Creating a Superposition of Unknown Quantum States

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The superposition principle is one of the landmarks of quantum mechanics. The importance of quantum superpositions provokes questions about the limitations that quantum mechanics itself imposes on the possibility of their generation. In this work, we systematically study the problem of the creation of superpositions of unknown quantum states. First, we prove a no-go theorem that forbids the existence of a universal probabilistic quantum protocol producing a superposition of two unknown quantum states. Second, we provide an explicit probabilistic protocol generating a superposition of two unknown states, each having a fixed overlap with the known referential pure state. The protocol can be applied to generate coherent superposition of results of independent runs of subroutines in a quantum computer. Moreover, in the context of quantum optics it can be used to efficiently generate highly nonclassical states or non-Gaussian states.

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The existence of superpositions of pure quantum states is one of the most intriguing consequences of the postulates of quantum mechanics. Quantum superpositions are responsible for numerous nonclassical phenomena that are considered to be the key features of quantum theory [1], with the prominent examples being the quantum interference [2–4] and quantum entanglement [5]. The coherent addition of wave functions is also responsible for quantum coherence, a feature of quantum states that recently received a lot of attention [6–8]. Quantum superpositions are not only important from the foundational point of view, but they also underpin the existence of ultrafast quantum algorithms (such as the Shor factoring algorithm [9] or Grover search algorithm [10]), quantum cryptography [11], and efficient quantum metrology [12].

The importance of quantum superpositions provokes questions about the restrictions that quantum mechanics itself imposes on the possibility of their generation. Studies of the limitations of quantum mechanics have a long tradition and are important both from the fundamental perspective as well as for the applications in quantum information theory. Quantum mechanics offers a number of protocols that either outperform all existing classical counterparts or even allow us to perform tasks that are impossible in the classical theory (such as quantum teleportation [13]). However, a number of no-go theorems [14–19] restrict a class of protocols that are possible to realize within quantum mechanics. Finally, such no-go theorems can be themselves useful for practical purposes. For instance, a no-cloning theorem can be used to certify the security of quantum cryptographic protocols [11].

In this Letter, we consider the scenario in which we are given two unknown pure quantum states and our task is to create, using the most general operations allowed by quantum mechanics, their superposition with some complex weights. Essentially, the same question was posed in a parallel work of Alvarez-Rodriguez *et al.* [20]: namely, the authors asked about the existence of *quantum adder*—a machine, that would superpose two registers with the plus sign.

Here, we first prove a no-go theorem, showing that it is impossible to create a superposition of two unknown states. We discuss the relation of our theorem with the no-go results of [20]. Subsequently, we provide a protocol that probabilistically creates the superposition of two states having fixed nonzero overlaps with some referential state.

We show that, by using appropriate encoding, the protocol can be used to generate superpositions of unknown vectors from the subspace perpendicular to the referential state, thus allowing for the generation of coherent superpositions of the results of quantum subroutines of a given quantum algorithm. This actually shows how to circumvent our no-go theorem to some extent. We find this surprising as intuitively the no-cloning theorem [14] should forbid results of this type. We also discuss optical implementation of the protocol, with the referential state being the vacuum state. Finally, we discuss the differences between our results, and analogous results concerning cloning.

Introduction.—Before we proceed we need to carefully analyze the concept of quantum superpositions. Recall first that the global phase of a wave function is not a physically accessible quantity. This redundancy is removed when one interprets pure states as one dimensional orthogonal projectors acting on the relevant Hilbert space [21]. In what follows, the pure state corresponding to a normalized vector $|\psi\rangle$ will be denoted by $\mathbb{P}_\psi$. Normalized vectors that rise to the same pure state $\mathbb{P}_\psi$ are called vector representatives of $\mathbb{P}_\psi$. They are defined up to a global phase, i.e., $\mathbb{P}_\psi = \mathbb{P}_{\psi'}$ if and only if $|\psi'\rangle = \exp(i\theta)|\psi\rangle$, for some phase θ. Let now α, β be complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$ and let $\mathbb{P}_\psi, \mathbb{P}_\phi$ be two pure states. By $\mathbb{P}_{\alpha,\beta}(|\psi\rangle, |\phi\rangle)$ we denote the projector onto the superposition of $|\psi\rangle$ and $|\phi\rangle$

