Critical Schwinger Pair Production

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We investigate Schwinger pair production in spatially inhomogeneous electric backgrounds. A critical point for the onset of pair production can be approached by fields that marginally provide sufficient electrostatic energy for an off-shell long-range electron-positron fluctuation to become a real pair. Close to this critical point, we observe features of universality which are analogous to continuous phase transitions in critical phenomena with the pair-production rate serving as an order parameter: electric backgrounds can be subdivided into universality classes and the onset of pair production exhibits characteristic scaling laws. An appropriate design of the electric background field can interpolate between power-law scaling, essential Berezinskii-Kosterlitz-Thouless-type scaling, and a power-law scaling with log corrections. The corresponding critical exponents only depend on the large-scale features of the electric background, whereas the microscopic details of the background play the role of irrelevant perturbations not affecting criticality.

Universality is an overarching concept in physics, signifying the independence of general gross properties of a physical system of the details of its microscopic realizations. Most prominently, critical phenomena near a continuous phase transition reveal a remarkably high degree of universality, such that different systems consisting microscopically of rather different building blocks exhibit quantitatively identical long-range behavior near the phase transition [1]. The quantification of universality by means of fixed points is one of the great successes of the renormalization group that provides a map from the microscopic details to the effective long-range properties [2].

As a consequence, critical systems can be associated with universality classes which are characterized by only a few properties such as the symmetries of the order parameter, the dimensionality, and the number and type of long-range degrees of freedom. It is therefore not surprising that universality and a notion of criticality can also be found beyond the realm of statistical physics. For instance, the onset of black-hole formation shows a surprising insensitivity to the initial data. Generically, the black hole mass as a function of a single control parameter parametrizing the initial data scales according to a power law with the universal Choptuik exponent [3,4]. Whereas universality in statistical physics is typically associated with the presence of fluctuations on all scales, the example of gravitational collapse is observed in a purely classical deterministic setting.

In the present work, we identify for the first time aspects of universality in the phenomenon of Schwinger pair production in quantum electrodynamics (QED). This sets a dual example as the phenomenon of pair production in strong external fields can be derived from the Dirac equation which—despite its quantum mechanical

interpretation—can be viewed as a classical deterministic field equation. In fact, the first observation of this phenomenon relied on this formulation [5], and so do many more modern approaches at least indirectly [6–11]. On the other hand Schwinger pair production is also encoded in the photon correlation functions derived from the full functional integral of QED, as seen from the derivations of Euler, Heisenberg [12], and Schwinger [13]. Again many variants of this fluctuation-based descriptions exist [14–16]. The fact that both descriptions are equivalent is a manifestation of the optical theorem (for a recent discussion in the context of pair production, see [17,18]).

In this work, we use the worldline formalism with background fields [19], as this method makes universality in the language of fluctuations of electrons in spacetime most transparent. We use the imaginary part of the QED effective action Γ as the order parameter for the onset of criticality. It is related to the probability of vacuum decay via $P = 1 - \exp(-2\,\mathrm{Im}\,\Gamma[E])$ in the presence of an electric field E; to lowest order, it is also related to the pair production rate [20,21]. The seminal Schwinger formula

$$\mathrm{Im}\,\Gamma = V_4 \frac{(eE)^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-n\frac{\pi m^2}{eE}\right) \qquad (1)$$

exhibits no signature of criticality, as it assumes the presence of an electric field being constant all throughout space and time. By contrast, a critical point can arise for spatially inhomogeneous fields, as can be read off from Nikishov's exact solution for the electric field with the localized Sauter-profile $E(x) = \mathcal{E}\,\mathrm{sech}^2 kx$ of inverse width k [21]. The order parameter for pair production $\mathrm{Im}\,\Gamma$ for such a spatial profile drops to zero at

$$e \int dx E(x) = \frac{2e\mathcal{E}}{k} \overset{!}{=} 2m \Rightarrow \gamma_{\mathrm{cr}} := \frac{km}{e\mathcal{E}} = 1. \qquad (2)$$

This equation has a simple meaning: the work done by the electric field on a particle of charge e propagating along the whole real axis has to be equal to the rest mass of the particle-antiparticle pair to be created. Translated into the language of fluctuations, a virtual pair created at some spacetime point in such a field with adiabaticity (Keldish) parameter $\gamma = (km/e\mathcal{E}) < \gamma_{\mathrm{cr}} \equiv 1$ can become real if particle and antiparticle separate from one another sufficiently far to acquire enough electrostatic energy. In all cases discussed below, the electrostatic energy becoming equal to $2m$, or $\gamma \to 1$, always characterizes the onset of critical Schwinger pair production, and a scaling behavior $\mathrm{Im}\,\Gamma \overset{?}{\sim} (1 - \gamma^2)^\beta$ with some critical exponent β seems already suggestive at this stage [22].

