Minimal noise subsystems

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A system subjected to noise contains a decoherence-free subspace or subsystem (DFS) only if the noise possesses an exact symmetry. Here we consider noise models in which a perturbation breaks a symmetry of the noise, so that if $\mathcal{S}$ is a DFS under a given noise process it is no longer so under the new perturbed noise process. We ask whether there is a subspace or subsystem that is more robust to the perturbed noise than $\mathcal{S}$. To answer this question we develop a numerical method that allows us to search for subspaces or subsystems that are maximally robust to arbitrary noise processes. We apply this method to a number of examples, and find that a subsystem that is a DFS is often not the subsystem that experiences minimal noise when the symmetry of the noise is broken by a perturbation. We discuss which classes of noise have this property.

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Introduction.—Techniques to reduce and correct errors are crucial for realizing scalable and fault-tolerant quantum information processing [1]. The key technique that enables noise reduction is the encoding of information in a way that includes redundancy. In *quantum error-correction codes* (QECC) [2–8] the encoded information is still affected by the noise, but errors can be detected and corrected by exploiting the redundancy. If the noise contains an appropriate symmetry then redundancy can be used to eliminate the noise entirely by encoding in a so-called decoherence free subspace or subsystem (DFS) [9–19]. Necessary and sufficient conditions for the existence of a DFS have been derived [10,20], as have numerical methods for finding DFS structures [21–23]. Moreover QECC and DFS's can be combined to implement fault-tolerant quantum computation [20]. Compared with QECC the implementation of a DFS is simpler and can save computational resources, but it is limited to noise that contains one or more symmetries, and this is often absent in real devices. In many cases a real noise process can be considered as a slight deviation from noise with a symmetry, and the corresponding DFS encoding is still useful for noise reduction. Here we address two open questions regarding such realistic noise. The first is, given an arbitrary noise process, how can one find a subspace or subsystem that is least affected by this noise? We will refer to such a subspace or subsystem as a *minimal-noise subspace or subsystem* (MNS). Secondly, if a subspace or subsystem $\mathcal{H}_0$ is a DFS for a given noise source $s(t)$, is $\mathcal{H}_0$ also the MNS for noise which deviates slightly from $s(t)$? We address these questions by developing a numerical method to search for minimal noise subspaces and subsystems.

Existence of a DFS.—All quantum systems $\mathcal{S}$ are subject to noise from their environments, and as a result their evolution is not unitary. Under the Born-Markov approximation, the reduced dynamics, excluding the evolution due to the Hamiltonian of the system, H, is given by $\dot{\rho} = \sum_i \mathcal{D}[V_i]\rho$, where $\mathcal{D}[V]\rho = V\rho V^\dagger - \frac{1}{2}(\rho V^\dagger V + V^\dagger V\rho)$. The operators V_i are called Lindblad operators and they characterize the noise source(s). The above dynamics is unitary if and only if $\sum_i \mathcal{D}[V_i]\rho = 0$, which is equivalent to $[V_i,\rho] = 0$ for each V_i [20]. Thus a DFS is a subspace or a subsystem $\mathcal{H}_0$ such that $[V_i,\rho] = 0$ for any $\rho \in \mathcal{H}_0$. Another way of describing noise is the operator-sum representation:

$$\mathcal{E}\rho \equiv \sum_{k=1}^{p} E_k \rho E_k^\dagger, \tag{1}$$

where the quantum channel $\mathcal{E}\colon \rho \to \mathcal{E}\rho$ is characterized by a set of noise operators $\{E_k\}$ satisfying $\sum_k E_k^\dagger E_k = \mathbb{I}$. A space $\mathcal{H}_0$ is a DFS if and only if $[E_k,\rho] = 0$ for each k and for any ρ in $\mathcal{H}_0$ [20]. By working in the interaction picture we can set $H = 0$, and this allows us to focus on the effect of the noise.