$$\mathbb{P}_{\alpha,\beta}(|\psi\rangle, |\phi\rangle) = \mathbb{P}_\Psi, \qquad |\Psi\rangle = \mathcal{N}^{-1}(\alpha|\psi\rangle + \beta|\phi\rangle), \quad (1)$$

where $\mathcal{N} = \sqrt{1 + 2\,\mathrm{Re}(\bar{\alpha}\beta\langle\psi|\phi\rangle)}$ is a normalization factor. The crucial observation is that $\mathbb{P}_{\alpha,\beta}(|\psi\rangle, |\phi\rangle)$ is not a well-defined function of the states $\mathbb{P}_\psi$ and $\mathbb{P}_\phi$. This is because $\mathbb{P}_{\alpha,\beta}(|\psi\rangle, |\phi\rangle)$ depends on vector representatives $|\psi\rangle$, $|\phi\rangle$, whose phases can be gauged independently. Consequently, we have the infinite family of pure states

$$\mathbb{P}_{\alpha,\beta}[|\psi\rangle, \exp(i\theta)|\phi\rangle], \quad \theta \in [0, 2\pi), \quad (2)$$

which can be legitimately called superpositions of $\mathbb{P}_\psi$ and $\mathbb{P}_\phi$. This phenomenon appears already in the simplest example of a qubit. For $\mathbb{P}_\psi = |0\rangle\langle 0|$, $\mathbb{P}_\phi = |1\rangle\langle 1|$, and $\alpha = \beta = 1/\sqrt{2}$, the family given by (2) can be identified with the equator on the Bloch ball. The analogous analysis was conducted in [20] and it was argued there that the ambiguity of the relative phase forbids the existence of the universal quantum adding machine. In our approach, we propose to relax the definition of superposing, so that it is not excluded from the very definition. Namely, we ask if there exists a machine, that for any given two states it prepares a state, that is the projector corresponding to a superposition of *some* representatives of the states with the required amplitudes and phases. This is a well-defined question, and it is not *a priori* clear whether it has a positive answer.

Let us set the notation. By $\mathrm{Herm}(\mathcal{H})$ we denote the set of Hermitian operators on Hilbert space $\mathcal{H}$. By $\mathcal{CP}(\mathcal{H}, \mathcal{K})$ we denote the set of completely positive (*CP*) maps $\Lambda: \mathrm{Herm}(\mathcal{H}) \to \mathrm{Herm}(\mathcal{K})$.

We can now formalize our scenario. We assume that we have access to two identical quantum registers (to each of them we associate a Hilbert space $\mathcal{H}$) and we know that the input state is a product of unknown pure states $\mathbb{P}_\psi \otimes \mathbb{P}_\phi$. Our aim is to generate the superposition $\mathbb{P}_{\alpha,\beta}(|\psi\rangle, |\phi\rangle)$ by some quantum protocol, i.e., a sequence of operations allowed by quantum mechanics. Since the superposition $\mathbb{P}_{\alpha,\beta}(|\psi\rangle, |\phi\rangle)$ is a state on a single quantum register, the protocol takes two states as an input and outputs a state on a single register. This can be physically achieved by the first application of the general quantum channel [22] to two systems and then disregarding (partial tracing) one of them. We also allow for postselection; i.e., we allow the possibility that the desired output state is obtained only probabilistically, depending on the outcome of some measurement on the disregarded register. Mathematically, such operations can be characterized [22] as *CP* maps $\Lambda \in \mathcal{CP}(\mathcal{H}^{\otimes 2}, \mathcal{H})$ that do not increase the trace, i.e., $\mathrm{tr}[\Lambda(\rho)] \leq \mathrm{tr}(\rho)$ for all states ρ. The number $\mathrm{tr}[\Lambda(\rho)]$ is the probability that the state ρ undergoes the transformation $\rho \to \Lambda(\rho)/\mathrm{tr}[\Lambda(\rho)]$.