An important difference to standard critical phenomena of the type mentioned in the beginning is the occurrence of an explicit finite mass scale: the electron mass. While universality arising near continuous phase transitions is related to a diverging correlation length, i.e., long-range interactions mediated by an excitation becoming exactly massless at the critical point, the electron mass remains as a finite scale in QED. This prevents us from associating the critical point with the notion of scale invariance and self-similarity in a straightforward way. We find that this leads to a reduced degree of universality, implying that critical pair production will not be characterized by a universal scaling law or exponent, but rather by a set of scaling laws for different large-scale properties of the spatial electric field profile. Still a rather large degree of universality, i.e., independence of the microscopic profile details, remains, such that electric fields fall into universality classes of field profiles.

In the present work, we confine ourselves to simple unidirectional electric fields that vary only in one spatial coordinate, which also specifies the direction of the field. More precisely, we assume that the x component of the electric field can be written as $E(x) = \mathcal{E}f'(u)$, where the potential function f is antisymmetric, monotonic, and normalized such that $\max f = 1$, and $u = kx$ is a dimensionless coordinate with $1/k$ being a suitable length scale of the spatial profile. With this restricted class of fields we avoid pathological cases where large microscopic details could dominate the pair production process. The latter type of fields would require a case by case study along the lines of fields with compact support included below, possibly accompanied by interference effects [23]. Still, the present class of fields is sufficiently general to illustrate aspects of universality and gives access to a variety of interesting universality classes.

We begin with the worldline representation of the effective action of scalar QED in an external field [19] (for the following discussion of universality, the difference to spinor QED consists only in an irrelevant prefactor [24])

$$\Gamma[A] = - \int_0^\infty \frac{ds}{s} e^{-im^2 s} \int_{x(s)=x(0)} \mathcal{D}x\, e^{i \int_0^s d\sigma(\dot{x}^2/4 - eA\cdot\dot{x})}, \qquad (3)$$

where the path integral can be interpreted as an average over all trajectories of electron fluctuations within the background field A. Though the electron mass m explicitly sets a scale, effectively constraining the (proper-)time s available for the fluctuations, the free path integral has a Gaussian velocity distribution such that the ensemble contains paths of arbitrary length scale [25]. This is the origin of universality for localized fields, as the near critical regime is dominated by the trajectories of largest relevant extent which become less and less sensitive to the microscopic details of the background field.

In the following, we study universality in the weak-field regime,

$$\left(\frac{e\mathcal{E}}{m^2} \right)^2 \ll 1 - \gamma^2 \ll 1. \qquad (4)$$

Although this prevents us from going all the way to $\gamma = 1$, it is experimentally relevant given the large value of the critical field strength $E_c = m^2/e$. This is precisely the regime, where the semiclassical approximation of the path integral as well as the propertime integral in Eq. (3) become exact. In this *semiclassical critical* limit, the path integral is dominated by the stationary points of the worldline action: the worldline instantons [24,26–31] which in general can be complex stationary paths [23]. Up to finite prefactors, the order parameter for pair production near semiclassical criticality scales as [27]

$$\mathrm{Im}\,\Gamma \sim \frac{\exp\left[-(\pi m^2/eE)g(\gamma^2)\right]}{(\gamma^2 g)' \sqrt{(\gamma^2 g)''}}, \qquad (\ldots)' \equiv \frac{d}{d\gamma^2}(\ldots), \quad (5)$$

where the field dependence is contained in a single function related to the worldline instanton action

$$g(\gamma^2) = \frac{1}{\gamma^2} \frac{4}{\pi} \int_0^{u_\gamma} du \sqrt{\gamma^2 - f^2}. \qquad (6)$$

Here, $\pm u_\gamma$ correspond to the semiclassical turning points defined by $f(u_\gamma) = \gamma$ [because of the anticipated antisymmetry of $f(u)$, it suffices to consider $u > 0$ here and in the following]. Heuristically, these turning points correspond to those points, where a separated virtual pair has acquired sufficient electrostatic energy to become real.

