Quasi-Long-Range Order and Vortex Lattice in the Three-State Potts Model

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We show that the order-disorder phase transition in the three-state Potts ferromagnet on a square lattice is driven by a coupled proliferation of domain walls and vortices. Raising the vortex core energy above a threshold value decouples the proliferation and splits the transition into two. The phase between the two transitions exhibits an emergent U(1) symmetry and quasi-long-range order. Lowering the core energy below a threshold value also splits the order-disorder transition but the system forms a vortex lattice in the intermediate phase.

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Phase transitions in a variety of systems are driven by the proliferation of topological defects [1–4]. Manipulation of defects, therefore, provides a natural route towards altering the nature and location of phase transitions, which in turn can significantly alter the phase diagram itself. The role played by a single type of defect, and the effect of manipulating it, has been studied in models of superfluids [5–7], liquid crystals [8,9], and Heisenberg ferromagnets [10–12]. In a large class of systems, however, the phase diagram is determined by the proliferation of not one but multiple types of defects [13–20]. We would like to identify a minimal model in which the interplay between two types of defects, and the effect of manipulating them, can be studied clearly.

The two-state (Ising) ferromagnet on a square lattice is one of the simplest spin models that exhibit a defect driven phase transition. Domain wall defects appear as small loops in the ordered phase of the model and drive a transition to the disordered phase upon proliferation [21,22]. The next simplest model, the three-state Potts ferromagnet, also exhibits an order-disorder transition [23,24]. This model, however, supports the formation of domain wall as well as $\mathbb{Z}_3$ vortex defects. An approximate energy versus entropy balance calculation suggests that the system disorders because vortex-antivortex pairs unbind as soon as the domain walls proliferate [14]. Apart from this calculation, the role played by the two types of defects in the model has remained largely unexplored.

In this Letter, we use Monte Carlo simulations to demonstrate that the interplay between domain walls and vortices in the three-state Potts model generates a rich phase diagram (Fig. 1). We show that the order-disorder transition in the model is driven by a coupled proliferation of the two types of defects. When we raise the core energy of the vortices by an amount λ, the model continues to exhibit the order-disorder transition up to a certain threshold $\lambda = \lambda_+$. Above λ_+, the vortices proliferate after the domain walls and the transition splits into two. The phase, which appears intermediate between the two transitions, exhibits an emergent $U(1)$ symmetry and quasi-long-range order (QLRO). When we lower the core energy using negative values of λ, the order-disorder transition of the pure Potts case becomes sharper. Below a threshold λ_-, the transition again splits into two. In this case, however, the intermediate phase is a vortex lattice in which the vortices and antivortices display sublattice ordering.

Before discussing the phase diagram in detail, we describe how the defects are identified for a given configuration of spins. Each spin σ_i, at vertex i on a square lattice Λ, can be in one of three states: $\sigma_i \in \{0, 1, 2\}$. Domain walls and vortices reside on the edges and vertices, respectively, of the dual lattice Λ' which is a square lattice displaced from Λ by half a lattice spacing along each axis. If two spins across an edge $\langle i, j\rangle \in \Lambda$ are in dissimilar states, then a domain wall is placed on the dual edge in Λ'. The vorticity at each dual vertex $i' \in \Lambda'$ is determined by calculating a discrete winding number $\omega_{i'}$ [19]. For $\mathbb{Z}_3$ vortices, $\omega_{i'} = (\Delta_{ba} + \Delta_{cb} + \Delta_{dc} + \Delta_{ad})/3$ where Δ_{ba} represents $(\sigma_b - \sigma_a)$ wrapped to lie in $[-1, +1]$ and σ_a, σ_b, σ_c, σ_d are the four spins on the square plaquette in Λ