No-go theorem.—We prove the no-go result in the strongest possible form. First, we impose the minimal assumptions on the generated superpositions, assuming only that vectors $|\psi\rangle$, $|\phi\rangle$ are vector representatives depending on the input states [in other words, we are not interested in the relative phase θ of the superposition appearing in (2)]. Second, we allow the probabilistic protocols; i.e., the superposition may be created with some probability.

Theorem 1.—Let α, β be nonzero complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$ and let $\dim\mathcal{H} \geq 2$. There exists no nonzero completely positive map $\Lambda \in \mathcal{CP}(\mathcal{H}^{\otimes 2}, \mathcal{H})$ such that for all pure states $\mathbb{P}_1$, $\mathbb{P}_2$

$$\Lambda(\mathbb{P}_1 \otimes \mathbb{P}_2) \propto |\Psi\rangle\langle\Psi|, \quad (3)$$

where

$$|\Psi\rangle = \alpha|\psi\rangle + \beta|\phi\rangle \quad (4)$$

and $|\psi\rangle\langle\psi| = \mathbb{P}_1$, $|\phi\rangle\langle\phi| = \mathbb{P}_2$ and the representatives $|\psi\rangle$, $|\phi\rangle$ may in general depend on both $\mathbb{P}_1$ and $\mathbb{P}_2$.

Remark.—In particular, for two pairs $(\mathbb{P}_1, \mathbb{P}_2)$ and $(\mathbb{P}_1, \mathbb{P}_2')$ the representative of $\mathbb{P}_1$ can be different for each pair.

Proof.—Assume that there exists a nonzero *CP* map Λ satisfying (3). Let the collection of operators $\{V_i\}_{i\in I}, V_i: \mathcal{H}^{\otimes 2} \to \mathcal{H}$, form the Kraus decomposition [22] of Λ, $\Lambda(\rho) = \sum_{i\in I} V_i \rho V_i^\dagger$. Since operators $\lambda|\Psi\rangle\langle\Psi|$, $\lambda \geq 0$, belong to the extreme ray of the cone of nonnegative operators on $\mathcal{H}$ we must have

$$V_i \mathbb{P}_1 \otimes \mathbb{P}_2 V_i^\dagger \propto |\Psi\rangle\langle\Psi|, \quad \text{for all } i \in I. \tag{5}$$

Consequently, it is enough to consider only CP maps that have one operator in their Kraus decomposition. In such a case (3) reduces to the investigation of a single linear operator. Equation (5) must necessarily hold for $\mathbb{P}_1$, $\mathbb{P}_2$ having support on two-dimensional subspaces of $\mathcal{H}$. Therefore, it suffices to show that in the qubit case only operators V_i that satisfy condition (5) are the null operators. We present the proof of this in the Supplemental Material [23].

Theorem 1 shows that, even if we allow for postselection, there exists no quantum operation that produces superpositions of all unknown pure quantum states with some probability (we allowed this probability to be zero for some pairs of input states and in general it can be different for different inputs). We would like to stress that the creation of superpositions is still impossible even if we allow for the arbitrary dependence of the relative phase of the input states. Namely, in our formulation of the problem we explicitly assumed that vector representatives $|\psi\rangle$, $|\phi\rangle$ of states $\mathbb{P}_\psi$ and $\mathbb{P}_\phi$ are some functions of these states. As a matter of fact, otherwise one would not be able to formulate the problem of generation of superpositions in a consistent manner. We emphasize that in that respect the problem of the creation of superpositions is different from quantum cloning [24]. Moreover, to our best knowledge, there is no immediate connection between the no-cloning theorem [14,15] and its generalized variants (such as no-deleting theorem [17] or no-anticloning theorem [18]) to our result. This is a consequence of the fact that Λ must be non-invertible and therefore cannot be used to obtain a cloning map. Moreover, in the formulation of the theorem we allow for situations in which for some input states $\mathbb{P}_\psi \otimes \mathbb{P}_\phi$ the probability of success is zero.