The relationship between the operators $\{E_k\}$ and the existence of a DFS is neatly characterized by the Wedderburn decomposition for block diagonalization [24–26]:

$$\mathcal{N} = \bigoplus_i^{\ell} \mathcal{N}_i \equiv \bigoplus_i^{\ell} \mathbb{I}_{n_i} \otimes \mathcal{M}_{m_i}, \tag{2a}$$

$$\mathcal{N}' = \bigoplus_i^{\ell} \mathcal{N}_i{}' \equiv \bigoplus_i^{\ell} \mathcal{M}_{n_i} \otimes \mathbb{I}_{m_i}. \tag{2b}$$

Here, $\mathcal{N}$ is the C^*-algebra generated by $\{E_k\}$, and $\mathcal{N}'$ is its commutant algebra [27]. The indices n_i and m_i represent dimensions: $\mathcal{M}_{n_i}$ denotes the C^* algebra of $n_i \times n_i$ matrices and $\mathbb{I}_{m_i}$ is the identity operator with dimension m_i [29,30]. Hence, any subsystem $\mathcal{M}_{n_i} \otimes \mathbb{I}_{m_i}$ with $n_i > 1$ corresponds to a DFS that can encode an n_i-dimensional quantum state ρ_{n_i} into $\rho^{(en)} = \rho_{n_i} \otimes (1/m_i)\mathbb{I}_{m_i}$. Note that Eq. (2) is valid for any N-dimensional system $\mathcal{S}$, including a collection of qubits or qudits.

Minimal noise subsystems.—Let us encode a state $\rho_1 = |\psi\rangle\langle\psi|$ of an N_1 dimensional system in another system $\mathcal{S}$ with dimension $N \geq N_1$. We can always write the state that encodes ρ_1 in the form $\rho = \rho_1 \otimes (1/N_2)\mathbb{I}_{N_2} \oplus 0_{N_3}$. Here $\mathbb{I}_{N_2}$ is the identity operator with dimension N_2, 0_{N_3} is the zero operator with dimension N_3, and $N = N_1 N_2 + N_3$. Here the Hilbert space $\mathcal{H}$ of $\mathcal{S}$ has been decomposed as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \oplus \mathcal{H}_3$ so that H_1 contains the state $|\psi\rangle$. If $N_2 = 0$ then ρ_1 is encoded purely in a subspace of $\mathcal{S}$, and if $N_3 = 0$ then it is encoded purely in a subsystem of $\mathcal{S}$.

We now note that every encoding of $|\psi\rangle$ with fixed values of N_2 and N_3 can be obtained by applying a unitary operator U to the state ρ as defined above. We thus wish to find the operator U for which the encoded state $|\psi\rangle$ experiences the least disturbance under the noise process in Eq. (1). The resulting U will give us an MNS encoding, and we will call U the encoding matrix.

To use a numerical search method to find the U that gives an MNS we need to obtain an explicit expression for the disturbance to the encoded state $|\psi\rangle$ as a function of U, and in doing so we will need to precisely quantify this disturbance. To obtain the desired expression we first note that under the transformation U, $\mathcal{E}\rho$ becomes

$$U\mathcal{E}\rho U^\dagger = \sum_{k=1}^{p} (UE_k U^\dagger)(U\rho U^\dagger)(UE_k U^\dagger)^\dagger = \bar{\mathcal{E}}\,\bar{\rho}, \quad (3)$$

where we have defined $\bar{\rho} = U\rho U^\dagger$. The reduced evolution of $|\psi\rangle$ induced by $\mathcal{E}$ is

$$\mathcal{E}_1\rho_1 = \bar{\mathcal{E}}\,\bar{\rho}\,|_{\mathcal{H}_1} \equiv \mathrm{Tr}_2(P\bar{\mathcal{E}}\,\bar{\rho}\,P) \quad (4)$$

where P is the projection operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and Tr_2 denotes the partial trace over $\mathcal{H}_2$. Now the action of $\mathcal{E}_1$ on ρ_1 can alternatively be written in the standard operator-sum representation as

$$\mathcal{E}_1\rho_1 = \sum_{j=1}^{\ell} A_j \rho_1 A_j^\dagger = p_1\rho_1 + \sum_{j=2}^{\ell} A_j \rho_1 A_j^\dagger \quad (5)$$

where $A_1 = \sqrt{p_1}\mathbb{I}_{N_1}$ and $0 < p_1 \leq 1$. Notice that p_1 is the probability that the state $|\psi\rangle$ remains undisturbed, and thus characterizes the ability of the coding scheme to protect $|\psi\rangle$ from the noise. If $p_1 = 1$ then $\mathcal{H}_1 \otimes \mathcal{H}_2$ corresponds to a perfect DFS encoding. We therefore define the MNS as that given by the choice of N_2, N_3, and U that gives the largest value of p_1. To obtain an explicit expression for p_1 in terms of E_k and U we proceed as follows. First, Eq. (4) gives