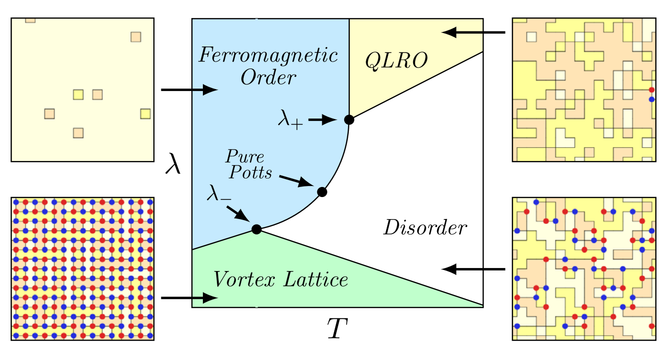

FIG. 1. Schematic phase diagram in the parameter space of temperature T and vortex suppression λ. The pure Potts model corresponds to $\lambda = 0$. Estimates for λ_+ and λ_- are given in the text. Domain walls (black lines), vortices (blue dots), and antivortices (red dots) are overlaid on typical spin configurations obtained in each phase.

surrounding i' in an anticlockwise sense. A vortex (antivortex) is present at i' if $\omega_{i'}$ is $+1$ (-1).

A standard method for suppressing the formation of vortices in models of superfluids involves raising the vortex core energy by an amount λ [5,6,25,26]. Upon inclusion of such a term for $\mathbb{Z}_3$ vortices, the three-state Potts Hamiltonian, with nearest neighbor ferromagnetic interaction $J > 0$, becomes

$$\mathcal{H} = J \sum_{\langle i,j \rangle \in \Lambda} (1 - \delta_{\sigma_i,\sigma_j}) + \lambda \sum_{i' \in \Lambda'} |\omega_{i'}|. \quad (1)$$

If the number of domain walls and vortex defects corresponding to a given spin configuration is denoted by N_{dw} and N_{vx}, respectively, then (1) can be rewritten as $\mathcal{H} = JN_{dw} + \lambda N_{vx}$, clearly indicating that the statistical behavior of the model depends solely on the number of defects. In particular, the behavior depends on the number of domain walls and their three branch intersections. Each $i' \in \Lambda'$ is visited by zero, two, three, or four domain walls. Out of these four scenarios, $\omega_{i'} \neq 0$ only when three domain walls intersect. The present model can, therefore, be equivalently expressed as a domain wall loop model [27] with a fugacity parameter λ controlling the density of three branch intersections.

We have determined the phase diagram of the model in the two-dimensional parameter space of λ and temperature T by simulating (1) on a $L \times L$ square lattice. As the plaquette-based λ term cannot be incorporated into currently known cluster algorithms, the spins were updated using a single spin-flip algorithm [28]. We measured the density of domain walls $\rho_{dw} = N_{dw}/2L^2$, the density of vortices $\rho_{vx} = N_{vx}/L^2$, and the Potts order parameter $\mathcal{S} = 3(\max\{n_0, n_1, n_2\} - 1/3)/2$, where n_σ represents the fraction of spins in state σ. Large autocorrelation times, arising from the use of the single spin-flip algorithm, were estimated for these observables and measurements were made over 10^5–10^6 uncorrelated configurations.

In the pure Potts case ($\lambda = 0$), the order parameter decays and the susceptibility $\chi_\mathcal{S} = L^2(\langle \mathcal{S}^2 \rangle - \langle \mathcal{S} \rangle^2)/T$ diverges (Fig. 2) close to the transition temperature $T_c = 1/\log(1 + \sqrt{3}) = 0.9949$ [24]. The transition is accompanied by a simultaneous proliferation of both types of defects, as indicated by an increase in their densities. The transition and defect proliferation shift to a higher temperature $T \approx 1.26$ when vortices are weakly suppressed using $\lambda = 2$ (Fig. 2). The location of the transition continues to shift in this manner, with increasing λ, up to a threshold value $\lambda = \lambda_+$ which we estimate to lie around $\lambda_+ \approx 8$.

Above λ_+, the order parameter clearly exhibits a two-step decay and the susceptibility shows two distinct peaks (Fig. 2). For $\lambda = 10$, the first decay from the ordered phase to the intermediate phase occurs around $T \approx 1.4$ and is accompanied by the proliferation of domain walls. The vortices proliferate near the second decay, which marks the transition from the intermediate phase to the disordered phase at $T \approx 2.7$. Extremely strong suppression of vortices keeps the first decay unchanged but shifts the second decay to $T \to \infty$, thus establishing the role of vortices in driving the disordering transition.