Constructive protocol.—Here, we study whether it is possible to create quantum superpositions under the knowledge about the input states. Except for specifying the class of input states for which a given protocol would work, it is also necessary to prescribe which superpositions will be generated [see the discussion before Eq. (2)]. In what follows, we present an explicit protocol that generates superpositions of unknown pure states $\mathbb{P}_\psi$, $\mathbb{P}_\phi$ having fixed nonzero overlaps with some referential pure state $\mathbb{P}_\chi$ (see Fig. 1). Let us describe the superpositions that will be generated by our protocol. Let $|\chi\rangle$ be a vector representative of $\mathbb{P}_\chi$. For every pair of normalized vectors $|\psi\rangle$, $|\phi\rangle$ satisfying $\langle\chi|\psi\rangle \neq 0$, $\langle\chi|\psi\rangle \neq 0$ we define their superposition

$$|\Psi\rangle = \alpha \frac{\langle\chi|\phi\rangle}{|\langle\chi|\phi\rangle|}|\psi\rangle + \beta \frac{\langle\chi|\psi\rangle}{|\langle\chi|\psi\rangle|}|\phi\rangle. \tag{6}$$

The norm of this vector is given by

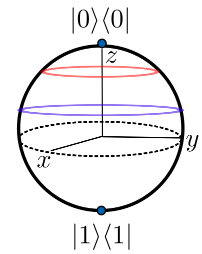

FIG. 1. Graphical representation of the class of input states satisfying $\mathrm{tr}(\mathbb{P}_\chi \mathbb{P}_\psi) = c_1$, $\mathrm{tr}(\mathbb{P}_\chi \mathbb{P}_\phi) = c_2$ for $\mathcal{H} = \mathbb{C}^2$. For convenience we set $\mathbb{P}_\chi = |0\rangle\langle 0|$.

$$\mathcal{N}_\Psi = \sqrt{1 + 2\,\mathrm{Re}\left(\bar{\alpha}\beta \frac{\mathrm{tr}(\mathbb{P}_{|\chi\rangle}\mathbb{P}_{|\psi\rangle}\mathbb{P}_{|\phi\rangle})}{|\langle\chi|\phi\rangle||\langle\chi|\psi\rangle|}\right)}. \tag{7}$$

The vector $|\Psi\rangle$ changes only by a global phase once any of the vectors $|\psi\rangle$, $|\phi\rangle$, $|\chi\rangle$ get multiplied by a phase factor. Consequently, $\mathbb{P}_\Psi$ is a well-defined function of the states $\mathbb{P}_{|\psi\rangle}, \mathbb{P}_{|\phi\rangle}$, provided they have nonzero overlap with $\mathbb{P}_\chi$. This can be also seen from the explicit formula

$$|\Psi\rangle\langle\Psi| = |\alpha|^2\mathbb{P}_\psi + |\beta|^2\mathbb{P}_\phi + \left(\alpha\beta^* \frac{\mathbb{P}_\psi\mathbb{P}_\chi\mathbb{P}_\phi}{\sqrt{\mathrm{tr}(\mathbb{P}_\psi\mathbb{P}_\chi)}\sqrt{\mathrm{tr}(\mathbb{P}_\phi\mathbb{P}_\chi)}} + \mathrm{H.c.}\right). \tag{8}$$

One could argue that the above choice of the superposition $|\Psi\rangle\langle\Psi|$ is somewhat arbitrary. However, the mapping $(\mathbb{P}_\psi, \mathbb{P}_\phi) \to |\Psi\rangle\langle\Psi|$ is related to the so-called Pancharatnam connection and appears in studies concerning the superposition rules from the perspective of geometric approach to quantum mechanics [25,26]. Moreover, it is shown in [27] that Eq. (6) has a strong connection with the concept of the geometric phase. Finally, from the purely operational grounds, Eq. (6) constitutes a rightful superposition of states $\mathbb{P}_\psi$, $\mathbb{P}_\phi$ and as we vary coefficients α, β we can recover all possible superpositions of $\mathbb{P}_\psi$, $\mathbb{P}_\phi$.

