.

For this aim, we focus on the Floquet Hamiltonian H_F, which plays a central role in periodically driven systems:

$$e^{-iH_F T} \equiv \mathcal{T} e^{-i\int_0^T dt H(t)}, \tag{1}$$

where $H(t)$ is the Hamiltonian of the system, $\mathcal{T}$ is the time-ordering operator, and T is the period of the driving ($\hbar = 1$ throughout this Letter). The Floquet Hamiltonian is an effective Hamiltonian that contains full information on the stroboscopic dynamics. The Floquet-Magnus (FM) expansion is a formal expression for the Floquet Hamiltonian: $H_F = \sum_{m=0}^{\infty} T^m \Omega_m$ [32,33]. The explicit form of Ω_m is given in Eq. (8) below. However, it has recently been recognized that using a full series expansion is problematic since it is not convergent in general. The convergence radius shrinks as the system size increases [33]. Instead, we here use the technique of truncation in the FM expansion, which was recently developed for describing the Floquet Hamiltonian for transient time scales [34,35]:

$$e^{-iH_F^{(n)} T} \simeq e^{-iH_F T}, \quad \text{where } H_F^{(n)} = \sum_{m=0}^{n} T^m \Omega_m. \tag{2}$$

Here, $H_F^{(n)}$ is the nth order truncated Floquet Hamiltonian. There are several studies which show that the time evolution by the truncated Floquet Hamiltonian is reliable up to a certain long time τ for a driving with the high frequency $\omega = 2\pi/T$ [33], $\tau \sim \omega^{1/2}$ for the Friedrichs model on the continuous space [34], and $\tau \sim \exp[O(\omega)]$ for lattice systems when driving is local [35] or interactions are short ranged [35–37]. In this Letter, we use the truncation technique for high-frequency driving.

With this technique, two main findings are presented. We show as the first result that in the case of a high-frequency driving, the truncated Floquet Hamiltonian is a quasiconserved quantity (a quantity that is almost conserved in a long time scale). We also show as the second result that energy-absorption rate per one site is bounded for an arbitrary amplitude of driving and for arbitrary initial states:

$$\dot{E}/N \lesssim (N_V/N)\exp[-O(\omega/g)], \tag{3}$$

where E and N are the total energy and the number of lattice sites, respectively, and g is the maximum energy per one site. The driving field is applied to N_V sites. This provides a criterion on the stability of transient quantum dynamics in experiments. These two findings lead to an intriguing classification on the relaxation processes, one of which is similar to the prethermalization phenomenon seen in nondriven nearly integrable systems [38–41]; see Refs. [42,43] for recent relevant numerical calculations.

Setup and numerical example.—We consider a quantum spin system defined on a lattice with N sites in an arbitrary dimension, whose Hamiltonian is written as

$$H(t) = H_0 + V(t). \tag{4}$$

The driving field $V(t)$ is applied to $N_V(\leq N)$ sites and satisfies the periodicity in time $V(t) = V(t+T)$ with zero average over the single period. We mainly focus on the regime of high frequency $\omega = 2\pi/T$. Each lattice site $i = 1, 2, \ldots, N$ has its own spin. The basic assumption on the Hamiltonian is that it is expressed in the form of

$$H(t) = \sum_{X:|X|\leq k} h_X(t), \tag{5}$$

where $X = \{i_1, i_2, \ldots, i_{|X|}\}$ is a set of the lattice sites, with $|X|$ being the number of sites in X and $h_X(t)$ an operator acting on the sites in X. In addition, we assume that the single-site energy is bounded in the sense that,

$$\text{for any site } i, \qquad \sum_{X:X\ni i} \|h_X(t)\| \leq g, \tag{6}$$

with some fixed positive constant g, where $\|\cdot\|$ denotes the operator norm.