Two-step decay of magnetization, driven by successive proliferation of domain walls and vortices, has been discussed in the context of $\mathbb{Z}_n$ vector Potts (clock) spin models with $n \geq 5$ [13,14,19,29–34]. These models exhibit a phase, intermediate between order and disorder, where

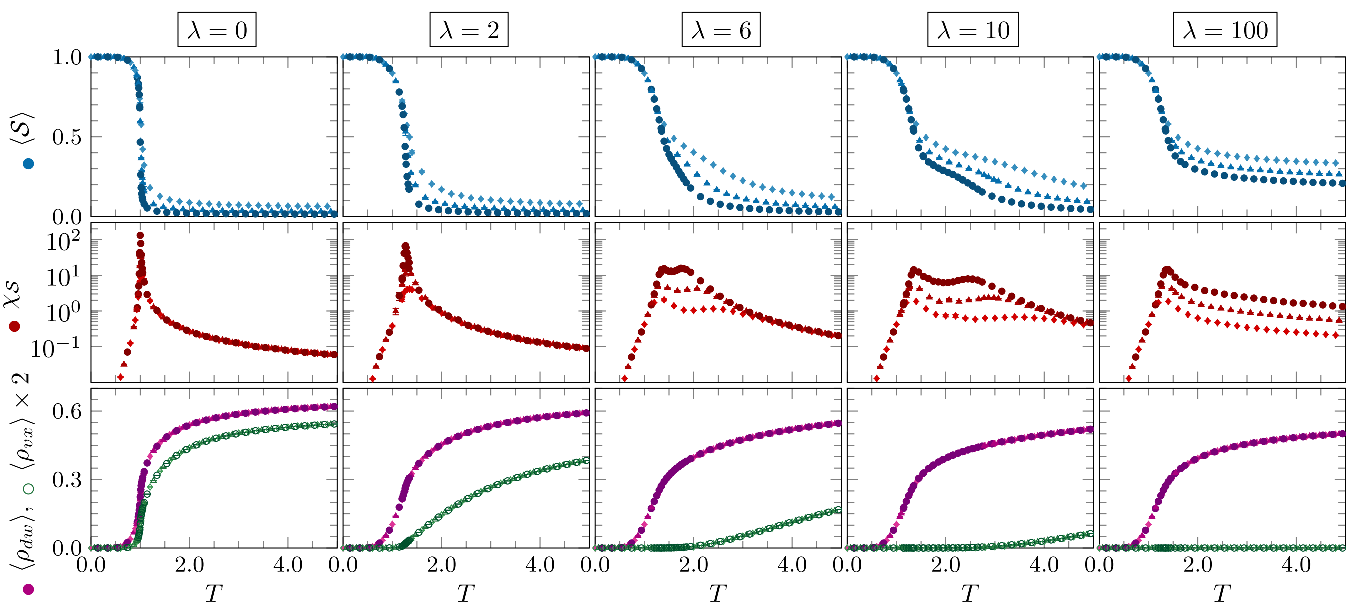

FIG. 2. The order-disorder transition of the pure Potts model ($\lambda = 0$), marked by the decay of $\mathcal{S}$ and divergence of $\chi_\mathcal{S}$, is accompanied by a simultaneous increase of ρ_{dw} and ρ_{vx}. Weak suppression of vortices ($\lambda = 2$) shifts the transition to a higher temperature. Stronger suppression decouples the simultaneous proliferation of the defects and splits the transition into two. The data have been obtained for $L = 16$ (diamonds), $L = 32$ (triangles), and $L = 64$ (circles).

domain walls proliferate but vortices do not. In the intermediate phase, the system fragments into numerous domains in a manner such that the spins fluctuate by arbitrary amounts over large distances and exhibit a $U(1)$ symmetry upon coarse-graining [14]. The emergent continuous symmetry destroys long-range order and gives rise to a quasi-long-range order which is characterized by a power-law decay of two-point correlation $\mathcal{C}(r) = \langle\cos(2\pi(\sigma_0 - \sigma_r)/3)\rangle \propto r^{-\eta}$, where σ_0 and σ_r are spins located at a Eucledian distance r apart on the lattice. The exponent η changes continuously with temperature throughout the quasi-long-range ordered phase until vortex proliferation disorders the system [19].