Theorem 2.—Let $\mathbb{P}_\chi$ be a fixed pure state on Hilbert space $\mathcal{H}$. There exists a CP map $\Lambda_{\mathrm{sup}} \in \mathcal{CP}(\mathbb{C}^2 \otimes \mathcal{H}^{\otimes 2}, \mathcal{H})$ such that for all pure states $\mathbb{P}_\psi$, $\mathbb{P}_\phi$ on $\mathcal{H}$ satisfying

$$\mathrm{tr}(\mathbb{P}_\chi\mathbb{P}_\psi) = c_1, \qquad \mathrm{tr}(\mathbb{P}_\chi\mathbb{P}_\phi) = c_2, \tag{9}$$

we have

$$\Lambda_{\mathrm{sup}}(\mathbb{P}_\nu \otimes \mathbb{P}_\psi \otimes \mathbb{P}_\phi) \propto |\Psi\rangle\langle\Psi|, \tag{10}$$

where $\mathbb{P}_\nu$, $|\nu\rangle = \alpha|0\rangle + \beta|1\rangle$, is an unknown qubit state and the vector $|\Psi\rangle$ is given by (6). Moreover, a CP map Λ_{sup} realizing (10) is unique up scaling.

Proof.—We first present a protocol that realizes (10). Let us define an auxiliary normalized qubit vector $|\mu\rangle = \mathcal{C}(\sqrt{c_1}|0\rangle + \sqrt{c_2}|1\rangle)$, where $\mathcal{C}$ is a normalization constant. We set $\Lambda_{\mathrm{sup}} = \Lambda_4 \circ \Lambda_3 \circ \Lambda_2 \circ \Lambda_1$, where

$$\Lambda_1(\rho) = V_1 \rho V_1^\dagger, \qquad V_1 = |0\rangle\langle 0| \otimes \mathbb{I} \otimes \mathbb{I} + |1\rangle\langle 1| \otimes \mathbb{S}, \tag{11}$$

$$\Lambda_2(\rho) = V_2 \rho V_2^\dagger, \qquad V_2 = \mathbb{I} \otimes \mathbb{I} \otimes |\chi\rangle\langle\chi|, \tag{12}$$

$$\Lambda_3(\rho) = V_3 \rho V_3, \qquad V_3 = \mathbb{P}_\mu \otimes \mathbb{I} \otimes \mathbb{I}, \tag{13}$$

$$\Lambda_4(\rho) = \mathrm{tr}_{13}(\rho). \tag{14}$$

In the above, $\mathbb{S}$ denotes the unitary operator that swaps between two copies of $\mathcal{H}$ and $\mathrm{tr}_{13}(\cdot)$ is the partial trace over the first and the third factor in the tensor product $\mathbb{C}^2 \otimes \mathcal{H} \otimes \mathcal{H}$. For a graphical presentation of the above protocol, see Fig. 2. Operation Λ_{sup} is completely positive and trace nonincreasing. Direct calculation shows that (10) indeed holds. We prove the uniqueness result in the Supplemental Material [23].

The probability that the above protocol will successfully create superpositions of states is given by

$$P_{\mathrm{succ}} = \mathrm{tr}[\Lambda_{\mathrm{sup}}(\mathbb{P}_\nu \otimes \mathbb{P}_\psi \otimes \mathbb{P}_\phi)] = \frac{c_1 c_2}{c_1 + c_2} \mathcal{N}_\Psi^2. \tag{15}$$

The map Λ_{sup} cannot be rescaled to increase the probability of success. This follows from the (tight) operator inequality $(V_3 V_2 V_1)^\dagger (V_3 V_2 V_1) \leq \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}$. Taking into account the uniqueness (up to scaling) of Λ_{sup}, we get that P_{succ} from (15) is the maximal achievable probability of success [for inputs specified in the assumptions of Theorem (2)]. However, for fixed coefficients α, β it is possible to design a *CP* map that can achieve a higher probability of success [23]. Moreover, it is possible to generalize the protocol Λ_{sup} to the situation when we have d input states (having nonzero overlap with $\mathbb{P}_\chi$) and coefficients of superposition are encoded in an unknown state of a qudit [23].

The existence of the map Λ_{sup} shows that the problem of creating superpositions of quantum states differs greatly from the cloning problem. Probabilistic quantum cloning of pure states is possible if and only if we have a promise that the input states belong to the family of states whose vector representatives form a linearly independent set [28]. Consequently, the aforementioned family of states must be discrete. Our protocol shows that it is possible to probabilistically create superpositions from unknown quantum states belonging to uncountable families of quantum states.