Equation (5) has the standard semiclassical form of an exponential tunneling amplitude arising from the action along a classical path, and a prefactor from the fluctuations about the classical path. The order parameter $\mathrm{Im}\,\Gamma$ vanishes if the prefactor vanishes, i.e., g', g'' diverge, or if the exponent $\sim g$ diverges.

Universality becomes already apparent from the dependence of g on the potential function f in Eq. (6): *any field strength profile that leads to the same divergence structure of g or its derivatives for $\gamma \to 1$ belongs to the same universality class.* In the limit $\gamma \to 1$, the turning point u_γ approaches the point u_0 where the electric field vanishes and f attains its maximum

$$f(u_\gamma) = \gamma \to 1 = \max f =: f(u_0). \qquad (7)$$

Substituting $\sin\theta = f(u)/\gamma$, we find

$$[\gamma^2 g(\gamma^2)]' = \frac{2}{\pi} \int_0^{\pi/2} d\theta \, \frac{1}{f'}, \qquad (8)$$

demonstrating that the divergence of g' comes from the region close to zero field strength and the maximum of the potential, $f' \to 0$. Actually, while g'' always diverges, g' can be finite for certain compact fields (see below). Further, for fields vanishing asymptotically, $u_0 \to \infty$, also g can diverge; otherwise, e.g., for fields with compact support, g is finite for regular fields and the tunneling amplitude [i.e. the exponent in Eq. (5)] cannot contribute to criticality.

The resulting scaling laws can analytically be extracted by expanding f near the leading-order divergence of $1/f'$. Let us consider several paradigmatic examples, starting with localized fields that decay asymptotically with a power, $E(x) \sim \mathcal{E}[c/(kx)^p]$ as $x \to \infty$, with some constant c. Then, $f \approx 1 - [c/(p-1)](1/u^{p-1})$ and $u_0 \to \infty$. Depending on the power p, three different cases occur:

(I) for $p > 3$, the function g stays finite and the scaling law arises purely from the fluctuation prefactor, yielding a standard power-law

$$\text{Im}\,\Gamma \sim (1 - \gamma^2)^\beta, \qquad \beta = \frac{5p+1}{4(p-1)}. \qquad (9)$$

We emphasize that all field profiles with the same power-law decay exhibit the same universal critical scaling independently of the details of the profile at finite x (at least within the class of fields specified above). Equation (9) also includes exponentially decaying fields: in the limit $p \to \infty$, we discover a unique exponent $\beta = \frac{5}{4}$. This agrees, for instance, with the exact result [21] for the $\text{sech}^2 kx$ profile which in the regime (4) reads

$$\text{Im}\,\Gamma = \frac{L^2 T m^3}{2(2\pi)^3} \left(\frac{e\mathcal{E}}{m^2}\right)^{3/2} (1 - \gamma^2)^{5/4} e^{-(2\pi m^2/e\mathcal{E})}. \qquad (10)$$

We emphasize that the exponent $\beta = \frac{5}{4}$ also holds for other exponentially localized fields; different examples are shown in Fig. 1.

(II) The powerlaw decay $p = 3$ is special, since the function g itself diverges logarithmically which—upon insertion into Eq. (5)—produces a field-dependent power in addition to $\beta = 2$,
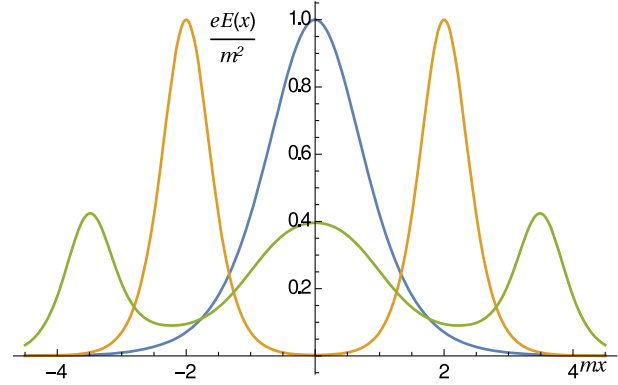

FIG. 1. Various examples for critical field profiles with exponent $\beta = \frac{5}{4}$. The onset of criticality is determined by the asymptotic behavior (exponential in these cases). The critical scaling law Eq. (9) is independent of the local details of the field profiles.

$$\text{Im}\,\Gamma \sim (1 - \gamma^2)^{2[1+\sqrt{c}(m^2/e\mathcal{E})]}. \qquad (11)$$

Since $e\mathcal{E}/m^2 \ll 1$, cf. Eq. (4), this field-dependent part dominates the exponent, indicating the approach to exponential scaling.