We have measured the distribution $P(m_x, m_y)$ of the $\mathbb{Z}_3$ order parameter $m = m_x + im_y$, where $m_x = \sum_\sigma n_\sigma \cos(2\pi\sigma/3)$ and $m_y = \sum_\sigma n_\sigma \sin(2\pi\sigma/3)$ [32,34]. The distribution (Fig. 3) clearly indicates a breaking of the threefold symmetry in the ordered phase and an enhancement to $U(1)$ symmetry in the intermediate phase. The $U(1)$ symmetry survives an increase in system size while the magnetization $(m_x^2 + m_y^2)^{1/2}$ tends to zero in accordance with the Mermin-Wagner theorem [35,36]. $\mathcal{C}(r)$ exhibits a power-law decay throughout the intermediate phase. η increases with temperature from $\eta \approx 0.35$ and appears to saturate around $\eta \approx 0.75$ at high temperatures (Fig. 3). This confirms that the intermediate phase is indeed a quasi-long-range ordered phase.

We now turn to the regime $\lambda < 0$. The formation of vortices is enhanced in this regime and the order-disorder transition of the pure Potts case shifts to lower temperatures when λ is made more negative (Fig. 4). Additionally, the decay of the order parameter grows sharper and the simultaneous rise in the densities of domain walls and vortices across the transition becomes more abrupt

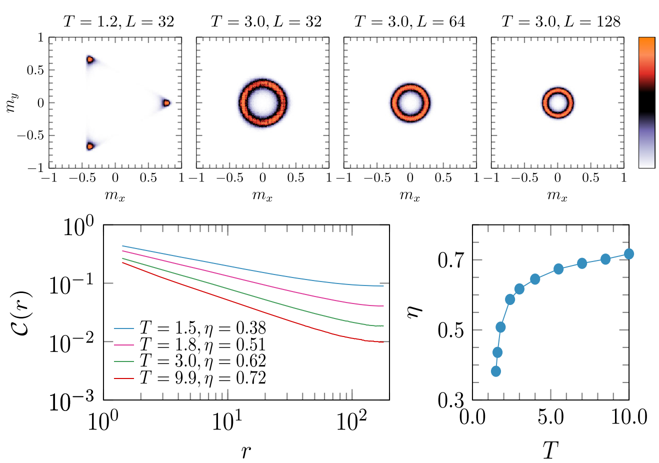

FIG. 3. Top panel shows $P(m_x, m_y)$ in the ordered $(T = 1.2)$ and quasi-long-range ordered $(T = 3.0)$ phases for $\lambda = 100$. Bottom panel shows power-law decay of $\mathcal{C}(r)$ at different temperatures in the latter phase for $L = 256$. The saturation at large r is due to the finite size of the system. η increases with temperature as shown on the right.

compared to the pure Potts case. When λ is decreased below a threshold value $\lambda_- \approx -1.3$, the model exhibits three distinct regimes (Fig. 4). The densities of domain walls and vortices are nearly zero in the ordered phase, show a sharp jump to a large value at intermediate temperatures and decrease gradually in the disordered phase. The order parameter, on the other hand, shows a sharp decay from the ordered phase to the intermediate regime, and remains zero thereafter. This might suggest disordered behavior in the intermediate regime but an inspection of spin configurations in that regime (Fig. 1) reveals that the spins are not disordered. Instead, they exhibit a weave pattern that corresponds to an ordering of the vortex defects: vortices reside on one sublattice and antivortices reside on the other.