Applications.—There exist deterministic circuits realizing classical arithmetic operations (like addition,

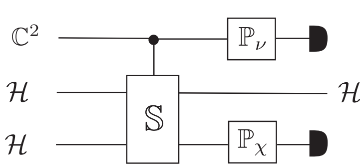

FIG. 2. Graphical representation of the circuit realizing the map Λ_{sup}.

multiplication, exponentiation, etc.) on a quantum computer [29]. However, to our best knowledge there exist no protocols realizing an addition on vectors belonging to the Hilbert space responsible for the independent quantum computations. We now present a method to generate a coherent superposition of results of quantum computations. Assume that $\alpha = \beta = \sqrt{c_1} = \sqrt{c_2} = 1/\sqrt{2}$. By setting the overlap of vector representatives of $\mathbb{P}_\psi$ and $\mathbb{P}_\phi$ with $|\chi\rangle$ to be positive we get

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\chi\rangle + \frac{1}{\sqrt{2}}|\psi^\perp\rangle, \qquad |\phi\rangle = \frac{1}{\sqrt{2}}|\chi\rangle + \frac{1}{\sqrt{2}}|\phi^\perp\rangle, \tag{16}$$

where unit vectors $|\psi^\perp\rangle$, $|\phi^\perp\rangle$ are perpendicular to $|\chi\rangle$. Input states $\mathbb{P}_\psi$, $\mathbb{P}_\phi$ are in one-to-one correspondence with the vectors $|\psi^\perp\rangle$, $|\phi^\perp\rangle$. By the application of Λ_{sup} it is possible to obtain a state having the (nonnormalized) vector representative

$$|\Psi\rangle = |\chi\rangle + \frac{1}{2}(|\psi^\perp\rangle + |\phi^\perp\rangle), \tag{17}$$

with the probability $P_{\mathrm{succ}} = \frac{1}{4}(1 + \frac{1}{4}\||\psi^\perp\rangle + |\phi^\perp\rangle\|^2) \geq \frac{1}{4}$. We have obtained a state encoding the superposition of *unknown* vectors $|\psi^\perp\rangle$, $|\phi^\perp\rangle$ encoded in states $\mathbb{P}_\psi$ and $\mathbb{P}_\phi$, respectively. The method presented above effectively superposes the wave functions coherently, provided one has access to the auxiliary one dimensional subspace (spanned by $|\chi\rangle$). It is highly unexpected but by changing the perspective and by treating as "primary" objects the vectors perpendicular to $|\chi\rangle$ we have managed to effectively get around the no-go result from Theorem 1. To apply the above protocol, one has to run quantum computation in a the perpendicular space. In the Supplemental Material [23], we present an exemplary scheme implementing such a computation.

The protocol Λ_{sup} can be also used to generate nonclassical sates in the context of quantum optics. Let the states $\mathbb{P}_\psi$, $\mathbb{P}_\phi$ describe quantum fields in two different optical modes. Hilbert spaces associated with each of the modes are isomorphic and can be identified with the single-mode bosonic Fock space. Moreover, let the auxiliary qubit be encoded in a polarization of a single photon in a different optical mode or in another two level physical system. In such a setting the natural choice of the state $\mathbb{P}_\chi$ is the Fock vacuum $|0_F\rangle\langle 0_F|$ describing the state of the field with no photons. As an input we can put coherent or pure Gaussian states [30] that have fixed overlaps with the vacuum. Then, the protocol Λ_{sup} generically creates highly nonclassical or, respectively, non-Gaussian states. Operations Λ_2, Λ_3, Λ_4 are relatively easy to realize in this setting. The most demanding operation is the conditional swap Λ_1. However, the conditional swap can be realized via the implementation of standard beam splitters of phase flip operation [31]. The latter can be in principle [32] obtained by coupling light to atoms inside the cavity, trapped ions, or by the usage of cross-Kerr nonlinearities in materials with an electromagnetically induced transparency. Despite the possible difficulties with the implementation, the map Λ_{sup} is worth realizing as it gives the maximal probability of success. Moreover, the protocol Λ_{sup} is universal and can be used in different physical scenarios.