The form of Eq. (5) means that the Hamiltonian contains at most k-body interactions. For most physical applications, we can consider the case of $k=2$. In the case of spin-(1/2) systems, the most general form of the Hamiltonian (5) with $k=2$ is

$$H(t) = \sum_{i=1}^{N} \boldsymbol{B}_i(t)\cdot\boldsymbol{\sigma}_i + \sum_{i<j}^{N} \sum_{\alpha,\gamma=x,y,z} J_{ij}^{\alpha\gamma}(t)\sigma_i^\alpha\sigma_j^\gamma, \tag{7}$$

where $\boldsymbol{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$ is the Pauli matrix of the ith spin, $\boldsymbol{B}_i(t)$ is the local magnetic field at the ith site, and $J_{ij}^{\alpha\gamma}(t)$ denotes the interaction between the ith and jth spins. We can explicitly confirm that this Hamiltonian can be brought into the form of Eq. (5) by putting $h_{\{i\}} = \boldsymbol{B}_i\cdot\boldsymbol{\sigma}_i$ and $h_{\{i,j\}} = \sum_{\alpha,\gamma=x,y,z} J_{ij}^{\alpha\gamma}\sigma_i^\alpha\sigma_j^\gamma$.

To make clear the physical phenomena that we address, we show a numerical example with a toy model that has been used to show the Floquet ETH in Ref. [31]. We consider the dynamics over one cycle, taking $H(t) = H_z$ for the first half period and $H(t) = H_x$ for the second half period, where $H_z = \sum_{i=1}^N [-J\sigma_i^z\sigma_{i+1}^z + B_z\sigma_i^z]$ with the periodic boundary condition and $H_x = B_x\sum_{i=1}^N \sigma_i^x$. We calculate the time evolution of the z component of the first spin σ_1^z setting each spin-down state as the initial state. Floquet ETH implies that a steady state in the longtime limit is a random state, and hence, when it is satisfied, the expectation value of any local spin operator eventually reaches zero. In Fig. 1, the time evolution for a sufficiently large system size is shown. Figs. 1(a), 1(b), and 1(c) are the time evolution of σ_1^z in the large time scale, the initial stage, and the transient time scale, respectively. Figure 1(a) shows a vanishing expectation value that is a clear indication of

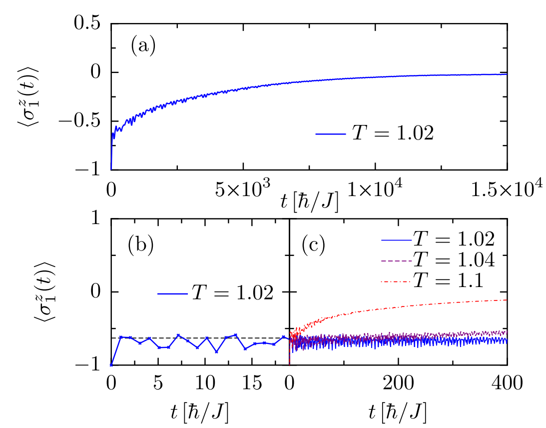

FIG. 1. Numerical demonstration of prethermalizationlike phenomenon. (a) Relaxation in the long time scale. (b) Initial relaxation. (c) Transient time evolution after the initial one. Parameters are $(J, B_x, B_z) = (1, 0.9045, 0.8090)$ and $N = 24$. The dotted line in (b) is the expectation value in the equilibrium state of $H_F^{(0)}$ at the inverse temperature $\beta = 0.85$, which is determined from the expectation value of $H_F^{(0)}$ at $t = 10$.

the Floquet EHT. Crucial observation is that after the initial relaxation [Fig. 1(b)], the expectation value is almost constant for finite transient time scales, and the time scales depend on the period T [Fig. 1(c)]. This implies that the heating process is seemingly suppressed during this time scale. This is somewhat similar to the prethermalization phenomenon in nondriven nearly integrable systems. In experimental situations, this transient time behavior is crucial, and hence we address the mechanism of the behavior and consider the period dependence on the time scale.