$$\mathcal{E}_1\rho_1 = \mathrm{Tr}_2\left[\sum_{k=1}^{p}(PUE_k U^\dagger P)\rho(PUE_k^\dagger U^\dagger P)\right] = \frac{1}{N_2}\sum_{k=1}^{p}\sum_{mm'n} a_{mn}^{(k)} a_{m'n}^{*(k)} \sigma_m^{(1)} \rho_1 \sigma_{m'}^{(1)} \quad (6)$$

where each $PUE_k U^\dagger P$ is decomposed into $a_{mn}^{(k)}\sigma_m^{(1)}\sigma_n^{(2)}$. The set of operators $\{\sigma_m^{(j)}\}$ is a generalized orthonormal Pauli basis for Hermitian operators on $\mathcal{H}_j$, $j = 1, 2$, and $\sigma_0^{(j)} = \mathbb{I}_{N_j}/\sqrt{N_j}$. Equating Eqs. (5) and (6) results in the expression we seek:

$$p_1 = \frac{1}{N_1 N_2}\sum_{k=1}^{p}\sum_{n=1}^{N_2^2} |a_{0n}^{(k)}|^2, \quad (7)$$

where $a_{0n}^{(k)} = \mathrm{Tr}[PUE_k U^\dagger P \sigma_0^{(1)}\sigma_n^{(2)}]$. We will denote the function in Eq. (7) that maps U to p_1 by $p_1 = \mathcal{L}(U)$.

Numerical method.—The analysis above provides the following numerical method for finding one or more MNS's. For a given problem, defined by N and the noise operators $\{E_k\}$, we first enumerate all the pairs (N_1, N_2) for which $N_1 N_2 \leq N$, and then for each of these pairs we use a gradient search method to search for unitaries U that maximize the function $p_1 = \mathcal{L}(U)$. The triple(s) (N_1, N_2, U) for which p_1 is maximal give the MNS('s).

A gradient search method performs a search over a space defined by a set of real parameters, and so in our case this will be the space of N^2 real parameters that define an N-dimensional unitary matrix U. For this purpose we use the parametrization devised in [31] in which U is written in terms of $\frac{1}{2}N(N+1)$ phase variables $\{\phi_n\}$ and $\frac{1}{2}N(N-1)$ angle variables $\{\theta_k\}$. We can thus write $p_1 = \mathcal{J}(v) = \mathcal{L}[U(\mathbf{v})]$ where the N^2 elements of the vector $\mathbf{v}$ are the parameters. The search runs over all values of the parameters and is thus "unconstrained." Two popular gradient search methods are the conjugate gradient (CG) and the quasi-Newton Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithms [32]. The CG method requires fewer evaluations of the function $\mathcal{J}(\mathbf{v})$ per iteration while BFGS usually requires fewer iterations to converge to a solution. When the objective function is relatively inexpensive to calculate, which is true in our case, the BFGS method is usually faster and so we use it here.

The gradient search must start at some point in the parameter space, and this point is usually chosen at random. The values of p_1 obtained by the search are only guaranteed to be locally maximal. To account for this the search can be performed multiple times, each time starting at a different random location. In this way one collects a set of locally maximal values of p_1. When enough searches have been performed that new searches provide no new local maxima, we obtain some confidence that all the local maxima have been enumerated. In this case the global maxima are simply

TABLE I. Algorithm to search for an MNS.

Step 1:	(a) choose N_1 and N_2 for the encoding subsystem;
	(b) parametrize $U[\boldsymbol{\alpha}] = U(\alpha_1, \ldots, \alpha_{N^2})$;
	(c) express $\mathcal{J}$ in terms of $\boldsymbol{\alpha}$;
Step 2:	(d) choose a random $\boldsymbol{\alpha}^{(0)}$ as the initial point;
	(e) at the kth iteration, BFGS method gives $\mathcal{J}^{(k)}$;
	(f) $\{\mathcal{J}^{(k)}\}$ converges to an optimal value $\mathcal{J}_{\rm opt}$;
Step 3:	(g) repeat Step 1 and Step 2 for other N_1 and N_2.

those local maxima that all achieve the highest value of p_1. The entire algorithm for locating MNS's is summarized in Table I.