In order to characterize the sublattice ordering of the vortices, we define a variable $\epsilon_{i'}$ which is either $+1$ or -1 depending on the sublattice of the dual vertex i'. Since some of the dual vertices are vacant ($\omega_{i'} = 0$), the sublattice vortex order parameter can be chosen to be of the same form as that for a site-diluted Ising antiferromagnet [37,38]: $m_{svx} = L^{-2}\sum_{i'\in\Lambda'}\epsilon_{i'}\omega_{i'}$. The magnitude of this order parameter becomes nonzero in the vortex lattice and clearly demarcates the intermediate phase from the ordered and disordered phases (Fig. 4). Since this order parameter, based on topological defects, is able to distinguish the vortex lattice phase from the disordered phase, while the symmetry-based order parameter $\mathcal{S}$ is unable to do so, the vortex lattice phase provides a simple example of classical topological order. Formation of vortex lattices in superfluids and superconductors has been extensively studied over the past few decades [7,38–41] In superconducting thin films, proliferation of dislocations and disclinations drive a two-step structural melting of the vortex lattice [15,42–44]. In the present model, the melting of the vortex lattice to the disordered phase occurs via a single step process as indicated by the decay of $\langle|m_{svx}|\rangle$ (Fig. 4). The sublimation of the vortex lattice to the ordered phase, which occurs at a lower temperature, is marked by a sharp decay of $\langle|m_{svx}|\rangle$. With a further decrease of λ, the melting shifts to higher temperatures (Fig. 4). The sublimation, on the other hand, shifts to lower temperatures and hits the zero temperature limit at $\lambda \approx -1.5$. Below this λ, the ordered phase is absent and the model exhibits a single melting transition from the vortex lattice to the disordered phase.

The nature of the new phases uncovered above is quite clear from the data obtained for small systems. A precise estimate of the temperature and nature of the transitions, on the other hand, requires detailed analysis of data from large systems and will be presented separately. Here, we make a few comments regarding the possible nature of the transitions. The order-disorder transition for $\lambda = 0$ is of the second order type [24,45]. This order-disorder transition continues to occur throughout the range $\lambda_- < \lambda < \lambda_+$ and is accompanied by a coupled proliferation of domain walls and vortices. However, the decay of the order parameter appears to grow weaker with increasing λ (Fig. 2) and sharper with