Time scale of the heating process.—We use the FM expansion for analyzing the energy absorption and the relaxation process. The FM expansion is the formal expansion of the Floquet Hamiltonian given by $H_F = \sum_{n=0}^{\infty} T^n \Omega_n$, with $\Omega_0 = H_0$ and the nth order coefficient Ω_n for $n \geq 1$ being given by [44]

$$\Omega_n = \sum_{\sigma} \frac{(-1)^{n-\theta[\sigma]}\theta[\sigma]!(n-\theta[\sigma])!}{i^n (n+1)^2 n! T^{n+1}} \int_0^T dt_{n+1} \cdots \times \int_0^{t_2} dt_1 [H(t_{\sigma(n+1)}), [H(t_{\sigma(n)}), \ldots, [H(t_{\sigma(2)}), H(t_{\sigma(1)})] \ldots]], \tag{8}$$

where σ is a permutation and $\theta[\sigma] = \sum_{i=1}^{n} \theta(\sigma(i+1) - \sigma(i))$ with $\theta(\cdot)$ is the step function. It is believed that the FM expansion is divergent in many-body interacting systems [19,29,33]. See the Supplemental Material for the numerical demonstration of the divergence [45]. This divergence is not merely a mathematical phenomenon but is now thought to be an indication of a heating process due to periodic driving [19,29,33].

We define the nth order truncation of the FM expansion as in Eq. (2) and show that, for general spin systems, the time scale of the heating is exponentially slow in frequency. To this end, we start with an intuitive explanation on our analysis. From Eq. (8), Ω_n has, at most, $(n+1)k$-spin effective interactions because of the multiple commutators in Eq. (8), which describes the collective flip of $(n+1)k$ spins. Since the energy exchange between a quantum system and the external periodic field is quantized into integer multiples of ω and the energy of each spin is bounded by g, $N^* \sim \omega/g$ spins must flip cooperatively in order to absorb or emit the single "energy quantum." Such a process is taken into account only in the terms higher than the n_0th order in the FM expansion, with $n_0 \sim N^*/k \sim \omega/(gk)$. Indeed, each term of the FM expansion is rigorously bounded from above as

$$\|\Omega_n\| T^n \leq 2gN_V \frac{(2gkT)^n n!}{(n+1)^2}. \tag{9}$$

This is given by estimating norms of the multiple commutators taking into account that the Hamiltonian has, at most, k-body interaction and the energy per site is bounded by g [35,45]. Equation (9) shows that the FM expansion (2) looks convergent up to $n \leq n_0 \sim \omega/(gk)$ and

$$\|H_F^{(n)} - H_F^{(n_0)}\| = N_V O(T^{n+1}) \quad (n < n_0), \tag{10}$$

but it grows rapidly for $n > n_0$. Therefore, we can eliminate the heating effect most efficiently by truncating the FM expansion at $n = n_0$. The time scale of the heating is thus evaluated by comparing the difference between the exact time evolution and the approximate time evolution under the n_0th order truncated Floquet Hamiltonian. It is expected that higher-order terms (i.e., simultaneous flip of a large number of spins) would matter only in the later stage of the time evolution.

We now make the above argument mathematically rigorous. We can prove the following theorem.

Theorem.—The n_0th order truncated Floquet Hamiltonian $H_F^{(n_0)}$ is almost conserved up to an exponentially long time in frequency in the sense that

$$\|H_F^{(n_0)}(t) - H_F^{(n_0)}\| \leq 16g^2 k 2^{-n_0} N_V t, \tag{11}$$

where $t = mT$ with a positive integer m, $n_0 = \lfloor 1/(8gkT) - 1 \rfloor$, and $H_F^{(n_0)}(t) = U^\dagger(t) H_F^{(n_0)} U(t)$ is the n_0th order truncated Floquet Hamiltonian at time t in the Heisenberg picture, with $U(t) = \mathcal{T} e^{-i\int_0^t dt' H(t')}$.