Performance of an MNS.—While we have defined the MNS as the encoding that preserves the encoded state with the highest probability, we can also ask to what extent a state is disturbed by the noise when it is encoded in the MNS. Such an average disturbance can be thought of as the performance of the MNS. There are many ways to quantify disturbance and we choose here to employ the fidelity between the state that is initially encoded, $|\psi\rangle$, and the encoded state after the noise has acted, which we will denote by σ [33–37]. Because the initial encoded state is pure, the fidelity reduces to the simple form $F = \langle\psi|\sigma|\psi\rangle = {\rm Tr}[\rho_1\sigma]$ with $\rho_1 = |\psi\rangle\langle\psi|$. We characterize the performance of the MNS as this fidelity F minimized over all pure states that could be encoded. Given an arbitrary density matrix w for system $\mathcal{S}$, let us define the operation $\mathcal{O}_{\rm en}$ by $\mathcal{O}_{\rm en}w = UwU^\dagger$ and its inverse by $\mathcal{O}_{\rm de}w = U^\dagger wU$. Note that $\mathcal{O}_{\rm en}$ is used to encode the state $|\psi\rangle$ and $\mathcal{O}_{\rm de}$ is used to decode it. With these definitions we can write the performance of an MNS as the *minimum fidelity*:

$$F_{\rm min} = \min_{|\psi\rangle}{\rm Tr}[\rho\mathcal{O}_{\rm de}\mathcal{E}\mathcal{O}_{\rm en}\rho], \tag{8}$$

where $\rho = |\psi\rangle\langle\psi| \otimes (1/N_2)\mathbb{I}_{N_2} \oplus 0_{N_3}$. The larger $F_{\rm min}$ the better the performance of the MNS. If $F_{\rm min} = 1$ the MNS is a DFS, giving perfect protection from the noise.

Applying the procedure to Lindblad evolution—As mentioned in the Introduction, noisy Markovian quantum dynamics is often expressed in terms of the Lindblad master equation. Since our numerical algorithm is based on the operator-sum representation, Eq. (1), we must translate from the Lindblad operators to this representation. Within an infinitesimal time-step dt, the Lindblad dynamics $\dot{\rho} = \sum_i \mathcal{D}[V_i]\rho$ is equivalent to

$$\mathcal{E}[\rho(0)] = \rho(dt) = \sum_k E_k\rho(0)E_k^\dagger \tag{9}$$

where $E_0 = \mathbb{I} - \frac{1}{2}\sum_i V_i^\dagger V_i dt$, $E_k = \sqrt{dt}V_k$, $k \geq 1$. This allows us to apply the numerical algorithm to any Lindblad master equation.

Finding a DFS.—As a test of our algorithm we apply it to a noise model under which the system contains a DFS, since in this case the MNS will coincide with this DFS. Notice that our algorithm requires no prior information of the Wedderburn decomposition, Eq. (2), so it is distinct from the previous methods given in [21,23]. We choose as our example an n_q-qubit system $\mathcal{S}_{cn}$ governed by the following dynamics:

$$\dot{\rho} = \gamma_x\mathcal{D}[S_x]\rho + \gamma_z\mathcal{D}[S_z]\rho \tag{10}$$

with $S_x = \sum_{k=1}^{n_q} X_k$, $S_z = \sum_{k=1}^{n_q} Z_k$ and decoherence rates $\gamma_{x,z}$. As shown above, we can rewrite the Lindblad evolution of $\rho(dt)$ in the operator-sum representation in which the operators are $E_0 = \mathbb{I} - E_1^2/2 - E_2^2/2$, $E_1 = \sqrt{dt}S_x$, and $E_2 = \sqrt{dt}S_z$. This system has a DFS, and the DFS structure is illustrated by the Wedderburn decomposition [20,38]. For example, for $n_q = 3$, the noise algebra $\mathcal{N}$ and its commutant $\mathcal{N}'$ are

$$\mathcal{N} = (\mathbb{I}_2 \otimes \mathcal{M}_2) \oplus \mathcal{M}_4, \qquad \mathcal{N}' = (\mathcal{M}_2 \otimes \mathbb{I}_2) \oplus \mathbb{I}_4. \tag{11}$$

The component $\mathcal{M}_2 \otimes \mathbb{I}_2$ corresponds to a DFS that can store one qubit of information. We now apply our algorithm to find this DFS. The encoded state has the form $\rho = (\rho_1 \otimes \mathbb{I}_2/2) \oplus 0_4$ and we denote the encoding matrix by U. The function to maximize is then

$$\mathcal{J}(U) = \frac{1}{8}\sum_{k=1}^{3}\sum_{n=1}^{4}|{\rm Tr}(PUE_kU^\dagger P\sigma_n^{(2)})|^2. \tag{12}$$

Choosing a random initial matrix for U, and following the algorithm in Table I, we obtain an encoding matrix that indeed encodes in the above DFS. Since there are many different matrices U that encode in the same subspace or subsystem, each run of the algorithm gives a different U that encodes in this DFS.