Discussion.—Let us conclude by stating some open problems. First of all, the relation of our no-go theorem to other no-go results in quantum mechanics is not clear and requires further investigation. The constructive protocol presented by us suggests a connection with the recent works concerning the problem of controlling an unknown unitary operation [33–37] (the state $\mathbb{P}_{\chi}$ can be regarded as an analogue of the known eigenvector of the "unknown" operation U allowing for its control). Second, it is interesting to study the problem of the approximate generation of quantum superpositions (in analogy to approximate quantum cloning [38]). Another possible line of research is to investigate protocols designed especially to generate superpositions of states naturally appearing in the experimental context (like pure coherent or Gaussian states).

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[1] E. Schrödinger, Naturwissenschaften **23**, 1 (1935).
[2] B. Anderson and M. A. Kasevich, Science **282**, 1686 (1998).
[3] A. Zeilinger, Rev. Mod. Phys. **71**, S288 (1999).
[4] D. J. Wineland, Rev. Mod. Phys. **85**, 1103 (2013).
[5] E. Schrödinger, in *Mathematical Proceedings of the Cambridge Philosophical Society* (Cambridge University Press, 1935), Vol. 31. pp. 555–563.
[6] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[7] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[8] A. Winter and D. Yang, arXiv:1506.07975 [Phys. Rev. Lett. (to be published)].
[9] P. W. Shor, SIAM J. Comput. **26**, 1484 (1997).
[10] L. K. Grover, in *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing* (ACM, New York, 1996), pp. 212–219.
[11] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).
[12] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photonics **5**, 222 (2011).
[13] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[14] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).
[15] D. Dieks, Phys. Lett. A **92**, 271 (1982).
[16] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. **76**, 2818 (1996).
[17] A. K. Pati and S. L. Braunstein, Nature (London) **404**, 164 (2000).
[18] A. K. Pati, Phys. Rev. A **66**, 062319 (2002).
[19] M. Piani, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **100**, 090502 (2008).
[20] U. Alvarez-Rodriguez, M. Sanz, L. Lamata, and E. Solano, Sci. Rep. **5**, 11983 (2015).
[21] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, 2006).
[22] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).
[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.110403 for technical details.
[24] In quantum cloning one seeks a physical transformation that would realize the (well-defined) mapping $|\psi\rangle\langle\psi| \to |\psi\rangle\langle\psi|^{\otimes k}$.
[25] E. Ercolessi, G. Marmo, and G. Morandi, Riv. Nuovo Cimento Soc. Ital. Fis. **033**, 401 (2010).
[26] V. I. Man'ko, G. Marmo, E. G. Sudarshan, and F. Zaccaria, J. Phys. A **35**, 7137 (2002).
[27] S. Chaturvedi, E. Ercolessi, G. Morandi, A. Ibort, G. Marmo, N. Mukunda, and R. Simon, J. Math. Phys. **54**, 062106 (2013).
[28] L.-M. Duan and G.-C. Guo, Phys. Rev. Lett. **80**, 4999 (1998).
[29] V. Vedral, A. Barenco, and A. Ekert, Phys. Rev. A **54**, 147 (1996).
[30] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).
[31] R. Filip, Phys. Rev. A **65**, 062320 (2002).
[32] H. Jeong, C. Noh, S. Bae, D. G. Angelakis, and T. C. Ralph, J. Opt. Soc. Am. B **31**, 3057 (2014).
[33] A. Y. Kitaev, arXiv:quant-ph/9511026.
[34] X.-Q. Zhou, T. C. Ralph, P. Kalasuwan, M. Zhang, A. Peruzzo, B. P. Lanyon, and J. L. O'Brien, Nat. Commun. **2**, 413 (2011).
[35] J. Thompson, M. Gu, K. Modi, and V. Vedral, arXiv:1310.2927.
[36] M. Araújo, A. Feix, F. Costa, and Č. Brukner, New J. Phys. **16**, 093026 (2014).
[37] A. Bisio, M. Dall'Arno, and P. Perinotti, arXiv:1509.01062.
[38] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).