This is derived by evaluating the norm of the Dyson expansion for the time-evolution operators on the left-hand side, taking into account that the Hamiltonian is written as Eq. (5) with Eq. (6). See the Supplemental Material for more details on the derivation [45]. Combined with Eq. (10), this theorem leads to

$$\|H_F^{(n)}(t) - H_F^{(n)}\| \leq 16g^2 k 2^{-n_0} N_V t + N_V O(T^{n+1}) \tag{12}$$

for any $n < n_0$. In particular, by substituting $n = 0$, we obtain

$$\frac{1}{N}\|H_0(t) - H_0\| \leq \frac{N_V}{N}[16g^2 k 2^{-n_0} t + O(T)]. \tag{13}$$

It is noted that the term $O(T)$ in Eq. (13) is independent of t. Thus, the energy density remains constant within a small fluctuation of $O(T)$ for an exponentially long time in frequency. Equation (11) provides a *lower bound* on the time scale during which $H_F^{(n_0)}$ can be an approximately conserved quantity. This quasiconserving property lasts during a time scale larger than $\tau \sim 2^{n_0} \sim e^{O(\omega)}$. Similarly, Eq. (13) implies that the lower bound of the time scale of heating is an exponentially long time in frequency. This is a main result (3). The exponentially long time scale of the energy relaxation was shown for short-range interacting spin systems in the linear-response regime in Ref. [46], but it should be emphasized that Eq. (13) has been obtained without assuming short-range interactions and the linear-response argument. See Ref. [47] for a recent numerical result.

It is remarked that for a local driving with $N_V \lesssim e^{O(\omega)}$, a much stronger result was shown in Ref. [35], i.e.,

$$\|\mathcal{T}e^{-i\int_0^t H(s)ds} - e^{-iH_F^{(n_0)}t}\| \leq \exp[-O(\omega)]t, \qquad (14)$$

for $t = mT$. This inequality implies that, for any bounded operator that may be highly nonlocal, the FM truncated Hamiltonian gives the accurate time evolution for an exponentially long time. In the case of the global driving $N_V \propto N$, this strong inequality (14) is not satisfied for sufficiently large systems, but, even in this case, we can utilize the finite order truncation of the FM expansion to discuss the relaxation process, as is argued below.

Relaxation process.—Our rigorous result enables us to discuss possible scenarios on the initial stage of relaxation. According to the Floquet ETH [19,29–31], the steady state in the longtime limit induced by the Floquet Hamiltonian (1) is a state of infinite temperature. A full FM series expansion, in general, diverges in large quantum systems and hence it is not useful [33,35]. However, the truncated Floquet Hamiltonian $H_F^{(n_0)}$ is a quasiconserved quantity and plays a crucial role in the relaxation process.

We make a remark on the degree of nonlocality on the quasiconserved quantity. The n_0th order truncated Floquet Hamiltonian has effective $(n_0+1)k$-body interactions, and hence the nonlocality looks large. However, for a high-frequency driving, higher-order contributions in the FM expansion are very small since T is small. The dominant contribution is in fact the original Hamiltonian H_0. Hence, nonlocality of the truncated Floquet Hamiltonian is not very strong. Eigenstates for the truncated Floquet Hamiltonian $H_F^{(n_0)}$ thus should satisfy the usual ETH, not the Floquet ETH. In addition, we should note that from Eq. (10), $H_F^{(n)} \approx H_F^{(n_0)}$ for any $n < n_0$. Hence, these truncated Floquet Hamiltonians are not independent but are almost the same. Practically, one can approximate the quasiconserved quantity $H_F^{(n_0)}$ by $H_0(= H_F^{(0)})$.