(III) The latter becomes manifest for a field decaying with $1 < p < 3$, since the instanton action $\sim g$ diverges as a power near criticality, resulting in the scaling law

$$\text{Im}\,\Gamma \sim (1 - \gamma^2)^\beta \exp\left(-\frac{\pi m^2}{e\mathcal{E}} \frac{C}{(1-\gamma^2)^\lambda}\right), \qquad (12)$$

where the *essential* exponent λ and the constant C,

$$\lambda = \frac{3-p}{2(p-1)}, \quad C = \frac{2}{\pi c}\left(\frac{2c}{p-1}\right)^{p/(p-1)} B\left(\frac{3}{2}, \frac{3-p}{2(p-1)}\right), \qquad (13)$$

are both universal. [For $p < 5/3$, the exponent in Eq. (12) can acquire universal subleading singularities e.g. $1/(1-\gamma^2)^{\lambda-1}$ or $\ln(1-\gamma^2)$.] In critical phenomena, a scaling of this type is known as essential scaling or (Berezinskii-Kosterlitz-Thouless) BKT (or Miransky) scaling [32]. It is known to occur in a wide range of systems, in particular those exhibiting a transition from a conformal to a nonconformal phase [33]. While our scaling law includes the BKT-scaling law with exponent $\lambda = \frac{1}{2}$ for an electric field decaying with power $p = 2$, any essential exponent $\lambda > 0$ can be realized for appropriate decay powers p. Equation (12) also has a universal powerlaw prefactor which is reminiscent to the many-flavor phase transition in gauge theories [34]. We also observe that λ diverges for $p \to 1$ where the electrostatic energy receives dominant contributions from long-range fluctuations. Essential scaling of critical pair production hence is obviously related to a dominance of electron-positron fluctuations at the largest length scales.

Let us now turn to electric fields of compact support in x direction. Within the class of fields considered here, this

implies that the potential function $f(u)$ attains its maximum at a finite value u_0. Correspondingly, $E(x) = 0$ for $|x| > u_0/k$. The worldline action (6) cannot become singular in this case, so the scaling law is solely determined by the fluctuation prefactor and thus by the way in which $f'(u)$ approaches zero for $u \to u_0$, cf. (8). Let us assume that the electric field drops to zero as $E(x) \sim (u_0/k - x)^n$. For $n > 1$, the order parameter satisfies power-law scaling (9) with exponent

$$\beta = \frac{5n - 1}{4(n + 1)}. \tag{14}$$

It is interesting to see that the universal exponent for exponential decay $\beta = \frac{5}{4}$ is rediscovered in the limit $n \to \infty$. Note also that (14) can be obtained by replacing $p \to -n$ in (9).

Another special case is $n = 1$, where we find a logarithmic divergence in the prefactor, $g' \sim -\ln(1 - \gamma^2)$, such that the scaling law receives log corrections

$$\mathrm{Im}\,\Gamma \sim \frac{(1 - \gamma^2)^{1/2}}{-\ln(1 - \gamma^2)}. \tag{15}$$

Again, this has an analog in statistical physics, as log corrections are known to arise in cases where marginal operators contribute to criticality [35], such as in the $2d$ 4-state Potts model [36]. For $n < 1$ only g'' diverges and we again find power-law scaling (9) with

$$\beta = \frac{3n + 1}{4(n + 1)}. \tag{16}$$

In the limit $n \to 0$, the electric field becomes steplike with exponent $\beta = \frac{1}{4}$.

The diversity of universality classes given above can be covered in a unified description using an implicit definition of the electric field in terms of differential equations. One such possible definition is

$$f' = (1 - f^2)^b, \tag{17}$$

with the implicit solution $u = f\,{}_2F_1(1/2, b, 3/2, f^2)$ in terms of a hypergeometric function. The case $b > 1$ covers all examples of asymptotically vanishing fields given above with power $p = b/(b - 1)$, and $b < 1$ corresponds to fields with compact support with $n = b/(1 - b)$. The cases $b = 1/2$, 1 and $3/2$ correspond to to the fields $E \sim \cos kx$, $\mathrm{sech}^2 kx$ and $(1 + kx^2)^{-3/2}$, respectively, studied explicitly in [27]. We have $g = {}_2F_1(1/2, b, 2, \gamma^2)$, and thus $g'' \sim (1 - \gamma^2)^{-(b+1/2)}$, cf. [37], which agrees with all the different scalings above.