Noise with symmetry-breaking perturbations.—As mentioned earlier, symmetry is crucial for the existence of a DFS, so when the noise model has no symmetry, the above algorithm is the only way to find the best subspace or subsystem encoding. However, if the noise model is highly asymmetric our results indicate that no MNS's provide significantly reduced noise, and this is not unexpected. Nevertheless, if the noise model can be considered to be a perturbation of a symmetric noise model, the DFS for the symmetric model will provide a relatively good encoding scheme for the perturbed noise model. Our primary question is whether there exists an MNS (for the perturbed noise) that can provide a better encoding (under the perturbed noise) than that provided by the subspace or subsystem that is the DFS for the symmetric noise. From now on when we refer to "the original DFS" we will mean the subspace or subsystem that is the DFS under the symmetric noise model (but that experiences noise under the perturbed noise model).

As our first example we consider an n_q-qubit system under the collective noise model $\dot{\rho} = \gamma_z \mathcal{D}[S_z]\rho$, which applies to trapped-ions [39,40]. To break the symmetry we add local dephasing noise for each qubit to give the perturbed noise model $\dot{\rho} = \gamma_z \mathcal{D}[S_z]\rho + \delta \sum_k \gamma_k \mathcal{D}[Z_k]\rho$, where δ is the small parameter. For $n_q = 3$, without the local noise terms the system has two DFS's that are generated by $\mathcal{H}_a = \mathrm{span}\{|001\rangle, |010\rangle, |100\rangle\}$ and $\mathcal{H}_b = \mathrm{span}\{|101\rangle, |110\rangle, |011\rangle\}$, and they can be used to encode two independent qutrits. When the local noise is included the collective symmetry is broken, there is no DFS, and we can apply the MNS algorithm to find the least-noise encoding scheme to encode a qutrit or a qubit. To encode a qutrit, we choose $\rho = (|\psi\rangle\langle\psi| \otimes \mathbb{1}_1) \oplus 0_5$ for the MNS algorithm. After the optimization routine in Table I, we find that the MNS is either $\mathcal{H}_a$ or $\mathcal{H}_b$. Similarly we search for an MNS encoding for a qubit using $\rho = (|\psi\rangle\langle\psi| \otimes \mathbb{1}_1) \oplus 0_6$, and the MNS found is a 2-D subspace of either $\mathcal{H}_a$ or $\mathcal{H}_b$, depending on the value of γ_k. For instance, for randomly chosen values, $\gamma_1 = 0.33, \gamma_2 = 0.47, \gamma_3 = 0.85$, we find the 2-D MNS is always a subspace of $\mathcal{H}_b$. Thus the MNS's correspond to the original DFS's.

As our second example we take the collective noise model in Eq. (10) and perturb it again by the local noise in the example above. Again we find that the MNS is the same as the DFS for the unperturbed noise as long as the perturbation amplitude δ is sufficiently small. These examples indicate that for symmetric noise perturbed by strictly local noise there is no better encoding than the original DFS.