Taking account of those, we discuss a scenario on the initial stage of relaxation. Since the quasiconserved quantity exists with a long lifetime, the system relaxes to a *quasistationary state* characterized by the quasiconserved quantity, which will be close to a state corresponding to the (micro-)canonical ensemble $\rho_{\rm eq}^{(n_0)}$ of the effective Hamiltonian $H_F^{(n_0)}$ set by the initial state. Approximately, one can use $\rho_{\rm eq}^{(0)}$ (the equilibrium ensemble of H_0) instead of $\rho_{\rm eq}^{(n_0)}$ because $H_F^{(n)} \approx H_F^{(n_0)}$ for any $n < n_0$.

The initial stage of relaxation can be classified into two cases: i.e., (i) the case where the relaxation to the quasistationary state is faster than the energy relaxation, and (ii) the case where both relaxation times are comparable. In case (i), the system first reaches the quasistationary state, then relaxes to the true steady state. This is highly related to the prethermalization phenomenon in the isolated nearly integrable systems [40,41], where the system first relaxes to a quasistationary state corresponding to the generalized Gibbs ensemble and then relaxes to the true steady state. This is what we numerically observed in Figs. 1(b) and 1(c). Remarkably, in Figs. 1(b) and 1(c), $\langle\sigma_1^z(t)\rangle \approx -0.65$ in the quasistationary state, which is close to ${\rm Tr}\rho_{\rm eq}^{(0)}\sigma_1^z$ [the dotted line in Fig. 1(b)] at the inverse temperature $\beta = 0.85$ that is determined from the expectation value of $H_F^{(0)}$ at $t = 10$. This fact indicates that the quasistationary state is actually described by $\rho_{\rm eq}^{(0)}$ in this model.

In case (ii), on the other hand, the relaxation process towards the quasistationary state and that towards the true steady state are indistinguishable, and hence stable quasistationary behavior is not observed at the initial stage of relaxation.

Because the time scale of energy relaxation becomes longer exponentially as the frequency increases, we expect to find case (i) for sufficiently high frequencies unless there is some special reason such as conservation laws [15,17], strong quenched disorder [18,19], diverging time scale due to quantum criticality [48], and so on.

Our analysis deals with general spin models, which makes clear why the heating is slow in a precise manner and leads us to the universal scenario of relaxation processes. However, in our evaluation, the single-site energy is overestimated and the effect of quantum interference is underestimated. Hence, the divergence of the FM expansion presumably begins at a higher order than our estimation $n_0 \approx 1/(8gkT)$. We stress that our estimation on the time scale is a *rigorous lower bound* that can be exponentially large in frequency, and hence the actual time scale of the heating will be longer than our estimation [49]. In order to obtain a quantitatively accurate estimate for a specific model, we will have to study the quantum dynamics of the given model numerically.

Related to the above remark, we emphasize that our result does not tell us about the true steady state. It should be an infinite-temperature state if the Floquet ETH holds. However, another possibility is not excluded: there might be an energy-localized phase [16] with a vanishing energy-absorption rate. It is an open problem to understand the precise condition of the Floquet ETH.

Summary.—In summary, we have considered the quantum dynamics of general driven spin systems that have at most k-body interactions and a bounded single-site energy g. We have rigorously shown the Theorem stating that the truncated Floquet Hamiltonian is a quasiconserved quantity and the rate of energy absorption is exponentially small in frequency. This finding enables us to classify the initial stage of relaxation. It is emphasized that we need not assume short-range interactions in the Hamiltonian (7). For instance, $J_{ij}^{\alpha,\gamma} = \delta_{\alpha,\gamma} J/N$, which corresponds to the Heisenberg all-to-all couplings, satisfies the condition of Eq. (6) with a fixed value of g even in the thermodynamic limit. Therefore, the result in this Letter is applicable to most physically relevant spin models. However, as seen in Eq. (6), our argument excludes bosonic systems. We expect that our analysis will help to understand even bosonic systems.

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Note added.—Recently, closely related results obtained with a different approach have appeared [36,37].

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