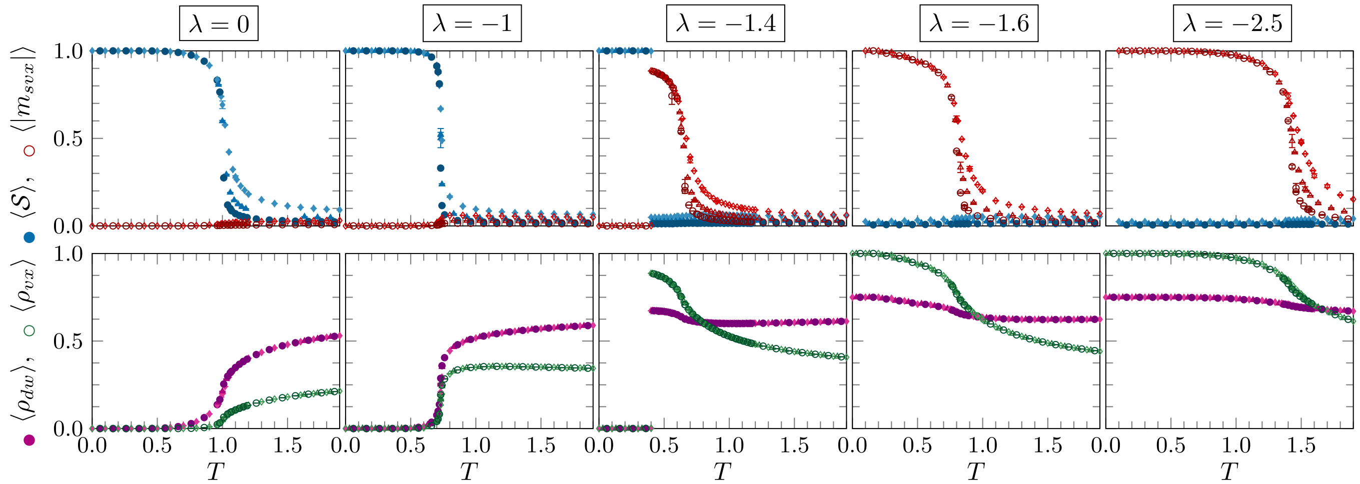

FIG. 4. Weak enhancement of vortices ($\lambda = -1$) results in a sharp decay of $\mathcal{S}$ accompanied by simultaneous increase of ρ_{dw} and ρ_{vx}. Stronger enhancement ($\lambda = -1.4$) opens up an intermediate vortex lattice phase characterized by a nonzero value of $\langle |m_{svx}| \rangle$. With further enhancement, the ordered phase vanishes while the vortex lattice melts at a higher temperature. Data correspond to system sizes mentioned in Fig. 2.

decreasing λ (Fig. 4). This observation leads us to conjecture that the critical exponents of the transition might vary with λ. The four-state (Ashkin-Teller) ferromagnet is known to exhibit continuously varying criticality along a line of order-disorder transitions due to an interplay between vortices and domain walls [46]. The effect of the coupled proliferation on the nature of the transition in the present model, therefore, seems to be an interesting problem. In this context, we note that claims of continuously varying criticality in the closely related three-state chiral Potts model continues to be a controversial issue [47–49].

For $\lambda > \lambda_+$, the order-disorder transition splits into two and the intermediate phase exhibits quasi-long-range order. In $\mathbb{Z}_n$ ferromagnets with $n \geq 5$, the two transitions bordering the quasi-long-range ordered phase are known to be of the Berezinskii-Kosterlitz-Thouless type [19,34]. We can expect the two transitions in the present model to be of that type as well. In $\mathbb{Z}_n$ models, the decay exponent η changes from $\eta = 4/n^2$ at the low temperature transition to $\eta = 1/4$ at the high temperature transition [34]. The values of $\eta \in (0.35, 0.75)$ obtained in the quasi-long-range ordered phase of the present model (Fig. 3) fall beyond that range. This deviation from the standard $\mathbb{Z}_n$ model scenario indicates that the bounds for η can change with vortex suppression.

Our estimate of $\lambda_+ \approx 8$ is an approximate one. For $\lambda = 6$, the intermediate phase is narrow in small systems and appears to shrink with increasing L (Fig. 2). On the other hand, for fixed L, the extent of the phase increases with increasing λ. This competing effect of L and λ offers two possibilities: (a) there exists a threshold λ_+, above which the intermediate region has a nonzero extent in the thermodynamic limit, or (b) the intermediate region shrinks to a point in the thermodynamic limit for all finite λ and the model exhibits an extended quasi-long-range ordered phase only in the $\lambda \to \infty$ limit. A precise estimate of λ_+, therefore, remains an open problem and is reminiscent of the problem regarding the location of the Lifshitz point in the three-state chiral Potts model [50–54].

In the range $\lambda_- < \lambda < 0$, the order-disorder transition grows sharper with decreasing λ, hinting at the possibility that the transition will become discontinuous before λ goes below λ_-. Such a scenario is quite plausible because an abrupt proliferation of vortices, similar to the behavior shown in Fig. 4, is known to induce discontinuous behavior [26,55]. For $\lambda < \lambda_-$, the transition from the ordered phase to the vortex lattice phase (Fig. 4) is clearly discontinuous. The gradual decay of $\langle |m_{svx}| \rangle$ between the vortex lattice phase and the disordered phase, on the other hand, suggests that the melting transition might be second order in nature.

The universality class of the three-state Potts transition has a ubiquitous presence in the physics of statistical [24,56–64], quantum [65–69], and gauge systems [34,70–73]. We have shown that the transition is driven by a coupled proliferation of domain walls and vortices. By manipulating the formation of the defects, we have uncovered two new phases in the model. Apart from the exciting possibility that these phases might be realizable in some of the systems, our work provides a step towards better understanding the role of topological defects and the presence of topological order in classical spin models.

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