As our third and fourth examples we consider a symmetric noise model in which one of the collective Lindblad operators is perturbed by (i) a randomly chosen global unitary, and (ii) a unitary that is the tensor product of single-qubit, independently selected random unitaries. In this case the noise remains collective, in that there is a single noise channel, but the symmetry is broken so that there is no longer a DFS. For the symmetric noise we use the noise model in Eq. (10). The noisy dynamics of the perturbed model we take as $\dot{\rho} = \gamma_1 \mathcal{D}[V_\epsilon S_x V_\epsilon^\dagger]\rho + \gamma_2 \mathcal{D}[S_z]\rho$, where V_ϵ represents the random global unitary perturbation satisfying $\|V_\epsilon - \mathbb{1}\| = \epsilon$, with ϵ the small parameter. One way to generate V_ϵ is to parametrize it using $\frac{1}{2}N(N+1)$ phase variables and $\frac{1}{2}N(N-1)$ angular variables. In this case we can set the phase variables to zero and choose the angle variables so that the sum of their squares is a small parameter δ. In this case δ can be used as the perturbative parameter since ϵ increases monotonically with δ and $V_\epsilon = \mathbb{1}$ when $\delta = 0$. We set $\gamma_1 = \gamma_2 = \gamma$ because we do not expect the relative values of γ_k to affect the existence of an MNS, and apply our algorithm to find, as a function of δ, the optimal encoding matrix U_{MNS}^δ along with its performance as characterized by F_{min}. For $\delta = 0$ we find that $F_{\mathrm{min}} = 1$ and the MNS is merely the original DFS. For nonzero δ, however, we find that the MNS is no longer equal to the original DFS. Further, the performance of the MNS for $\delta > 0$ is strictly better than that of the original DFS when subjected to the perturbed noise. In Fig. 1 we display and compare the minimum fidelity for the MNS to that for the original DFS subjected to the perturbed noise. As δ tends to zero the performances of the MNS and DFS both have a flat plateau, confirming that the original DFS encoding is robust against perturbations [41]. As δ increases the difference between the performance of the MNS and the original DFS increases, showing that the original DFS encoding becomes increasingly less optimal as the perturbation increases. Analyzing case (ii) in which V_ϵ is a tensor product of independently selected local random unitaries, we find the same behavior as for the global random unitary. However in this case there is less difference between the performance of the MNS and that of the original DFS. These results are shown in Fig. 2.

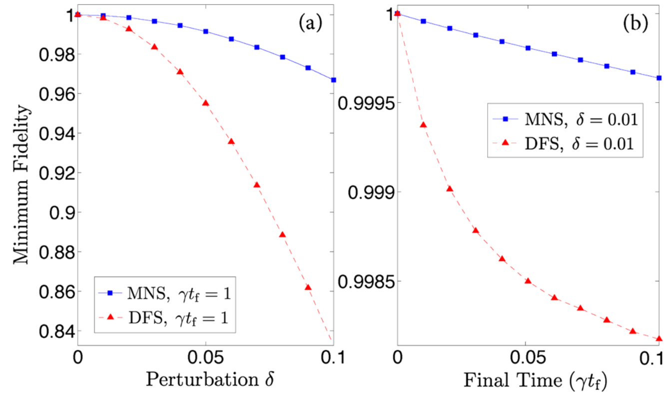

FIG. 1. Minimum fidelity curves for two different encodings U_{MNS} and U_{DFS} for a system subjected to noise whose symmetry is perturbed by a global random unitary: (a) $\delta \in [0, 0.1]$ and $\gamma t_f = 1$; (b) $\gamma t_f \in [0, 0.1]$ and $\delta = 0.01$ (the parameters δ, γ, and t_f are defined in the text).

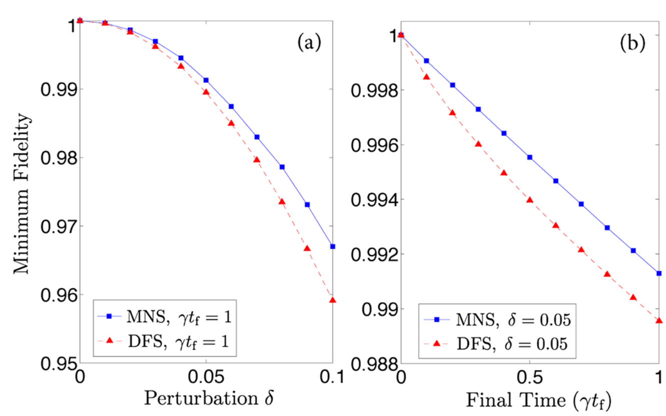

FIG. 2. Minimum fidelity curves for two different encodings U_{MNS} and U_{DFS} for a system subjected to noise whose symmetry is perturbed by local random unitaries: (a) $\delta \in [0, 0.1]$ and $\gamma t_f = 1$; (b) $\gamma t_f \in [0, 0.1]$ and $\delta = 0.05$ (the parameters δ, γ, and t_f are defined in the text).

Conclusion.—The above examples illustrate the ability of our numerical method to find both decoherence-free and minimal-noise subsystems or subspaces given a set of noise operators $\{E_k\}$. It is important to note that this is true even when the symmetry is not exact. In the examples we have examined, when a collective noise model is perturbed by noise that is local to each subsystem, the minimal-noise subsystem is merely the DFS for the unperturbed system. However, when a collective noise model is perturbed by a random unitary transformation, while no DFS exists there is a minimal-noise subsystem that is distinct from the DFS for the unperturbed system, providing an improvement over known methods of identification.

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