This unified analysis also verifies the general trend of the relation between field profile and scaling: field profiles which are more spread out exhibit a steeper scaling. So, the order parameter $\mathrm{Im}\,\Gamma$ vanishes faster for an asymptotically decaying field than for a compact field. This is in agreement

with the Euclidean worldline picture, since the contributions from large-scale fluctuations (large propertimes) are suppressed by the electron mass scale.

The similarity of critical Schwinger pair production to critical phenomena discovered in this work appears to call for a renormalization group description. It is conceivable that the critical point corresponds to a fixed point of a suitable coarse-graining procedure involving the worldlines, the background field, or both. Such a description would be rewarding as it could give access to potential further aspects of criticality such as (hyper)scaling relations.

Our results can straightforwardly be generalized to field profiles with only asymptotic symmetry as well as to an arbitrary number of translation invariant transversal directions (y, z, t in the present work). As the latter only influences the propertime integrand, $1/s \to 1/s^{(d-2)/2}$, the order parameter receives additional scaling factors according to $\mathrm{Im}\,\Gamma_d \sim (\gamma^2 g)^{\prime(4-d)/2}\mathrm{Im}\,\Gamma_4$ with corresponding consequences for the scaling exponents.

From the underlying picture in terms of virtual fluctuations needing to acquire sufficient energy to become real, we expect that our results analogously persist also for static fields that are localized in more than one space dimension, even though the analysis can become rather involved for nonunidirectional fields [28]. For time dependencies slower than the Compton scale, the existence of a critical point $\gamma_{\mathrm{cr}} \simeq 1$ has been observed in [38,39]. We expect though a qualitative change for fields varying rapidly in time. Indeed, there are no critical points for fields depending on lightfront time $t + x$ [40,41]. As soon as the fields vary in time, pair production does not have to rely only on (instantaneous) electrostatic energy, but can also be supported by finite (multi)photon energies of the varying field. Hence, we expect the critical point in the spatial adiabaticity parameter to shift to values larger than $\gamma_{\mathrm{cr}} = 1$ for increasing time variations. This claim is supported by a recent paper [42] which shows that for electric fields depending on a coordinate q that interpolates continuously between x, $t + x$, and t, the critical point increases from $\gamma_{\mathrm{cr}} = 1$ at $q = x$ to infinity at $q = t + x$ (i.e. effectively vanishes); for timelike q there is no critical point.

For rapidly varying fields, pair production via multiphoton effects may remove any singularity associated with criticality. The would-be critical point due to the spatial profile may then still be visible as a cross-over in the production rate.

We emphasize that our results for universality hold in the regime defined by Eq. (4). We expect analogous features also in the deeply critical regime where $1 - \gamma^2 \ll (e\mathcal{E}/m^2)^2 \ll 1$, even though the values of the exponents might differ. For instance, the exact solution for the $\mathrm{sech}^2 kx$ case scales differently in this regime, $\mathrm{Im}\,\Gamma \sim (1 - \gamma^2)^3$ [21]. Determining the degree of universality in this regime remains an interesting problem. As the electron mass scale is less dominant, universality could even be substantially enhanced.

Radiative corrections will also take a subleading quantitative influence on our results. For example, the two-loop correction includes mass-shift effects [43,44]; a resummation could also account for a production into a positronium bound state. As these effects modify the invariant mass of the final state, they may primarily lead to a deviation of the critical point $\gamma_{cr} \simeq 1$ but could preserve the universal critical exponents.

In summary, we have discovered an analogy between Schwinger pair production and continuous phase transitions. This analogy is quantitatively manifest in universal scaling laws for the onset of pair production in spatially inhomogeneous electric backgrounds. The scaling laws show a high degree of universality as the corresponding critical exponents only depend on the large-scale properties of the background (for monotonic potentials) but become insensitive to the microscopic details. Hence, localized electric backgrounds fall into universality classes each giving rise to a characteristic scaling law. As a particularly fascinating aspect, we discovered universality classes covering essentially all types of scaling laws familiar from continuous phase transitions.

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